INNER PRODUCT INEQUALITIES FOR THE ČEBYŠEV FUNCTIONAL IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For two continuous functions $f,\ g:[a,b]\rightarrow H$ we define the *Čebyšev functional*

$$D\left(f,g\right):=\left(b-a\right)\int_{a}^{b}\left\langle f\left(t\right),g\left(t\right)\right\rangle dt-\left\langle \int_{a}^{b}f\left(t\right)dt,\int_{a}^{b}g\left(t\right)dt\right\rangle .$$

In this paper we show among others that if $f,\ g:[a,b]\rightarrow H$ are strongly differentiable functions on the interval (a,b), then

$$\left\vert D\left(f,g\right)\right\vert \leq \frac{1}{4}\left(b-a\right)^{2}\times\left\{ \begin{array}{l} \left\Vert f^{\prime}\right\Vert_{[a,b],1}\left\Vert g^{\prime}\right\Vert_{[a,b],1}, \\ \\ \frac{1}{3}\left(b-a\right)^{2}\left\Vert f^{\prime}\right\Vert_{[a,b],\infty}\left\Vert g^{\prime}\right\Vert_{[a,b],\infty}, \\ \\ \frac{1}{2}\left(b-a\right)\left\Vert f^{\prime}\right\Vert_{[a,b],1}\left\Vert g^{\prime}\right\Vert_{[a,b],\infty}, \end{array}\right.$$

where $\left\Vert h^{\prime}\right\Vert_{[a,b],1}:=\int_{a}^{b}\left\Vert h^{\prime}\left(u\right)\right\Vert du$ and $\left\Vert h^{\prime}\right\Vert_{[a,b],\infty}:=\sup_{t\in(a,b)}\left\Vert h^{\prime}\left(u\right)\right\Vert$ for a strongly differentiable function h on (a,b). Some applications for operator monotone functions with examples for power and logarithmic functions are also given.

1. Introduction

For two Lebesgue integrable functions $f,\ g:[a,b]\rightarrow\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D\left(f,g\right):=\left(b-a\right)\int_{a}^{b}f\left(t\right)g\left(t\right)dt-\int_{a}^{b}f\left(t\right)dt\int_{a}^{b}g\left(t\right)dt.$$

In 1934, G. Grüss [15] showed that

$$\left\vert D\left(f,g\right)\right\vert \leq\frac{1}{4}\left(b-a\right)^{2}\left(M-m\right)\left(N-n\right), \tag{1.1}$$

provided $m,\ M,\ n,\ N$ are real numbers with the property that

$$-\infty<m\leq f\leq M<\infty,\quad -\infty<n\leq g\leq N<\infty\quad\text{a.e. on}\quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $D\left(f,g\right)$ was derived in 1882 by Čebyšev [5] under the assumption that $f^{\prime},\ g^{\prime}$ exist and are continuous on $[a,b]$, and is given by

$$\left\vert D\left(f,g\right)\right\vert \leq\frac{1}{12}\left\Vert f^{\prime}\right\Vert_{\infty}\left\Vert g^{\prime}\right\Vert_{\infty}\left(b-a\right)^{4}, \tag{1.3}$$

where $\left\Vert f^{\prime}\right\Vert_{\infty}:=\sup_{t\in[a,b]}\left\vert f^{\prime}\left(t\right)\right\vert<\infty.$

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The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $f,\ g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f',\ g' \in L_\infty [a,b]$.

In 1970, A. M. Ostrowski [18] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(f,g)| \leq \frac{1}{8} (b-a)^3 (M-m) \|g'\|_\infty , \tag{1.4}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_\infty [a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [16] (see also [17, p. 210]) obtained the following result as well:

$$|D(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a)^3 , \tag{1.5}$$

provided $f,\ g$ are absolutely continuous and $f',\ g' \in L_2 [a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [2], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(f,g)| \leq (b-a) \times \begin{cases} \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt, \\ \\ \inf_{\gamma \in \mathbb{R}} \|g-\gamma\|_q \left(\int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}} \\ \qquad \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(f,g)| \leq (b-a) \|g\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_\infty \leq \frac{1}{2} (M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [4]

$$|D(f,g)| \leq \frac{1}{2} (b-a) (M-m) \int_a^b \left| f(t) - \frac{1}{b-a} \int_a^b f(s)\,ds \right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [3].

The following result holds [14].

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(f,g)| \leq \frac{1}{2} (b-a) \bigvee_a^b (f) \int_a^b \left| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right| dt, \tag{1.9}$$

where $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{2}$ is best possible in (1.9).

For more recent upper bounds related to the Čebyšev functional see [2], [3] and [9]-[14].

An extension of this classical result to real or complex inner product spaces has been obtained by S S. Dragomir in [6]:

Theorem 2. *Let $(H;\langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$ and $e\in H$, $\|e\|=1$. If φ, ϕ, γ, $\Gamma\in\mathbb{C}$ and x, $y\in H$ are such that*

$$\operatorname{Re}\langle \phi e-x,x-\varphi e\rangle \geq 0 \quad \text{and} \quad \operatorname{Re}\langle \Gamma e-y,y-\gamma e\rangle\geq 0 \tag{1.10}$$

or, equivalently (see [8])

$$\left\|x-\frac{\varphi+\phi}{2}e\right\|\leq\frac{1}{2}|\phi-\varphi| \quad \text{and} \quad \left\|y-\frac{\gamma+\Gamma}{2}e\right\|\leq\frac{1}{2}|\Gamma-\gamma|, \tag{1.11}$$

then

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leq\frac{1}{4}|\phi-\varphi||\Gamma-\gamma|. \tag{1.12}$$

The constant $\frac{1}{4}$ is best possible in (1.12).

A further extension for Bochner integrals of vector-valued functions in real or complex Hilbert spaces was obtained by S. S. Dragomir in 2001, [7].

Theorem 3. *Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex Hilbert space, $\Omega\subset\mathbb{R}^n$ be a Lebesgue measurable set and $\rho:\Omega\to[0,\infty)$ a Lebesgue measurable function with $\int_\Omega\rho(s)\,ds=1$. We denote by $L_{2,\rho}(\Omega,H)$ the set of all Bochner measurable functions f on Ω such that $\|f\|_{2,\rho}^2:=\int_\Omega\rho(s)\|f(s)\|^2\,ds<\infty$. If f, g belong to $L_{2,\rho}(\Omega,H)$ and there exist the vectors x, X, y, $Y\in H$ such that*

$$\begin{aligned}&\int_\Omega\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle\,dt\geq 0,\\ &\int_\Omega\rho(t)\operatorname{Re}\langle Y-g(t),g(t)-y\rangle\,dt\geq 0,\end{aligned} \tag{1.13}$$

then we have the inequality

$$\begin{aligned}&\left|\int_\Omega\rho(t)\langle f(t),g(t)\rangle\,dt-\left\langle\int_\Omega\rho(t)f(t)\,dt,\int_\Omega\rho(t)g(t)\,dt\right\rangle\right|\\ &\leq\frac{1}{4}\|X-x\|\|Y-y\|.\end{aligned} \tag{1.14}$$

The constant $\frac{1}{4}$ is sharp in the sense mentioned above.

Remark 1. *A practical sufficient condition for (1.13) to hold is*

$$\operatorname{Re}\langle X-f(t),f(t)-x\rangle\geq 0,\ \operatorname{Re}\langle Y-g(t),g(t)-y\rangle\geq 0$$

or, equivalently

$$\left\|f(t)-\frac{X+x}{2}\right\|\leq\frac{1}{2}\|X-x\| \quad \text{and} \quad \left\|g(t)-\frac{Y+y}{2}\right\|\leq\frac{1}{2}\|Y-y\|,$$

for a.e. $t\in\Omega$.

An interesting particular inequality that has not been mentioned in [7] can be obtained by considering $H=\mathbb{C}$, $\langle x,y\rangle:=x\cdot\bar{y}$ and $g=\bar{f}$, to give

$$\left|\int_\Omega\rho(s)f^2(s)\,ds-\left(\int_\Omega\rho(s)f(s)\,ds\right)^2\right|\leq\frac{1}{4}|A-a|^2, \tag{1.15}$$

provided

$$\int_{\Omega}\rho(s)\operatorname{Re}\left[(A-f(s))\left(\overline{f(s)}-\overline{a}\right)\right]ds\geq 0 \tag{1.16}$$

or, sufficiently,

$$\operatorname{Re}\left[(A-f(s))\left(\overline{f(s)}-\overline{a}\right)\right]\geq 0 \tag{1.17}$$

for a.e. $s\in\Omega$.

Note that the alternative result

$$0\leq\int_{\Omega}\rho(s)|f(s)|^{2}ds-\left|\int_{\Omega}\rho(s)f(s)ds\right|^{2}\leq\frac{1}{4}|A-a|^{2}, \tag{1.18}$$

provided (1.16) or (1.17) hold, has been stated in [7].

Let H be a complex Hilbert space. For two continuous functions $f,g:[a,b]\to H$ we define the *Čebyšev functional*

$$D(f,g):=(b-a)\int_{a}^{b}\langle f(t),g(t)\rangle dt-\left\langle\int_{a}^{b}f(t)dt,\int_{a}^{b}g(t)dt\right\rangle.$$

In this paper we show among other that if $f,g:[a,b]\to H$ are strongly differentiable functions on the interval (a,b), then

$$|D(f,g)|\leq\frac{1}{4}(b-a)^{2}\times\begin{cases}\|f'\|_{[a,b],1}\|g'\|_{[a,b],1}, \\ \\ \frac{1}{3}(b-a)^{2}\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\infty}, \\ \\ \frac{1}{2}(b-a)\|f'\|_{[a,b],1}\|g'\|_{[a,b],\infty},\end{cases}$$

where $\|h'\|_{[a,b],1}:=\int_{a}^{b}\|h'(u)\|du$ and $\|h'\|_{[a,b],\infty}:=\sup_{t\in(a,b)}\|h'(u)\|$ for a strongly differentiable function h on (a,b). Some applications for operator monotone function with examples for power and logarithmic functions are also given.

2. Main Results

We have the following result of interest:

Theorem 4. *Let $f,g:[a,b]\to H$ be a strongly differentiable functions on the interval (a,b). Then*

$$\begin{aligned}|D(f,g)|&\leq D\left(\int_{a}^{\cdot}\|f'(u)\|du,\int_{a}^{\cdot}\|g'(u)\|du\right)\\&\leq\frac{1}{4}(b-a)^{2}\|f'\|_{[a,b],1}\|g'\|_{[a,b],1},\end{aligned} \tag{2.1}$$

where $\|h'\|_{[a,b],1}:=\int_{a}^{b}\|h'(u)\|du$.

Proof. Observe that

$$\begin{aligned}
&\int_a^b \int_a^b \langle f(t) - f(s), g(t) - g(s) \rangle \, dtds \\
&= \int_a^b \int_a^b \left(\langle f(t), g(t) \rangle - \langle f(s), g(t) \rangle - \langle f(t), g(s) \rangle + \langle f(s), g(s) \rangle \right) dtds \\
&= (b-a) \int_a^b \langle f(t), g(t) \rangle \, dt - \left\langle \int_a^b f(s) \, ds, \int_a^b g(t) \, dt \right\rangle \\
&- \left\langle \int_a^b f(t) \, dt, \int_a^b g(s) \, ds \right\rangle + (b-a) \int_a^b \langle f(s), g(s) \rangle \, ds \\
&= 2(b-a) \int_a^b \langle f(t), g(t) \rangle \, dt - 2 \left\langle \int_a^b f(t) \, dt, \int_a^b g(t) \, dt \right\rangle = 2D(f,g),
\end{aligned}$$

which give the Korkine's identity for functions with values in Hilbert spaces:

$$D(f,g) = \frac{1}{2} \int_a^b \int_a^b \langle f(t) - f(s), g(t) - g(s) \rangle \, dtds.$$

For Korkine's classical identity for real-valued functions, see [17, p. 242].

If we take the norm and use the integral's properties, we get by Schwarz inequality

$$\begin{aligned}
|D(f,g)| &\leq \frac{1}{2} \int_a^b \int_a^b |\langle f(t) - f(s), g(t) - g(s) \rangle| \, dtds \\
&\leq \frac{1}{2} \int_a^b \int_a^b \|f(t) - f(s)\| \, \|g(t) - g(s)\| \, dtds.
\end{aligned} \tag{2.2}$$

Observe that for $s,\ t \in [a,b]$

$$f(t) - f(s) = \int_s^t f'(u) \, du, \ \ g(t) - g(s) = \int_s^t g'(u) \, du,$$

which implies that

$$\begin{aligned}
\|f(t) - f(s)\| \, \|g(t) - g(s)\| &= \left\| \int_s^t f'(u) \, du \right\| \left\| \int_s^t g'(u) \, du \right\| \\
&\leq \left| \int_s^t \|f'(u)\| \, du \right| \left| \int_s^t \|g'(u)\| \, du \right| \\
&= \left| \int_s^t \|f'(u)\| \, du \int_s^t \|g'(u)\| \, du \right| \\
&= \int_s^t \|f'(u)\| \, du \int_s^t \|g'(u)\| \, du,
\end{aligned}$$

for all $s,\ t \in [a,b]$.

By (2.2) we get

$$|D(f,g)| \leq \frac{1}{2} \int_a^b \int_a^b \left(\int_s^t \|f'(u)\| \, du \right) \left(\int_s^t \|g'(u)\| \, du \right) dtds. \tag{2.3}$$

Since

$$\int_s^t \|f'(u)\|\,du \int_s^t \|g'(u)\|\,du$$
$$= \left(\int_a^t \|f'(u)\|\,du - \int_a^s \|f'(u)\|\,du\right)\left(\int_a^t \|g'(u)\|\,du - \int_a^s \|g'(u)\|\,du\right),$$

hence by Korkine's identity for real valued functions $f(t) = \int_a^t \|f'(u)\|\,du$ and $g(t) = \int_a^t \|g'(u)\|\,du$, we have

$$\begin{aligned}(2.4)\qquad &\frac{1}{2}\int_a^b\int_a^b\left(\int_a^t \|f'(u)\|\,du - \int_a^s \|f'(u)\|\,du\right)\\ &\times\left(\int_a^t \|g'(u)\|\,du - \int_a^s \|g'(u)\|\,du\right)\\ &= (b-a)\int_a^b\left(\int_a^t \|f'(u)\|\,du\right)\left(\int_a^t \|g'(u)\|\,du\right)dt\\ &- \int_a^b\left(\int_a^t \|f'(u)\|\,du\right)dt\int_a^b\left(\int_a^t \|g'(u)\|\,du\right)dt\\ &= D\left(\int_a^{\cdot} \|f'(u)\|\,du, \int_a^{\cdot} \|g'(u)\|\,du\right).\end{aligned}$$

By utilising (2.3) and (2.4), we deduce the first inequality in (2.1).

Observe that

$$0 \le \int_a^t \|f'(u)\|\,du \le \int_a^b \|f'(u)\|\,du$$

and

$$0 \le \int_a^t \|g'(u)\|\,du \le \int_a^b \|g'(u)\|\,du$$

for all $t \in [a,b]$, then by Grüss's inequality for the functions $f(t) = \int_a^t \|f'(u)\|\,du$ and $g(t) = \int_a^t \|g'(u)\|\,du$, $t \in [a,b]$, we get the last part of (2.1). □

We have the noncommutative Čebyšev's inequality:

Corollary 1. *Let $f, g : [a,b] \to H$ be strongly differentiable functions on the interval (a,b) with*

$$\|f'\|_{[a,b],\infty} := \sup_{u\in(a,b)} \|f'(u)\|,\ \ \|g'\|_{[a,b],\infty} < \infty,$$

then

$$(2.5)\qquad |D(f,g)| \le \frac{1}{12}(b-a)^4 \|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\infty}.$$

Proof. If we use Čebyšev's inequality (1.3) for $f(t) = \int_a^t \|f'(u)\|\,du$ and $g(t) = \int_a^t \|g'(u)\|\,du$, $t \in [a,b]$, then we get

$$\begin{aligned}0 &\le D\left(\int_a^{\cdot} \|f'(u)\|\,du, \int_a^{\cdot} \|g'(u)\|\,du\right)\\ &\le \frac{1}{12}(b-a)^4\|f'\|_\infty\|g'\|_\infty = \frac{1}{12}(b-a)^4\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\infty},\end{aligned}$$

which by the first inequality in (2.1) gives the desired result (2.5). □

By the use of Ostrowski's inequality (1.4) we derive:

Corollary 2. *Let $f,\ g:[a,b]\rightarrow H$ be strongly differentiable functions on the interval (a,b) with $\|g'\|_{[a,b],\infty}<\infty$, then*

$$|D(f,g)|\leq\frac{1}{8}(b-a)^{3}\|f'\|_{[a,b],1}\|g'\|_{[a,b],\infty}. \tag{2.6}$$

For a strongly differentiable function h on (a,b), we define

$$\|h'\|_{[a,b],2}:=\left(\int_{a}^{b}\|h'(u)\|^{2}du\right)^{1/2}.$$

By the use of Lupaş inequality for $f(t)=\int_{a}^{t}\|f'(u)\|du$ and $g(t)=\int_{a}^{t}\|g'(u)\|du$, $t\in[a,b]$, we get:

Corollary 3. *Let $f,\ g:[a,b]\rightarrow H$ be strongly differentiable functions on the interval (a,b) with $\|f'\|_{[a,b],2},\ \|g'\|_{[a,b],2}<\infty$, then*

$$|D(f,g)|\leq\frac{1}{\pi^{2}}(b-a)^{3}\|f'\|_{[a,b],2}\|g'\|_{[a,b],2}. \tag{2.7}$$

Observe that for $f(t)=\int_{a}^{t}\|f'(u)\|du$, we get integrating by parts that

$$\begin{aligned}
&\int_{a}^{b}\left|f(t)-\frac{1}{b-a}\int_{a}^{b}f(s)ds\right|dt\\
&=\int_{a}^{b}\left|\int_{a}^{t}\|f'(u)\|du-\frac{1}{b-a}\int_{a}^{b}\left(\int_{a}^{s}\|f'(u)\|du\right)ds\right|dt\\
&=\int_{a}^{b}\left|\int_{a}^{t}\|f'(u)\|du-\frac{1}{b-a}\left(\left(\int_{a}^{b}\|f'(u)\|du\right)b-\int_{a}^{b}\|f'(s)\|sds\right)\right|dt\\
&=\int_{a}^{b}\left|\int_{a}^{t}\|f'(u)\|du-\frac{1}{b-a}\left(\int_{a}^{b}(b-u)\|f'(u)\|du\right)\right|dt\\
&=\frac{1}{b-a}\int_{a}^{b}\left|(b-a)\int_{a}^{t}\|f'(u)\|du-\int_{a}^{b}(b-u)\|f'(u)\|du\right|dt\\
&=\frac{1}{b-a}\int_{a}^{b}\left|\int_{a}^{t}(u-a)\|f'(u)\|du-\int_{t}^{b}(b-u)\|f'(u)\|du\right|dt.
\end{aligned}$$

By utilising (1.8) for $f(t)=\int_{a}^{t}\|f'(u)\|du$ and $g(t)=\int_{a}^{t}\|g'(u)\|du$, $t\in[a,b]$, we get:

Corollary 4. *Let $f,\ g:[a,b]\rightarrow H$ be strongly differentiable functions on the interval (a,b), then*

$$\begin{aligned}|D(f,g)|&\leq\frac{1}{2}\|g'\|_{[a,b],1}\\
&\times\int_{a}^{b}\left|\int_{a}^{t}(u-a)\|f'(u)\|du-\int_{t}^{b}(b-u)\|f'(u)\|du\right|dt.\end{aligned} \tag{2.8}$$

Remark 2. *We observe that*

$$\int_a^b \left| \int_a^t (u-a) \|f'(u)\| \, du - \int_t^b (b-u) \|f'(u)\| \, du \right| dt$$
$$\le \int_a^b \left[\left| \int_a^t (u-a) \|f'(u)\| \, du \right| + \left| \int_t^b (b-u) \|f'(u)\| \, du \right| \right] dt$$
$$\le \int_a^b \left[\int_a^t (u-a) \|f'(u)\| \, du + \int_t^b (b-u) \|f'(u)\| \, du \right] dt$$

$$= \left[\int_a^t (u-a) \|f'(u)\| \, du + \int_t^b (b-u) \|f'(u)\| \, du \right] t \Big|_a^b$$
$$- \int_a^b t \left((t-a) \|f'(t)\| - (b-t) \|f'(t)\| \right) dt$$

$$= b \int_a^b (u-a) \|f'(u)\| \, du - a \int_a^b (b-u) \|f'(u)\| \, du$$
$$- \int_a^b t \left((t-a) \|f'(t)\| - (b-t) \|f'(t)\| \right) dt$$
$$= 2 \int_a^b (b-t)(t-a) \|f'(t)\| \, dt$$

and by (2.8) we get

$$|D(f,g)| \le \|g'\|_{[a,b],1} \int_a^b (b-t)(t-a) \|f'(t)\| \, dt. \tag{2.9}$$

Theorem 5. *Let f, $g : [a,b] \to H$ be continuous functions on the interval (a,b), then*

$$|D(f,g)| \le \begin{cases} \inf_{v\in H} \|f-v\|_{[a,b],\infty} \int_a^b \left\| (b-a) g(t) - \int_a^b g(s) \, ds \right\| dt, \\ \\ \inf_{v\in H} \|f-v\|_{[a,b],q} \left(\int_a^b \left\| (b-a) g(t) - \int_a^b g(s) \, ds \right\|^p dt \right)^{1/p}, \\ \\ \inf_{v\in H} \|f-v\|_{[a,b],1} \sup_{t\in[a,b]} \left\| (b-a) g(t) - \int_a^b g(s) \, ds \right\| \end{cases} \tag{2.10}$$

for all p, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. For all $v \in H$ we have

$$\int_a^b [f(t) - v] \left[(b-a) g(t) - \int_a^b g(s) \, ds \right] dt$$

$$= \int_a^b f(t)\left[(b-a)g(t) - \int_a^b g(s)\,ds\right]dt$$
$$- v\int_a^b \left[(b-a)g(t) - \int_a^b g(s)\,ds\right]dt$$
$$= (b-a)\int_a^b f(t)g(t)\,dt - \int_a^b f(t)\,dt\int_a^b g(s)\,ds$$
$$- v\left[(b-a)\int_a^b g(t)\,dt - (b-a)\int_a^b g(s)\,ds\right] = D(f,g).$$

Taking the norm in this equality, we get by Hölder's inequality that

$$|D(f,g)| \le \int_a^b \left\|[f(t)-v]\left[(b-a)g(t) - \int_a^b g(s)\,ds\right]\right\|dt$$

$$\le \int_a^b \|f(t)-v\|\left\|(b-a)g(t) - \int_a^b g(s)\,ds\right\|dt$$
$$\le \begin{cases} \sup_{t\in[a,b]}\|f(t)-v\|\int_a^b \left\|(b-a)g(t) - \int_a^b g(s)\,ds\right\|dt, \\ \left(\int_a^b \|f(t)-v\|^q\,dt\right)^{1/q}\left(\int_a^b \left\|(b-a)g(t) - \int_a^b g(s)\,ds\right\|^p dt\right)^{1/p}, \\ \int_a^b \|f(t)-v\|\sup_{t\in[a,b]}\left\|(b-a)g(t) - \int_a^b g(s)\,ds\right\| \end{cases}$$

for all $v \in H$.

By taking the infimum over $v \in H$, we obtain the desired result (2.10). □

Corollary 5. *With the assumptions of Theorem 5 and if there exists $v \in H$ and $M > 0$ such that*

$$\|f(t)-v\| \le M \text{ for all } t \in [a,b],$$

then

$$|D(f,g)| \le M\int_a^b \left\|(b-a)g(t) - \int_a^b g(s)\,ds\right\|dt. \tag{2.11}$$

Remark 3. *If there exists x, $X \in H$ such that*

$$\operatorname{Re}\langle X - f(t), f(t) - x\rangle \ge 0,$$

or, equivalently

$$\left\|f(t) - \frac{X+x}{2}\right\| \le \frac{1}{2}\|X-x\|$$

for all $t \in [a,b]$, then by (2.11) we get

$$|D(f,g)| \le \frac{1}{2}\|X-x\|\int_a^b \left\|(b-a)g(t) - \int_a^b g(s)\,ds\right\|dt. \tag{2.12}$$

The proof is obvious from the first branch of (2.10).

Corollary 6. *Let f, $g : [a,b] \to H$ be continuous functions on the interval $[a,b]$ and f strongly differentiable on (a,b), then for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$
\begin{aligned}
(2.13)\qquad |D(f,g)| &\le \sup_{t\in[a,b]} \left\| (b-a)\, g(t) - \int_a^b g(s)\, ds \right\| \\
&\times \left[\int_a^{\frac{a+b}{2}} (t-a) \left\| f'(t) \right\| dt + \int_{\frac{a+b}{2}}^b (b-t) \left\| f'(t) \right\| dt \right] \\
&\le \sup_{t\in[a,b]} \left\| (b-a)\, g(t) - \int_a^b g(s)\, ds \right\| \\
&\times \begin{cases} \frac{1}{8}(b-a)^2 \left[\left\| f' \right\|_{\left[a,\frac{a+b}{2}\right],\infty} + \left\| f' \right\|_{\left[\frac{a+b}{2},b\right],\infty} \right], \\ \\ \frac{1}{(q+1)^{1/q} 2^{1+1/q}} \left[\left\| f' \right\|_{\left[a,\frac{a+b}{2}\right],p} + \left\| f' \right\|_{\left[\frac{a+b}{2},b\right],p} \right], \\ \\ \frac{1}{2}(b-a) \left\| f' \right\|_{[a,b],1}. \end{cases}
\end{aligned}
$$

Proof. We have

$$
\begin{aligned}
&\int_a^b \left\| f(t) - f\left(\frac{a+b}{2}\right) \right\| dt \\
&= \int_a^b \left\| \int_{\frac{a+b}{2}}^t f'(s)\, ds \right\| dt \le \int_a^b \left| \int_{\frac{a+b}{2}}^t \left\| f'(s) \right\| ds \right| dt \\
&= \int_a^{\frac{a+b}{2}} \left(\int_t^{\frac{a+b}{2}} \left\| f'(s) \right\| ds \right) dt + \int_{\frac{a+b}{2}}^b \left(\int_{\frac{a+b}{2}}^t \left\| f'(s) \right\| ds \right) dt \\
&\quad = \left(\int_t^{\frac{a+b}{2}} \left\| f'(s) \right\| ds \right) t \Bigg|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} t \left\| f'(t) \right\| dt \\
&\quad + \left(\int_{\frac{a+b}{2}}^t \left\| f'(s) \right\| ds \right) t \Bigg|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b t \left\| f'(t) \right\| dt \\
&\quad = \int_a^{\frac{a+b}{2}} t \left\| f'(t) \right\| dt - a \int_a^{\frac{a+b}{2}} \left\| f'(s) \right\| ds \\
&\quad + b \int_{\frac{a+b}{2}}^b \left\| f'(s) \right\| ds - \int_{\frac{a+b}{2}}^b t \left\| f'(t) \right\| dt \\
&\quad = \int_a^{\frac{a+b}{2}} (t-a) \left\| f'(t) \right\| dt + \int_{\frac{a+b}{2}}^b (b-t) \left\| f'(t) \right\| dt,
\end{aligned}
$$

which, by the third branch of (2.10), gives the first part of (2.13).

The last part follows by Hölder's inequality. □

3. Applications for Operator Monotone Functions

Assume that $f : I \to \mathbb{R}$ is continuous on the interval I and the selfadjoint operators A, B with spectra $\mathrm{Sp}(A)$, $\mathrm{Sp}(A) \subset I$. Using the continuos functional calculus for selfadjoint operators in Hilbert spaces, we can define the functions

$$f_{A,B}(t) := f((1-t)A + tB), \ g_{A,B}(t) := g((1-t)A + tB)$$

for $t \in [0,1]$. For $x, y \in H$ we define the Čebyšev functional

$$D(f,g,A,B;x,y) := \int_0^1 \langle f((1-t)A+tB)x, g((1-t)A+tB)y\rangle dt - \left\langle \int_0^1 f((1-t)A+tB)x dt, \int_0^1 g((1-t)A+tB)y dt \right\rangle.$$

A real valued continuous function f on $[0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B \geq 0$.

We have the following representation of operator monotone functions, see for instance [1, p. 144-145]:

Theorem 6. *A function* $f : [0,\infty) \to \mathbb{R}$ *is operator monotone in* $[0,\infty)$ *if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{3.1}$$

where $b \geq 0$ *and a positive measure* μ *on* $[0,\infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{3.2}$$

Lemma 1. *Let* $f : [0,\infty) \to \mathbb{R}$ *be operator monotone in* $[0,\infty)$*. Assume that* $U \geq 0$*, then for all selfadjoint operators* V *we have*

$$Df(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1}\right] d\mu(\lambda). \tag{3.3}$$

Proof. From (3.1) we get

$$f(t) = f(0) + bt + \int_0^\infty \left(\lambda - \frac{\lambda^2}{t+\lambda}\right) d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, then by the representation of f we have for t in a small open interval around 0 that

$$\begin{aligned}
&f(U+tV) - f(U) \\
&= btV + \int_0^\infty \left(\lambda - \lambda^2 (U+tV+\lambda)^{-1}\right) d\mu(\lambda) - \int_0^\infty \left(\lambda - \lambda^2 (U+\lambda)^{-1}\right) d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} - (\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} (\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\
&= btV + t\int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda).
\end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{f(U+tV) - f(U)}{t} = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Df(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} V (\lambda + U)^{-1} \right] d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Theorem 7. *Let $f : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators V we have*

$$\|Df(U)(V)\| \leq f'(u) \|V\|. \tag{3.4}$$

Proof. From (3.3) we get

$$\begin{aligned} \|Df(U)(V) - bV\| &\leq \int_0^\infty \lambda^2 \left\| (\lambda + U)^{-1} V (\lambda + U)^{-1} \right\| d\mu(\lambda) \\ &\leq \|V\| \int_0^\infty \lambda^2 \left\| (\lambda + U)^{-1} \right\|^2 d\mu(\lambda). \end{aligned} \tag{3.5}$$

Observe that $\lambda + U \geq \lambda + u > 0$ for $\lambda \in [0,\infty)$. Then $0 < (\lambda + U)^{-1} \leq (\lambda + u)^{-1}$, which implies that $\left\| (\lambda + U)^{-1} \right\| \leq (\lambda + u)^{-1}$, namely $\left\| (\lambda + U)^{-1} \right\|^2 \leq (\lambda + u)^{-2}$ for $\lambda \in [0,\infty)$.

Therefore by (3.5) we get

$$\|Df(U)(V) - bV\| \leq \|V\| \int_0^\infty \lambda^2 (\lambda + u)^{-2} d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$f'(t) = b + \int_0^\infty \frac{\lambda(t+\lambda) - \lambda t}{(t+\lambda)^2} d\mu(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) \tag{3.7}$$

for $t > 0$.

From (3.7) we get

$$\int_0^\infty \lambda^2 (\lambda + u)^{-2} d\mu(\lambda) = f'(u) - b,$$

and by (3.6) we derive

$$\|Df(U)(V) - bV\| \leq \|V\| f'(u) - b\|V\|.$$

Finally, by the triangle inequality and by the fact that $b \geq 0$, we obtain that

$$\|Df(U)(V)\| - b\|V\| \leq \|Df(U)(V) - bV\|,$$

which proves the desired result (3.4). □

For a continuous function f on $(0,\infty)$ and A, $B > 0$ we consider the auxiliary function $f_{A,B} : [0,1] \to \mathbb{R}$ defined by

$$f_{A,B}(t) := f((1-t)A + tB), \; t \in [0,1].$$

We have the following representations of the derivatives:

Lemma 2. *Assume that the operator function generated by f is Fréchet differentiable in each $A \geq 0$, then for $B \geq 0$ we have that $f_{A,B}$ is differentiable on $[0,1]$ and*

$$f'_{A,B}(t) = D(f)((1-t)A + tB)(B - A) \tag{3.8}$$

for $t \in [0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t+h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned}&\frac{f_{A,B}(t+h)-f(t)}{h}\\&=\frac{f((1-(t+h))A+(t+h)B)-f((1-t)A+tB)}{h}\\&=\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}f'_{A,B}(t)&=\lim_{h\to 0}\frac{f_{A,B}(t+h)-f(t)}{h}\\&=\lim_{h\to 0}\left[\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\right]\\&=D(f)((1-t)A+tB)(B-A),\end{aligned}$$

which proves (3.8). □

Corollary 7. *Let $f:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A\geq a>0$, $B\geq b>0$ we have*

$$\|D(f)((1-t)A+tB)(B-A)\| \leq f'((1-t)a+tb)\|B-A\| \tag{3.9}$$

for all $t\in[0,1]$.

The proof follows by Theorem 7 and Lemma 2.

One can observe that the inequality (3.9) remains valid for operator monotone functions on $(0,\infty)$. This follows by considering the function $f_\varepsilon(t):=f(t+\varepsilon)$ for $\varepsilon>0$, which is operator monotone on $[0,\infty)$ and then by letting $\varepsilon\to 0+$ and using the continuity of f and f'.

We have the following result:

Theorem 8. *Let $f,\ g:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A\geq a>0$, $B\geq b>0$ with $a\neq b$, we have*

$$\begin{aligned}&|D(f,g,A,B;x,y)|\\&\leq\frac{1}{4}\|B-A\|^2\|x\|\|y\|\\&\times\begin{cases}\frac{[f(b)-f(a)][g(b)-g(a)]}{(b-a)^2},\\[2ex]\frac{1}{3}\sup_{t\in[0,1]}f'((1-t)a+tb)\sup_{t\in[0,1]}g'((1-t)a+tb),\\[2ex]\frac{1}{2}\frac{f(b)-f(a)}{b-a}\sup_{t\in[0,1]}g'((1-t)a+tb)\end{cases}\end{aligned}\tag{3.10}$$

for all $x,\ y\in H$.

Also, we have

$$\begin{aligned}&|D(f,g,A,B;x,y)|\\&\leq\frac{1}{\pi^2}\|B-A\|^2\|x\|\|y\|\\&\times\left(\int_0^1[f'((1-t)a+tb)]^2dt\right)^{1/2}\left(\int_0^1[g'((1-t)a+tb)]^2dt\right)^{1/2}\end{aligned}\tag{3.11}$$

for all x, $y \in H$.

Proof. Observe that

$$f'_{A,B}(t)\,x = D(f)\,((1-t)\,A + tB)\,(B-A)\,x,$$
$$g'_{A,B}(t)\,y = D(g)\,((1-t)\,A + tB)\,(B-A)\,y$$

and by (3.9) we get

$$\left\|f'_{A,B}(t)\,x\right\| \le f'\,((1-t)\,a + tb)\,\|B-A\|\,\|x\|$$

and

$$\left\|g'_{A,B}(t)\,y\right\| \le g'\,((1-t)\,a + tb)\,\|B-A\|\,\|y\|\,,$$

for all x, $y \in H$.

If we use inequality (2.1), then we get

$$\begin{aligned}
&|D(f,g,A,B;x,y)| \\
&\le \frac{1}{4}\int_0^1 \left\|f'_{A,B}(t)\,x\right\|dt \int_0^1 \left\|g'_{A,B}(t)\,y\right\|dt \\
&\le \frac{1}{4}\|B-A\|^2\,\|x\|\,\|y\| \int_0^1 f'\,((1-t)\,a+tb)\,dt \int_0^1 g'\,((1-t)\,a+tb)\,dt \\
&= \frac{1}{4}\|B-A\|^2\,\|x\|\,\|y\|\,\frac{[f(b)-f(a)]\,[g(b)-g(a)]}{(b-a)^2},
\end{aligned}$$

which proves the first inequality in (3.10).

The second inequality in (3.10) follows by (2.5) while the third inequality follows by (2.6). Finally, the inequality (3.11) follows by (2.7). □

Remark 4. *If $b = a$, then we derive from the proof of Theorem 8 that*

$$|D(f,g,A,B;x,y)| \le \frac{1}{4}\|B-A\|^2\,\|x\|\,\|y\| \times \begin{cases} f'(a)\,g'(a)\,, \\ \frac{1}{3}f'(a)\,g'(a)\,, \\ \frac{1}{2}f'(a)\,g'(a)\,, \\ \frac{4}{\pi^2}f'(a)\,g'(a)\,, \end{cases} \tag{3.12}$$

which shows that second inequality in (3.12) is best.

Corollary 8. *Let $f : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A \ge a > 0$, $B \ge b > 0$, $a \ne b$, we have*

$$\begin{aligned}
0 &\le \int_0^1 \|f((1-t)\,A+tB)\,x\|^2\,dt - \left\|\int_0^1 f((1-t)\,A+tB)\,x dt\right\|^2 \\
&\le \frac{1}{4}\|B-A\|^2\,\|x\|^2 \\
&\times \begin{cases} \frac{[f(b)-f(a)]^2}{(b-a)^2} \\ \frac{1}{3}\sup_{t\in[0,1]}\left[f'\,((1-t)\,a+tb)\right]^2, \\ \frac{1}{2}\frac{f(b)-f(a)}{b-a}\sup_{t\in[0,1]} f'\,((1-t)\,a+tb) \end{cases}
\end{aligned} \tag{3.13}$$

for all $x \in H$.

Also, we have

$$0 \leq \int_0^1 \|f((1-t)A+tB)x\|^2 dt - \left\|\int_0^1 f((1-t)A+tB)xdt\right\|^2 \leq \frac{1}{\pi^2}\|B-A\|^2\|x\|^2\int_0^1 [f'((1-t)a+tb)]^2 dt \tag{3.14}$$

for all $x \in H$.

For $a = b$, we get the best bound to be

$$\begin{aligned} 0 &\leq \int_0^1 \|f((1-t)A+tB)x\|^2 dt - \left\|\int_0^1 f((1-t)A+tB)xdt\right\|^2 \\ &\leq \frac{1}{12}\|B-A\|^2\|x\|^2[f'(a)]^2, \end{aligned} \tag{3.15}$$

for all $x \in H$.

The proof follows by Theorem 8 for $g = f$ and $y = x$.

In the following we assume that $A \geq a > 0$, $B \geq b > 0$ with $a \neq b$. Consider the power function $\ell^r(t) = t^r$, $r \in (0,1)$ which is operator monotone on $[0,\infty)$, then for r, $p \in (0,1)$ we have by Theorem 8 that

$$\begin{aligned} &|D(\ell^r,\ell^p,A,B;x,y)| \\ &\leq \frac{1}{4}\|B-A\|^2\|x\|\|y\| \times \begin{cases} \frac{(b^r-a^r)(b^p-a^p)}{(b-a)^2}, \\ \frac{1}{3}\frac{rp}{\min\{a^{1-r},b^{1-r}\}\min\{a^{1-p},b^{1-p}\}}, \\ \frac{1}{2}\frac{p}{\min\{a^{1-p},b^{1-p}\}}\frac{b^r-a^r}{b-a} \end{cases} \end{aligned} \tag{3.16}$$

for all x, $y \in H$.

Also, we have

$$\begin{aligned} &|D(\ell^r,\ell^p,A,B;x,y)| \\ &\leq \frac{1}{\pi^2}\frac{rp}{\sqrt{(3-2r)(3-2p)}}\|B-A\|^2\|x\|\|y\| \\ &\times\left(\frac{b^{3-2r}-a^{3-2r}}{b-a}\right)^{1/2}\left(\frac{b^{3-2p}-a^{3-2p}}{b-a}\right)^{1/2} \end{aligned} \tag{3.17}$$

for all x, $y \in H$.

For $b = a$ we derive the bound

$$|D(\ell^r,\ell^p,A,B;x,y)| \leq \frac{1}{12}rp\|B-A\|^2\|x\|\|y\|a^{r+p-2} \tag{3.18}$$

for all x, $y \in H$.

By taking $f=g=\ln$ which is operator monotone on $(0,\infty)$, we also get for $b\neq a$

$$|D(\ln,\ln,A,B;x,y)| \leq \frac{1}{4}\|B-A\|^2\|x\|\|y\|\times\begin{cases}\frac{(\ln b-\ln a)^2}{(b-a)^2},\\ \\ \frac{1}{3\min\{a^2,b^2\}},\\ \\ \frac{1}{2}\frac{\ln b-\ln a}{(b-a)\min\{a,b\}}\end{cases} \tag{3.19}$$

for all x, $y\in H$.

Also, we have

$$|D(\ln,\ln,A,B;x,y)|\leq\frac{1}{\pi^2 ab}\|B-A\|^2\|x\|\|y\| \tag{3.20}$$

for all x, $y\in H$.

If we take $y=x$, then we obtain

$$\begin{aligned} 0 &\leq \int_0^1\|\ln((1-t)A+tB)x\|^2dt-\left\|\int_0^1\ln((1-t)A+tB)xdt\right\|^2\\ &\leq \frac{1}{4}\|B-A\|^2\|x\|^2\times\begin{cases}\frac{(\ln b-\ln a)^2}{(b-a)^2},\\ \\ \frac{1}{3\min\{a^2,b^2\}},\\ \\ \frac{1}{2}\frac{\ln b-\ln a}{(b-a)\min\{a,b\}},\\ \\ \frac{4}{\pi^2 ab}\end{cases}\end{aligned}$$

for all $x\in H$.

Finally, if $b=a$, then

$$\begin{aligned} 0 &\leq \int_0^1\|\ln((1-t)A+tB)x\|^2dt-\left\|\int_0^1\ln((1-t)A+tB)xdt\right\|^2\\ &\leq \frac{1}{12a^2}\|B-A\|^2\|x\|^2, \end{aligned}\tag{3.21}$$

for all $x\in H$.

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