DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA A REVERSE OF JENSEN'S INTEGRAL INEQUALITY

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Abstract. In this paper we prove among others that, if A, C are positive definite matrices and $p \geq 1$, then

$$\begin{aligned}
0 &\leq \frac{\det(C)}{\det(C+pA)} - \left(\frac{\det(C)}{\det(C+A)}\right)^p \\
&\leq p\left[\frac{\det(C)}{\det(C+pA)} - \frac{\det(C)}{\det(C+(p-1)A)}\frac{\det(C)}{\det(C+A)}\right] \\
&\leq \frac{1}{2}p\left[\frac{\det(C)}{\det(C+2A)} - \left(\frac{\det(C)}{\det(C+A)}\right)^2\right]^{\frac{1}{2}} \leq \frac{1}{4}p.
\end{aligned}$$

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the Lebesgue space $L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f$ is μ-measurable and $\int_\Omega w(x)|f(x)|\,d\mu(x) < \infty\}$. For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$. We also assume that $\int_\Omega w d\mu = 1$.

An useful result that is used to provide simpler upper bounds for the difference in Jensen's inequality is the Grüss' inequality. We recall now some facts related to this famous result.

If $f, g : \Omega \to \mathbb{R}$ are μ-measurable functions and $f, g, fg \in L_w(\Omega, \mu)$, then we may consider the *Čebyšev functional*

$$T_w(f,g) := \int_\Omega wfgd\mu - \int_\Omega wfd\mu \int_\Omega wgd\mu. \tag{1.1}$$

The following result is known in the literature as the *Grüss inequality*

$$|T_w(f,g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.2}$$

provided

$$-\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty \tag{1.3}$$

for μ -a.e. a. $x \in \Omega$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

1991 *Mathematics Subject Classification.* 47A63, 26D15, 46C05.

Key words and phrases. Positive definite matrices, Determinants, Inequalities.

With the above assumptions, if $f \in L_w(\Omega,\mu)$ then we may define

$$D_w(f) := D_{w,1}(f) := \int_\Omega w \left| f - \int_\Omega wf d\mu \right| d\mu. \tag{1.4}$$

In 2002, Cerone & Dragomir [2] obtained the following refinement of the Grüss inequality (1.2):

Theorem 1 (Cerone & Dragomir, 2002 [2])**.** *Let $w, f, g : \Omega \to \mathbb{R}$ be μ-measurable functions with $w \geq 0$ μ-a.e. (almost everywhere) on Ω and $\int_\Omega w d\mu = 1$. If f, g, $fg \in L_w(\Omega,\mu)$ and there exists the constants δ, Δ such that*

$$-\infty < \delta \leq g(x) \leq \Delta < \infty \quad \text{for } \mu\text{-a.e. } x \in \Omega, \tag{1.5}$$

then we have the inequality

$$|T_w(f,g)| \leq \frac{1}{2}(\Delta - \delta) D_w(f). \tag{1.6}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Remark 1. *The inequality (1.6) was obtained for the particular case $\Omega = [a,b]$ and the uniform weight $w(t) = 1$, $t \in [a,b]$ by X. L. Cheng and J. Sun in* [12]*. However, in that paper the authors did not prove the sharpness of the constant $\frac{1}{2}$.*

For $f \in L_{p,w}(\Omega,\mathcal{A},\mu) := \left\{ f : \Omega \to \mathbb{R}, \int_\Omega w |f|^p d\mu < \infty \right\}$, $p \geq 1$ we may also define

$$D_{w,p}(f) := \left[\int_\Omega w \left| f - \int_\Omega wf d\mu \right|^p d\mu \right]^{\frac{1}{p}} = \left\| f - \int_\Omega wf d\mu \right\|_{\Omega,p} \tag{1.7}$$

where $\|\cdot\|_{\Omega,p}$ is the usual p-norm on $L_{p,w}(\Omega,\mathcal{A},\mu)$, namely,

$$\|h\|_{\Omega,p} := \left(\int_\Omega w |h|^p d\mu \right)^{\frac{1}{p}}, \quad p \geq 1.$$

Using Hölder's inequality we get

$$D_{w,1}(f) \leq D_{w,p}(f) \quad \text{for } p \geq 1,\ f \in L_{p,w}(\Omega,\mathcal{A},\mu); \tag{1.8}$$

and, in particular for $p = 2$

$$D_{w,1}(f) \leq D_{w,2}(f) := \left[\int_\Omega wf^2 d\mu - \left(\int_\Omega wf d\mu \right)^2 \right]^{\frac{1}{2}}, \tag{1.9}$$

if $f \in L_{2,w}(\Omega,\mathcal{A},\mu)$.

For $f \in L_\infty(\Omega,\mathcal{A},\mu) := \left\{ f : \Omega \to \mathbb{R},\ \|f\|_{\Omega,\infty} := \operatorname{essup}_{x\in\Omega} |f(x)| < \infty \right\}$ we also have

$$D_{w,p}(f) \leq D_{w,\infty}(f) := \left\| f - \int_\Omega wf d\mu \right\|_{\Omega,\infty}. \tag{1.10}$$

The following corollary may be useful in practice.

Corollary 1. *With the assumptions of Theorem 1, we have*

$$\begin{aligned} |T_w(f,g)| &\le \frac{1}{2}(\Delta-\delta)\,D_w(f) \\ &\le \frac{1}{2}(\Delta-\delta)\,D_{w,p}(f) \quad \text{if } f\in L_p(\Omega,\mathcal{A},\mu),\ 1<p<\infty; \\ &\le \frac{1}{2}(\Delta-\delta)\,D_{w,\infty}(f) \quad \text{if } f\in L_\infty(\Omega,\mathcal{A},\mu). \end{aligned} \tag{1.11}$$

Remark 2. *The inequalities in (1.11) are in order of increasing coarseness. If we assume that $-\infty<\gamma\le f(x)\le\Gamma<\infty$ for μ-a.e. $x\in\Omega$, then by the Grüss inequality for $g=f$ we have for $p=2$*

$$\left[\int_\Omega wf^2d\mu-\left(\int_\Omega wfd\mu\right)^2\right]^{\frac{1}{2}} \le \frac{1}{2}(\Gamma-\gamma). \tag{1.12}$$

By (1.11), we deduce the following sequence of inequalities

$$\begin{aligned} |T_w(f,g)| &\le \frac{1}{2}(\Delta-\delta)\int_\Omega w\left|f-\int_\Omega wfd\mu\right|d\mu \\ &\le \frac{1}{2}(\Delta-\delta)\left[\int_\Omega wf^2d\mu-\left(\int_\Omega wfd\mu\right)^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma) \end{aligned} \tag{1.13}$$

for $f,\ g:\Omega\to\mathbb{R}$, μ-measurable functions and so that $-\infty<\gamma\le f(x)<\Gamma<\infty$, $-\infty<\delta\le g(x)\le\Delta<\infty$ for μ-a.e. $x\in\Omega$. Thus, the inequality (1.13) is a refinement of Grüss' inequality (1.2).

In order to provide a reverse of the celebrated Jensen's integral inequality for convex functions, S. S. Dragomir obtained in 2002 [4] the following result:

Theorem 2 (Dragomir, 2002 [4])**.** *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f,\ \Phi'\circ f,\ (\Phi'\circ f)f\in L_w(\Omega,\mu)$, where $w\ge 0$ μ-a.e. on Ω with $\int_\Omega wd\mu=1$. Then we have the inequality:*

$$\begin{aligned} 0 &\le \int_\Omega w(\Phi\circ f)\,d\mu-\Phi\left(\int_\Omega wfd\mu\right) \\ &\le \int_\Omega w(\Phi'\circ f)\,fd\mu-\int_\Omega w(\Phi'\circ f)\,d\mu\int_\Omega wfd\mu \\ &\le \frac{1}{2}[\Phi'(M)-\Phi'(m)]\int_\Omega w\left|f-\int_\Omega wfd\mu\right|d\mu. \end{aligned} \tag{1.14}$$

Remark 3. *On making use of (1.14) and (1.13), one can state the following string of reverse inequalities for the Jensen's difference*

$$
\begin{aligned}
0 &\le \int_\Omega w\left(\Phi\circ f\right)d\mu-\Phi\left(\int_\Omega wfd\mu\right) \\
&\le \int_\Omega w\left(\Phi'\circ f\right)fd\mu-\int_\Omega w\left(\Phi'\circ f\right)d\mu\int_\Omega wfd\mu \\
&\le \frac{1}{2}\left[\Phi'\left(M\right)-\Phi'\left(m\right)\right]\int_\Omega w\left|f-\int_\Omega wfd\mu\right|d\mu \\
&\le \frac{1}{2}\left[\Phi'\left(M\right)-\Phi'\left(m\right)\right]\left[\int_\Omega wf^2d\mu-\left(\int_\Omega wfd\mu\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{4}\left[\Phi'\left(M\right)-\Phi'\left(m\right)\right]\left(M-m\right).
\end{aligned}
\tag{1.15}
$$

We notice that the inequality between the first, second and last term from (1.15) was proved in the general case of positive linear functionals in 2001 by S. S. Dragomir in [3].

2. Some Facts for Determinants of Positive Definite Matrices

A real square matrix $A=(a_{ij})$, $i,j=1,...,n$ is *symmetric* provided $a_{ij}=a_{ji}$ for all $i,j=1,...,n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x)=\sum_{i,j=1}^n a_{ij}x_ix_j$ is positive for all $x=(x_1,...,x_n)\in\mathbb{R}^n\setminus\{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A>0$, is that all determinants

$$\det(A_k)=\det\left(a_{ij}\right),\ i,\ j=1,...,k;\ k=1,...,n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [11, pp. 211-212]

$$
\begin{aligned}
J_n\left(A\right) &:= \int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle)dx := \int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}\exp(-\langle Ax,x\rangle)dx \\
&= \frac{\pi^{n/2}}{\left[\det\left(A\right)\right]^{1/2}},
\end{aligned}
\tag{2.1}
$$

where A is a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.14) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [11, p. 212]), namely

$$\det\left(\left(1-\lambda\right)A+\lambda B\right)\ge\left[\det\left(A\right)\right]^{1-\lambda}\left[\det\left(B\right)\right]^{\lambda} \tag{2.2}$$

for any positive definite matrices A, B and $\lambda\in[0,1]$.

By mathematical induction we can get a generalization of (2.2) which was obtained by L. Mirsky in [10], see also [11, p. 212]

$$\det\left(\sum_{j=1}^m\lambda_jA_j\right)\ge\prod_{j=1}^m\left[\det\left(A_j\right)\right]^{\lambda_j},\ m\ge 2, \tag{2.3}$$

where $\lambda_j > 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} \lambda_j = 1$ and $A_j > 0$, $j = 1, ..., m$.

If we write (2.3) for $A_j = B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m} \lambda_j B_j^{-1}\right) \geq \prod_{j=1}^{m} \left[\det\left(B_j^{-1}\right)\right]^{\lambda_j} = \left(\prod_{j=1}^{m} \left[\det\left(B_j\right)\right]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m} \left[\det\left(A_j\right)\right]^{\lambda_j} \geq \det\left[\left(\sum_{j=1}^{m} \lambda_j A_j^{-1}\right)^{-1}\right], \tag{2.4}$$

where $\lambda_j > 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m} \lambda_j = 1$ and $A_j > 0$, $j = 1, ..., m$.

Using the representation (1.14) one can also prove the result, see [11, p. 212],

$$\det(A) = \det(A_{1n}) \leq \det(A_{1k}) \det\left(A_{(k+1)n}\right), \ k = 1, ..., n; \tag{2.5}$$

where the determinant $\det(A_{rs})$ is defined by

$$\det(A_{rs}) = \det(a_{ij}), i, \ j = r, ..., s.$$

In particular,

$$\det(A) \leq a_{11} a_{22} ... a_{nn}. \tag{2.6}$$

We recall also the Minkowski's type inequality,

$$[\det(A+B)]^{1/n} \geq [\det(A)]^{1/n} + [\det(B)]^{1/n} \tag{2.7}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [11]. For some recent results see [5]-[9].

Motivated by the above results, in this paper we prove among others that, if A, C are positive definite matrices and $p \geq 1$, then

$$\begin{aligned} 0 &\leq \frac{\det(C)}{\det(C+pA)} - \left(\frac{\det(C)}{\det(C+A)}\right)^{p} \\ &\leq p\left[\frac{\det(C)}{\det(C+pA)} - \frac{\det(C)}{\det(C+(p-1)A)} \frac{\det(C)}{\det(C+A)}\right] \\ &\leq \frac{1}{2} p\left[\frac{\det(C)}{\det(C+2A)} - \left(\frac{\det(C)}{\det(C+A)}\right)^{2}\right]^{\frac{1}{2}} \leq \frac{1}{4} p. \end{aligned}$$

3. Inequalities Via Simple Integral Representation

Our first result is as follows:

Theorem 3. *Assume that A, C are positive definite matrices and $p \geq 1$, then*

$$\begin{aligned} 0 &\leq \left(\frac{\det(C)}{\det(C+pA)}\right)^{1/2} - \left(\frac{\det(C)}{\det(C+A)}\right)^{p/2} \\ &\leq p\left[\left(\frac{\det(C)}{\det(C+pA)}\right)^{1/2} - \left(\frac{\det(C)}{\det(C+(p-1)A)}\right)^{1/2} \left(\frac{\det(C)}{\det(C+A)}\right)^{1/2}\right] \\ &\leq \frac{1}{2} p\left[\left(\frac{\det(C)}{\det(C+2A)}\right)^{1/2} - \frac{\det(C)}{\det(C+A)}\right]^{\frac{1}{2}} \leq \frac{1}{4} p. \end{aligned} \tag{3.1}$$

In particular, for $C = I_n$, we derive

$$0 \leq \frac{1}{[\det(I_n + pA)]^{1/2}} - \frac{1}{[\det(I_n + A)]^{p/2}} \tag{3.2}$$
$$\leq p\left[\frac{1}{[\det(I_n + pA)]^{1/2}} - \frac{1}{[\det(I_n + (p-1)A)]^{1/2}} \frac{1}{[\det(I_n + A)]^{1/2}}\right]$$
$$\leq \frac{1}{2}p\left[\frac{1}{[\det(I_n + 2A)]^{1/2}} - \frac{1}{\det(I_n + A)}\right]^{\frac{1}{2}} \leq \frac{1}{4}p.$$

Proof. If we take $w = h/\int_\Omega h d\mu$ and $\Phi : [m, M] \subset [0, \infty) \to [0, \infty)$, $\Phi(t) = t^p$, $p \geq 1$, then we get by (1.15) that

$$0 \leq \frac{\int_\Omega h f^p d\mu}{\int_\Omega h d\mu} - \left(\frac{\int_\Omega h f d\mu}{\int_\Omega h d\mu}\right)^p \tag{3.3}$$
$$\leq p\left[\frac{\int_\Omega h f^p d\mu}{\int_\Omega h d\mu} - \frac{\int_\Omega h f^{p-1} d\mu}{\int_\Omega h d\mu} \frac{\int_\Omega h f d\mu}{\int_\Omega h d\mu}\right]$$
$$\leq \frac{1}{2}p\left(M^{p-1} - m^{p-1}\right)\left[\frac{\int_\Omega h f^2 d\mu}{\int_\Omega h d\mu} - \left(\frac{\int_\Omega h f d\mu}{\int_\Omega h d\mu}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}p\left(M^{p-1} - m^{p-1}\right)(M - m).$$

Take $\Omega = \mathbb{R}^n$, $h(x) = \exp(-\langle Cx, x\rangle)$ and $f(x) = \exp(-\langle Ax, x\rangle)$, $x \in \mathbb{R}^n$. Observe that $0 < f(x) \leq 1$ for $x \in \mathbb{R}^n$ and by (3.3) we get

$$0 \leq \frac{\int_{\mathbb{R}^n} \exp(-\langle(C + pA)x, x\rangle) dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle) dx} - \left(\frac{\int_{\mathbb{R}^n} \exp(-\langle(C + A)x, x\rangle) dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle) dx}\right)^p$$
$$\leq p\left[\frac{\int_{\mathbb{R}^n} \exp(-\langle(C + pA)x, x\rangle) dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle) dx}\right.$$
$$\left.- \frac{\int_{\mathbb{R}^n} \exp(-\langle(C + (p-1)A)x, x\rangle) dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle) dx} \frac{\int_{\mathbb{R}^n} \exp(-\langle(C + A)x, x\rangle) dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle) dx}\right]$$
$$\leq \frac{1}{2}p\left[\frac{\int_{\mathbb{R}^n} \exp(-\langle(C + 2A)x, x\rangle) dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle) dx} - \left(\frac{\int_{\mathbb{R}^n} \exp(-\langle(C + A)x, x\rangle) dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle) dx}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}p,$$

which by representation (2.1) gives

$$0 \leq \frac{J_n(C + pA)}{J_n(C)} - \left(\frac{J_n(C + A)}{J_n(C)}\right)^p$$
$$\leq p\left[\frac{J_n(C + pA)}{J_n(C)} - \frac{J_n(C + (p-1)A)}{J_n(C)} \frac{J_n(C + A)}{J_n(C)}\right]$$
$$\leq \frac{1}{2}p\left[\frac{J_n(C + 2A)}{J_n(C)} - \left(\frac{J_n(C + A)}{J_n(C)}\right)^2\right]^{\frac{1}{2}} \leq \frac{1}{4}p.$$

By using the second identity in (2.1) we derive the desired result (3.1). □

Remark 4. *For $p=2$ we get*

$$0 \le \left(\frac{\det(C)}{\det(C+2A)} \right)^{1/2} - \frac{\det(C)}{\det(C+A)} \le \frac{1}{4}$$

for A, C positive definite matrices. In particular,

$$0 \le \left(\frac{1}{\det(I_n+2A)} \right)^{1/2} - \frac{1}{\det(I_n+A)} \le \frac{1}{4}.$$

Theorem 4. *Let $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Assume that A, B are positive definite matrices with $A-(q-1)B>0$, then*

$$\begin{aligned}
(3.4)\quad 0 &\le \left[\frac{\det(qB)}{\det(pA)} \right]^{1/2} - \left(\frac{\det(qB)}{\det(A+B)} \right)^{p/2} \\
&\le p \left[\left[\frac{\det(qB)}{\det(pA)} \right]^{1/2} - \left[\frac{\det(qB)}{\det((p-1)A+(q-1)B)} \right]^{1/2} \left[\frac{\det(qB)}{\det(A+B)} \right]^{1/2} \right] \\
&\le \frac{1}{2} p \left[\left[\frac{\det(qB)}{\det[2A+(2-q)B]} \right]^{1/2} - \frac{\det(qB)}{\det(A+B)} \right]^{1/2} \\
&\le \frac{1}{4} p.
\end{aligned}$$

Proof. If we replace h by g^q and f by $\frac{f}{g^{q-1}}$ in (3.3), then we get

$$\begin{aligned}
(3.5)\qquad 0 &\le \frac{\int_\Omega g^q \left(\frac{f}{g^{q-1}} \right)^p d\mu}{\int_\Omega g^q d\mu} - \left(\frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu} \right)^p \\
&\le p \left[\frac{\int_\Omega g^q \left(\frac{f}{g^{q-1}} \right)^p d\mu}{\int_\Omega g^q d\mu} - \frac{\int_\Omega g^q \left(\frac{f}{g^{q-1}} \right)^{p-1} d\mu}{\int_\Omega g^q d\mu} \frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu} \right] \\
&\le \frac{1}{2} p \left(\Gamma^{p-1} - \gamma^{p-1} \right) \\
&\times \left[\frac{\int_\Omega g^q \left(\frac{f}{g^{q-1}} \right)^2 d\mu}{\int_\Omega g^q d\mu} - \left(\frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu} \right)^2 \right]^{\frac{1}{2}} \\
&\le \frac{1}{4} p \left(\Gamma^{p-1} - \gamma^{p-1} \right) (\Gamma - \gamma),
\end{aligned}$$

provided that there exists the constants γ, $\Gamma>0$ and such that

$$\gamma \le \frac{f}{g^{q-1}} \le \Gamma \ \mu\text{-a.e on } \Omega.$$

The inequality (3.5) is equivalent to

$$0 \leq \frac{\int_\Omega f^p d\mu}{\int_\Omega g^q d\mu} - \left(\frac{\int_\Omega fg d\mu}{\int_\Omega g^q d\mu}\right)^p \tag{3.6}$$
$$\leq p\left[\frac{\int_\Omega f^p d\mu}{\int_\Omega g^q d\mu} - \frac{\int_\Omega f^{p-1}g^{q-1} d\mu}{\int_\Omega g^q d\mu}\frac{\int_\Omega fg d\mu}{\int_\Omega g^q d\mu}\right]$$
$$\leq \frac{1}{2}p\left(\Gamma^{p-1}-\gamma^{p-1}\right)\left[\frac{\int_\Omega f^2 g^{2-q} d\mu}{\int_\Omega g^q d\mu} - \left(\frac{\int_\Omega fg d\mu}{\int_\Omega g^q d\mu}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}p\left(\Gamma^{p-1}-\gamma^{p-1}\right)(\Gamma-\gamma).$$

Consider $f(x) = \exp(-\langle Ax, x\rangle)$, $g(x) = \exp(-\langle Bx, x\rangle)$. Then

$$\frac{f(x)}{g^{q-1}(x)} = \frac{\exp(-\langle Ax, x\rangle)}{\exp(-\langle (q-1)Bx, x\rangle)} = \exp(-\langle (A-(q-1)B)x, x\rangle) \leq 1$$

since $A-(q-1)B > 0$.

Then by (3.6) we get

$$0 \leq \frac{\int_{\mathbb{R}^n}\exp(-\langle pAx, x\rangle)dx}{\int_{\mathbb{R}^n}\exp(-\langle qBx, x\rangle)dx} - \left(\frac{\int_{\mathbb{R}^n}\exp(-\langle (A+B)x, x\rangle)dx}{\int_{\mathbb{R}^n}\exp(-\langle qBx, x\rangle)dx}\right)^p$$
$$\leq p\left[\frac{\int_{\mathbb{R}^n}\exp(-\langle pAx, x\rangle)dx}{\int_{\mathbb{R}^n}\exp(-\langle qBx, x\rangle)dx}\right.$$
$$-\frac{\int_{\mathbb{R}^n}\exp(-\langle ((p-1)A+(q-1)B)x, x\rangle)dx}{\int_{\mathbb{R}^n}\exp(-\langle qBx, x\rangle)dx}$$
$$\left.\times\frac{\int_{\mathbb{R}^n}\exp(-\langle (A+B)x, x\rangle)dx}{\int_{\mathbb{R}^n}\exp(-\langle qBx, x\rangle)dx}\right]$$
$$\leq \frac{1}{2}p\left[\frac{\int_{\mathbb{R}^n}\exp(-\langle [2A+(2-q)B]x, x\rangle)dx}{\int_{\mathbb{R}^n}\exp(-\langle qBx, x\rangle)dx}\right.$$
$$\left.-\left(\frac{\int_{\mathbb{R}^n}\exp(-\langle (A+B)x, x\rangle)dx}{\int_{\mathbb{R}^n}\exp(-\langle qBx, x\rangle)dx}\right)^2\right]^{1/2}$$
$$\leq \frac{1}{4}p,$$

which by the first equality in (2.1) gives

$$0 \leq \frac{J_n(pA)}{J_n(qB)} - \left(\frac{J_n(A+B)}{J_n(qB)}\right)^p$$
$$\leq p\left[\frac{J_n(pA)}{J_n(qB)} - \frac{J_n((p-1)A+(q-1)B)}{J_n(qB)}\frac{J_n(A+B)}{J_n(qB)}\right]$$
$$\leq \frac{1}{2}p\left[\frac{J_n(2A+(2-q)B)}{J_n(qB)} - \left(\frac{J_n(A+B)}{J_n(qB)}\right)^2\right]^{1/2}$$
$$\leq \frac{1}{4}p.$$

By utilizing the second equality in (2.1) we derive the desired result (3.4). □

Remark 5. *If A, B are positive definite matrices with $A-B>0$, then*

$$0\leq\left[\frac{\det(B)}{\det(A)}\right]^{1/2}-\frac{\det(2B)}{\det(A+B)}\leq\frac{1}{4}. \tag{3.7}$$

4. Inequalities Via Double Integrals Representation

If we take the square in the representation (1.14), then we get

$$\left(\int_{\mathbb{R}^n}\exp(-\langle x,Ax\rangle)dx\right)^2=\frac{\pi^n}{\det(A)}.$$

Since

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle)dx\right)^2&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle)\exp(-\langle Ay,y\rangle)dxdy\\&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle))dxdy,\end{aligned}$$

hence

$$K_n(A):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle))dxdy=\frac{\pi^n}{\det(A)} \tag{4.1}$$

for A a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

Utilizing this representation we can get the following results as well:

Theorem 5. *Assume that A, C are positive definite matrices and $p\geq1$, then*

$$\begin{aligned}0&\leq\frac{\det(C)}{\det(C+pA)}-\left(\frac{\det(C)}{\det(C+A)}\right)^p \\&\leq p\left[\frac{\det(C)}{\det(C+pA)}-\frac{\det(C)}{\det(C+(p-1)A)}\frac{\det(C)}{\det(C+A)}\right]\\&\leq\frac{1}{2}p\left[\frac{\det(C)}{\det(C+2A)}-\left(\frac{\det(C)}{\det(C+A)}\right)^2\right]^{\frac{1}{2}}\leq\frac{1}{4}p.\end{aligned} \tag{4.2}$$

In particular, for $C=I_n$, we derive

$$\begin{aligned}0&\leq\frac{1}{\det(I_n+pA)}-\frac{1}{[\det(I_n+A)]^p}\\&\leq p\left[\frac{1}{\det(I_n+pA)}-\frac{1}{\det(I_n+(p-1)A)}\frac{1}{\det(I_n+A)}\right]\\&\leq\frac{1}{2}p\left[\frac{1}{\det(I_n+2A)}-\frac{1}{[\det(I_n+A)]^2}\right]^{\frac{1}{2}}\leq\frac{1}{4}p.\end{aligned} \tag{4.3}$$

Proof. Take $\Omega=\mathbb{R}^n\times\mathbb{R}^n$, $h(x,y)=\exp(-\langle Cx,x\rangle-\langle Cy,y\rangle)$ and $f(x,y)=\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle)$, $x\in\mathbb{R}^n$. Observe that $0<f(x,y)\leq1$ for $(x,y)\in$

$\mathbb{R}^n \times \mathbb{R}^n$ and by (4.1) we get

$$\begin{aligned}
0 &\le \frac{\int_{\mathbb{R}^n} \exp\left(-\left\langle (C+pA)\,x,x\right\rangle - \left\langle (C+pA)\,y,y\right\rangle\right) dxdy}{\int_{\mathbb{R}^n} \exp\left(-\left\langle Cx,x\right\rangle - \left\langle Cy,y\right\rangle\right) dxdy} \\
&- \left(\frac{\int_{\mathbb{R}^n} \exp\left(-\left\langle (C+A)\,x,x\right\rangle - \left\langle (C+A)\,y,y\right\rangle\right) dxdy}{\int_{\mathbb{R}^n} \exp\left(-\left\langle Cx,x\right\rangle - \left\langle Cy,y\right\rangle\right) dxdy}\right)^p \\
&\le p\left[\frac{\int_{\mathbb{R}^n} \exp\left(-\left\langle (C+pA)\,x,x\right\rangle - \left\langle (C+pA)\,y,y\right\rangle\right) dxdy}{\int_{\mathbb{R}^n} \exp\left(-\left\langle Cx,x\right\rangle - \left\langle Cy,y\right\rangle\right) dxdy}\right. \\
&- \frac{\int_{\mathbb{R}^n} \exp\left(-\left\langle (C+(p-1)\,A)\,x,x\right\rangle - \left\langle (C+(p-1)\,A)\,y,y\right\rangle\right) dxdy}{\int_{\mathbb{R}^n} \exp\left(-\left\langle Cx,x\right\rangle - \left\langle Cy,y\right\rangle\right) dxdy} \\
&\left.\times \frac{\int_{\mathbb{R}^n} \exp\left(-\left\langle (C+A)\,x,x\right\rangle - \left\langle (C+A)\,y,y\right\rangle\right) dxdy}{\int_{\mathbb{R}^n} \exp\left(-\left\langle Cx,x\right\rangle - \left\langle Cy,y\right\rangle\right) dxdy}\right] \\
&\le \frac{1}{2}p\left[\frac{\int_{\mathbb{R}^n} \exp\left(-\left\langle (C+2A)\,x,x\right\rangle - \left\langle (C+2A)\,y,y\right\rangle\right) dxdy}{\int_{\mathbb{R}^n} \exp\left(-\left\langle Cx,x\right\rangle - \left\langle Cy,y\right\rangle\right) dxdy}\right. \\
&\left.- \left(\frac{\int_{\mathbb{R}^n} \exp\left(-\left\langle (C+A)\,x,x\right\rangle - \left\langle (C+A)\,y,y\right\rangle\right) dxdy}{\int_{\mathbb{R}^n} \exp\left(-\left\langle Cx,x\right\rangle - \left\langle Cy,y\right\rangle\right) dxdy}\right)^2\right]^{1/2} \\
&\le \frac{1}{4}p,
\end{aligned}$$

which by representation (2.1) gives

$$\begin{aligned}
0 &\le \frac{K_n\,(C+pA)}{K_n\,(C)} - \left(\frac{K_n\,(C+A)}{K_n\,(C)}\right)^p \\
&\le p\left[\frac{K_n\,(C+pA)}{K_n\,(C)} - \frac{K_n\,(C+(p-1)\,A)}{K_n\,(C)}\frac{K_n\,(C+A)}{K_n\,(C)}\right] \\
&\le \frac{1}{2}p\left[\frac{K_n\,(C+2A)}{K_n\,(C)} - \left(\frac{K_n\,(C+A)}{K_n\,(C)}\right)^2\right]^{\frac{1}{2}} \le \frac{1}{4}p.
\end{aligned}$$

By using the second identity in (2.1) we derive the desired result (4.2). □

Remark 6. *For $p=2$ we get*

$$0 \le \frac{\det(C)}{\det(C+2A)} - \left(\frac{\det(C)}{\det(C+A)}\right)^2 \le \frac{1}{4} \tag{4.4}$$

where A, C are positive definite matrices. In particular,

$$0 \le \frac{1}{\det(I_n+2A)} - \left(\frac{1}{\det(I_n+A)}\right)^2 \le \frac{1}{4}. \tag{4.5}$$

By making use of a similar argument to the one in the proof of Theorem 4 for the representation K_n, we can also prove the following result:

Theorem 6. *Let* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$*. Assume that* $A,\ B$ *are positive definite matrices with* $A-(q-1)B>0$*, then*

$$\begin{aligned}0 &\le \frac{\det(qB)}{\det(pA)}-\left(\frac{\det(qB)}{\det(A+B)}\right)^p \\ &\le p\left[\frac{\det(qB)}{\det(pA)}-\frac{\det(qB)}{\det((p-1)A+(q-1)B)}\frac{\det(qB)}{\det(A+B)}\right] \\ &\le \frac{1}{2}p\left[\frac{\det(qB)}{\det[2A+(2-q)B]}-\left[\frac{\det(qB)}{\det(A+B)}\right]^2\right]^{1/2} \\ &\le \frac{1}{4}p.\end{aligned} \tag{4.6}$$

Remark 7. *If* $A,\ B$ *are positive definite matrices with* $A-B>0$*, then*

$$0\le\frac{\det(B)}{\det(A)}-\left(\frac{\det(2B)}{\det(A+B)}\right)^2\le\frac{1}{4}.$$

A complex square matrix $H=(h_{ij}),\ i,j=1,...,n$ is said to be Hermitian provided $h_{ij}=\overline{h_{ji}}$ for all $i,\ j=1,...,n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z)=\sum_{i,j=1}^{n}a_{ij}z_i\overline{z_j}$ is positive for all $z=(z_1,...,z_n)\in\mathbb{C}^n\setminus\{0\}$.

It is known that, see for instance [11, p. 215], for a positive definite Hermitian matrix H, we have

$$K_n(H):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle\overline{z},Hz\rangle\right)dxdy=\frac{\pi^n}{\det(H)}, \tag{4.7}$$

where $z=x+iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x,y\rangle$ is understood in the real sense, i.e. $\langle x,y\rangle=\sum_{k=1}^{n}x_ky_k$.

On making use of a similar argument to the one in Theorem 5 and Theorem 6 for the representation $K_n(\cdot)$ we can state the same inequalities for positive definite Hermitian matrices H and K.

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