DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA A DIVIDED DIFFERENCE REVERSE OF JENSEN'S INTEGRAL INEQUALITY

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Abstract. In this paper we prove among others that, if A, C are positive definite matrices and $p \geq 1$, then

$$\begin{aligned} 0 &\leq \frac{\det(C)}{\det(C+pA)} - \left(\frac{\det(C)}{\det(C+A)}\right)^p \\ &\leq \frac{1}{2}\left[\frac{1-\left(\frac{\det(C)}{\det(C+A)}\right)^{p-1}}{1-\frac{\det(C)}{\det(C+A)}}\right]\left[\frac{\det(C)}{\det(C+2A)} - \left(\frac{\det(C)}{\det(C+A)}\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}\left[\frac{1-\left(\frac{\det(C)}{\det(C+A)}\right)^{p-1}}{1-\frac{\det(C)}{\det(C+A)}}\right]. \end{aligned}$$

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the Lebesgue space $L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f$ is μ-measurable and $\int_\Omega w(x)|f(x)|\,d\mu(x) < \infty\}$. For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$. We also assume that $\int_\Omega w d\mu = 1$.

An useful result that is used to provide simpler upper bounds for the difference in Jensen's inequality is the Grüss' inequality. We recall now some facts related to this famous result.

If $f, g : \Omega \to \mathbb{R}$ are μ-measurable functions and $f, g, fg \in L_w(\Omega, \mu)$, then we may consider the *Čebyšev functional*

$$T_w(f,g) := \int_\Omega wfgd\mu - \int_\Omega wfd\mu \int_\Omega wgd\mu. \tag{1.1}$$

The following result is known in the literature as the *Grüss inequality*

$$|T_w(f,g)| \leq \frac{1}{4}(\Gamma - \gamma)(\Delta - \delta), \tag{1.2}$$

provided

$$-\infty < \gamma \leq f(x) \leq \Gamma < \infty, \quad -\infty < \delta \leq g(x) \leq \Delta < \infty \tag{1.3}$$

for μ-a.e. a. $x \in \Omega$. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

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With the above assumptions, if $f \in L_w(\Omega,\mu)$ then we may define

$$D_w(f) := D_{w,1}(f) := \int_\Omega w \left| f - \int_\Omega wf d\mu \right| d\mu. \tag{1.4}$$

In 2002, Cerone & Dragomir [2] obtained the following refinement of the Grüss inequality (1.2):

Theorem 1 (Cerone & Dragomir, 2002 [2])**.** *Let $w,\ f,\ g : \Omega \to \mathbb{R}$ be μ-measurable functions with $w \geq 0$ μ-a.e. (almost everywhere) on Ω and $\int_\Omega w d\mu = 1$. If $f,\ g,\ fg \in L_w(\Omega,\mu)$ and there exists the constants $\delta,\ \Delta$ such that*

$$-\infty < \delta \leq g(x) \leq \Delta < \infty \quad \text{for} \quad \mu\text{-a.e. } x \in \Omega, \tag{1.5}$$

then we have the inequality

$$|T_w(f,g)| \leq \frac{1}{2}(\Delta - \delta) D_w(f). \tag{1.6}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

Remark 1. *The inequality (1.6) was obtained for the particular case $\Omega = [a,b]$ and the uniform weight $w(t) = 1$, $t \in [a,b]$ by X. L. Cheng and J. Sun in* [13]*. However, in that paper the authors did not prove the sharpness of the constant $\frac{1}{2}$.*

For $f \in L_{p,w}(\Omega,\mathcal{A},\mu) := \left\{ f : \Omega \to \mathbb{R}, \int_\Omega w |f|^p d\mu < \infty \right\}$, $p \geq 1$ we may also define

$$D_{w,p}(f) := \left[\int_\Omega w \left| f - \int_\Omega wf d\mu \right|^p d\mu \right]^{\frac{1}{p}} = \left\| f - \int_\Omega wf d\mu \right\|_{\Omega,p} \tag{1.7}$$

where $\|\cdot\|_{\Omega,p}$ is the usual p-norm on $L_{p,w}(\Omega,\mathcal{A},\mu)$, namely,

$$\|h\|_{\Omega,p} := \left(\int_\Omega w |h|^p d\mu \right)^{\frac{1}{p}}, \quad p \geq 1.$$

Using Hölder's inequality we get

$$D_{w,1}(f) \leq D_{w,p}(f) \quad \text{for } p \geq 1,\ f \in L_{p,w}(\Omega,\mathcal{A},\mu); \tag{1.8}$$

and, in particular for $p = 2$

$$D_{w,1}(f) \leq D_{w,2}(f) := \left[\int_\Omega wf^2 d\mu - \left(\int_\Omega wf d\mu \right)^2 \right]^{\frac{1}{2}}, \tag{1.9}$$

if $f \in L_{2,w}(\Omega,\mathcal{A},\mu)$.

For $f \in L_\infty(\Omega,\mathcal{A},\mu) := \left\{ f : \Omega \to \mathbb{R},\ \|f\|_{\Omega,\infty} := \text{essup}_{x\in\Omega} |f(x)| < \infty \right\}$ we also have

$$D_{w,p}(f) \leq D_{w,\infty}(f) := \left\| f - \int_\Omega wf d\mu \right\|_{\Omega,\infty}. \tag{1.10}$$

The following corollary may be useful in practice.

Corollary 1. *With the assumptions of Theorem 1, we have*

$$
\begin{aligned}
|T_w(f,g)| &\le \frac{1}{2}(\Delta-\delta)\, D_w(f) \\
&\le \frac{1}{2}(\Delta-\delta)\, D_{w,p}(f) \quad \text{if } f\in L_p(\Omega,\mathcal{A},\mu),\ 1<p<\infty; \\
&\le \frac{1}{2}(\Delta-\delta)\, D_{w,\infty}(f) \quad \text{if } f\in L_\infty(\Omega,\mathcal{A},\mu).
\end{aligned} \tag{1.11}
$$

Remark 2. *The inequalities in (1.11) are in order of increasing coarseness. If we assume that $-\infty<\gamma\le f(x)\le\Gamma<\infty$ for μ-a.e. $x\in\Omega$, then by the Grüss inequality for $g=f$ we have for $p=2$*

$$
\left[\int_\Omega wf^2d\mu-\left(\int_\Omega wfd\mu\right)^2\right]^{\frac{1}{2}}\le\frac{1}{2}(\Gamma-\gamma). \tag{1.12}
$$

By (1.11), we deduce the following sequence of inequalities

$$
\begin{aligned}
|T_w(f,g)| &\le \frac{1}{2}(\Delta-\delta)\int_\Omega w\left|f-\int_\Omega wfd\mu\right|d\mu \\
&\le \frac{1}{2}(\Delta-\delta)\left[\int_\Omega wf^2d\mu-\left(\int_\Omega wfd\mu\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma)
\end{aligned} \tag{1.13}
$$

for $f,\ g:\Omega\to\mathbb{R}$, μ-measurable functions and so that $-\infty<\gamma\le f(x)<\Gamma<\infty$, $-\infty<\delta\le g(x)\le\Delta<\infty$ for μ-a.e. $x\in\Omega$. Thus, the inequality (1.13) is a refinement of Grüss' inequality (1.2).

For a real function $g:[m,M]\to\mathbb{R}$ and two distinct points $\alpha,\ \beta\in[m,M]$ we recall that the *divided difference* of g in these points is defined by

$$
[\alpha,\beta;g]:=\frac{g(\beta)-g(\alpha)}{\beta-\alpha}.
$$

In what follows, we assume that $w:\Omega\to\mathbb{R}$, with $w(x)\ge0$ for μ-a.e. $x\in\Omega$, is a μ-measurable function with $\int_\Omega wd\mu=1$.

In 2011 we obtained the following refinement and reverse of Jensen's integral inequality:

Theorem 2 (Dragomir, 2011 [5])**.** *Let $\Phi:I\to\mathbb{R}$ be a continuous convex function on the interval of real numbers I and $m,\ M\in\mathbb{R}$, $m<M$ with $[m,M]\subset\mathring{I}$, $\mathring{I}$ the interior of I. If $f:\Omega\to\mathbb{R}$, is μ-measurable, satisfying the bounds*

$$
-\infty<m\le f(x)\le M<\infty \quad \text{for } \mu\text{-a.e. } x\in\Omega \tag{1.14}
$$

and such that $f,\ \Phi\circ f\in L_w(\Omega,\mu)$, then by denoting

$$
\overline{f}_{\Omega,w}:=\int_\Omega wfd\mu\in[m,M]
$$

and assuming that $\overline{f}_{\Omega,w} \neq m,\ M,$ *we have*

$$\begin{aligned} &\left|\int_{\Omega}\left|\Phi\left(f\right)-\Phi\left(\overline{f}_{\Omega,w}\right)\right|\operatorname{sgn}\left[f-\overline{f}_{\Omega,w}\right]wd\mu\right| \\ &\leq\int_{\Omega}\left(\Phi\circ f\right)wd\mu-\Phi\left(\overline{f}_{\Omega,w}\right) \\ &\leq\frac{1}{2}\left(\left[\overline{f}_{\Omega,w},M;\Phi\right]-\left[m,\overline{f}_{\Omega,w};\Phi\right]\right)D_{w}\left(f\right) \\ &\leq\frac{1}{2}\left(\left[\overline{f}_{\Omega,w},M;\Phi\right]-\left[m,\overline{f}_{\Omega,w};\Phi\right]\right)D_{w,2}\left(f\right) \\ &\leq\frac{1}{4}\left(\left[\overline{f}_{\Omega,w},M;\Phi\right]-\left[m,\overline{f}_{\Omega,w};\Phi\right]\right)\left(M-m\right), \end{aligned} \tag{1.15}$$

where sgn *is the sign function, i.e.* $\operatorname{sgn}(x)=\frac{x}{|x|}$ *for* $x\neq 0$ *and* $\operatorname{sgn}(0)=0$*. The constant* $\frac{1}{2}$ *in the second inequality from (1.15) is best possible.*

2. Some Facts for Determinants of Positive Definite Matrices

A real square matrix $A=(a_{ij})\,,\ i,\ j=1,...,n$ is *symmetric* provided $a_{ij}=a_{ji}$ for all $i,\ j=1,...,n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x)=\sum_{i,j=1}^{n}a_{ij}x_ix_j$ is positive for all $x=(x_1,...,x_n)\in\mathbb{R}^n\setminus\{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A>0$, is that all determinants

$$\det(A_k)=\det\left(a_{ij}\right),\ i,\ j=1,...,k;\ k=1,...,n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [12, pp. 211-212]

$$\begin{aligned} J_n\left(A\right) &:= \int_{\mathbb{R}^n}\exp(-\left\langle Ax,x\right\rangle)dx := \int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}\exp(-\left\langle Ax,x\right\rangle)dx \\ &= \frac{\pi^{n/2}}{\left[\det\left(A\right)\right]^{1/2}}, \end{aligned} \tag{2.1}$$

where A is a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (2.1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [12, p. 212]), namely

$$\det\left(\left(1-\lambda\right)A+\lambda B\right)\geq\left[\det\left(A\right)\right]^{1-\lambda}\left[\det\left(B\right)\right]^{\lambda} \tag{2.2}$$

for any positive definite matrices $A,\ B$ and $\lambda\in[0,1]$.

By mathematical induction we can get a generalization of (2.2) which was obtained by L. Mirsky in [11], see also [12, p. 212]

$$\det\left(\sum_{j=1}^{m}\lambda_jA_j\right)\geq\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j},\ m\geq 2, \tag{2.3}$$

where $\lambda_j>0,\ j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0,\ j=1,...,m.$

If we write (2.3) for $A_j = B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m}\lambda_j B_j^{-1}\right) \geq \prod_{j=1}^{m}\left[\det\left(B_j^{-1}\right)\right]^{\lambda_j} = \left(\prod_{j=1}^{m}\left[\det\left(B_j\right)\right]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}\left[\det\left(A_j\right)\right]^{\lambda_j} \geq \det\left[\left(\sum_{j=1}^{m}\lambda_j A_j^{-1}\right)^{-1}\right], \tag{2.4}$$

where $\lambda_j > 0$, $j = 1, ..., m$ with $\sum_{j=1}^{m}\lambda_j = 1$ and $A_j > 0$, $j = 1, ..., m$.

Using the representation (2.1) one can also prove the result, see [12, p. 212],

$$\det(A) = \det(A_{1n}) \leq \det(A_{1k})\det\left(A_{(k+1)n}\right), \ k = 1, ..., n; \tag{2.5}$$

where the determinant $\det(A_{rs})$ is defined by

$$\det(A_{rs}) = \det(a_{ij}), i, \ j = r, ..., s.$$

In particular,

$$\det(A) \leq a_{11}a_{22}...a_{nn}. \tag{2.6}$$

We recall also the Minkowski's type inequality,

$$\left[\det(A+B)\right]^{1/n} \geq \left[\det(A)\right]^{1/n} + \left[\det(B)\right]^{1/n} \tag{2.7}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [12]. For some recent results see [6]-[10].

Motivated by the above results, in this paper we prove among others that, if A, C are positive definite matrices and $p \geq 1$, then

$$\begin{aligned}
0 &\leq \frac{\det(C)}{\det(C+pA)} - \left(\frac{\det(C)}{\det(C+A)}\right)^p \\
&\leq \frac{1}{2}\left[\frac{1-\left(\frac{\det(C)}{\det(C+A)}\right)^{p-1}}{1-\frac{\det(C)}{\det(C+A)}}\right]\left[\frac{\det(C)}{\det(C+2A)} - \left(\frac{\det(C)}{\det(C+A)}\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left[\frac{1-\left(\frac{\det(C)}{\det(C+A)}\right)^{p-1}}{1-\frac{\det(C)}{\det(C+A)}}\right].
\end{aligned}$$

3. Main Results

Our first main result is as follows:

Theorem 3. *Assume that A, C are positive definite matrices and $p \geq 1$, then*

$$0 \leq \left(\frac{\det(C)}{\det(C+pA)}\right)^{1/2} - \left(\frac{\det(C)}{\det(C+A)}\right)^{p/2} \tag{3.1}$$
$$\leq \frac{1}{2}\left[\frac{1-\left(\frac{\det(C)}{\det(C+A)}\right)^{(p-1)/2}}{1-\left(\frac{\det(C)}{\det(C+A)}\right)^{1/2}}\right]\left[\left(\frac{\det(C)}{\det(C+2A)}\right)^{1/2} - \frac{\det(C)}{\det(C+A)}\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left[\frac{1-\left(\frac{\det(C)}{\det(C+A)}\right)^{(p-1)/2}}{1-\left(\frac{\det(C)}{\det(C+A)}\right)^{1/2}}\right].$$

In particular, for $C = I_n$, we have

$$0 \leq \left(\frac{1}{\det(I_n+pA)}\right)^{1/2} - \left(\frac{1}{\det(I_n+A)}\right)^{p/2} \tag{3.2}$$
$$\leq \frac{1}{2}\left[\frac{1-\left(\frac{1}{\det(I_n+A)}\right)^{(p-1)/2}}{1-\left(\frac{1}{\det(I_n+A)}\right)^{1/2}}\right]\left[\left(\frac{1}{\det(I_n+2A)}\right)^{1/2} - \frac{1}{\det(I_n+A)}\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left[\frac{1-\left(\frac{1}{\det(I_n+A)}\right)^{(p-1)/2}}{1-\left(\frac{1}{\det(I_n+A)}\right)^{1/2}}\right].$$

Proof. If we take $w = h/\int_\Omega h d\mu$ and $\Phi : [m, M] \subset [0, \infty) \to [0, \infty)$, $\Phi(t) = t^p$, $p \geq 1$, then we get by (1.15) that

$$0 \leq \frac{\int_\Omega h f^p d\mu}{\int_\Omega h d\mu} - \left(\frac{\int_\Omega h f d\mu}{\int_\Omega h d\mu}\right)^p \tag{3.3}$$
$$\leq \frac{1}{2}\left[\frac{M^p - \left(\frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu}\right)^p}{M - \frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu}} - \frac{\left(\frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu}\right)^p - m^p}{\frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu} - m}\right]$$
$$\times\left[\frac{\int_\Omega h f^2 d\mu}{\int_\Omega h d\mu} - \left(\frac{\int_\Omega h f d\mu}{\int_\Omega h d\mu}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left[\frac{M^p - \left(\frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu}\right)^p}{M - \frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu}} - \frac{\left(\frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu}\right)^p - m^p}{\frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu} - m}\right](M-m).$$

Take $\Omega = \mathbb{R}^n$, $h(x) = \exp(-\langle Cx, x\rangle)$ and $f(x) = \exp(-\langle Ax, x\rangle)$, $x \in \mathbb{R}^n$. Observe that $0 < f(x) \leq 1$ for $x \in \mathbb{R}^n$ and by (3.3) for $m = 0$, $M = 1$ we get

$$0 \leq \frac{\int_{\mathbb{R}^n} \exp(-\langle (C+pA)x, x\rangle)\,dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle)\,dx} - \left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x, x\rangle)\,dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle)\,dx}\right)^p$$
$$\leq \frac{1}{2}\left[\frac{1-\left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x,x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx,x\rangle)dx}\right)^p}{1-\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x,x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx,x\rangle)dx}} - \left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x, x\rangle)\,dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle)\,dx}\right)^{p-1}\right]$$
$$\times\left[\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+2A)x, x\rangle)\,dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle)\,dx} - \left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x, x\rangle)\,dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle)\,dx}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left[\frac{1-\left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x,x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx,x\rangle)dx}\right)^p}{1-\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x,x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx,x\rangle)dx}} - \left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x, x\rangle)\,dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle)\,dx}\right)^{p-1}\right],$$

namely

$$0 \leq \frac{\int_{\mathbb{R}^n} \exp(-\langle (C+pA)x, x\rangle)\,dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle)\,dx} - \left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x, x\rangle)\,dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle)\,dx}\right)^p$$
$$\leq \frac{1}{2}\left[\frac{1-\left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x,x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx,x\rangle)dx}\right)^{p-1}}{1-\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x,x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx,x\rangle)dx}}\right]$$
$$\times\left[\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+2A)x, x\rangle)\,dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle)\,dx} - \left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x, x\rangle)\,dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx, x\rangle)\,dx}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left[\frac{1-\left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x,x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx,x\rangle)dx}\right)^{p-1}}{1-\frac{\int_{\mathbb{R}^n} \exp(-\langle (C+A)x,x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle Cx,x\rangle)dx}}\right].$$

Using the first identity in (2.1), we get

$$0 \leq \frac{J_n(C+pA)}{J_n(C)} - \left(\frac{J_n(C+A)}{J_n(C)}\right)^p$$
$$\leq \frac{1}{2}\left[\frac{1-\left(\frac{J_n(C+A)}{J_n(C)}\right)^{p-1}}{1-\frac{J_n(C+A)}{J_n(C)}}\right]\left[\frac{J_n(C+2A)}{J_n(C)} - \left(\frac{J_n(C+A)}{J_n(C)}\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}\left[\frac{1-\left(\frac{J_n(C+A)}{J_n(C)}\right)^{p-1}}{1-\frac{J_n(C+A)}{J_n(C)}}\right].$$

By utilizing the second equality in (2.1), we deduce the desired result (3.1). □

Remark 3. *If we take $p = 2$, then we get*

$$0 \leq \left(\frac{\det(C)}{\det(C+2A)}\right)^{1/2} - \frac{\det(C)}{\det(C+A)} \leq \frac{1}{4} \tag{3.4}$$

for positive definite matrices A, C.

In particular, for $C=I_n$, we have

$$0\le\left(\frac{1}{\det(I_n+2A)}\right)^{1/2}-\frac{1}{\det(I_n+A)}\le\frac{1}{4}. \tag{3.5}$$

If we take the square in the representation (2.1), then we get

$$\left(\int_{\mathbb{R}^n}\exp(-\langle x,Ax\rangle)dx\right)^2=\frac{\pi^n}{\det(A)}.$$

Since

$$\begin{aligned}\left(\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle)dx\right)^2&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle)\exp(-\langle Ay,y\rangle)dxdy\\&=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle))dxdy,\end{aligned}$$

hence

$$K_n(A):=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle))dxdy=\frac{\pi^n}{\det(A)} \tag{3.6}$$

for A a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

Utilizing this representation we can get the following results as well:

Theorem 4. *Assume that A, C are positive definite matrices and $p\ge1$, then*

$$\begin{aligned}0&\le\frac{\det(C)}{\det(C+pA)}-\left(\frac{\det(C)}{\det(C+A)}\right)^p \\&\le\frac{1}{2}\left[\frac{1-\left(\frac{\det(C)}{\det(C+A)}\right)^{p-1}}{1-\frac{\det(C)}{\det(C+A)}}\right]\left[\frac{\det(C)}{\det(C+2A)}-\left(\frac{\det(C)}{\det(C+A)}\right)^2\right]^{\frac{1}{2}}\\&\le\frac{1}{4}\left[\frac{1-\left(\frac{\det(C)}{\det(C+A)}\right)^{p-1}}{1-\frac{\det(C)}{\det(C+A)}}\right].\end{aligned} \tag{3.7}$$

In particular, for $C=I_n$, we have

$$\begin{aligned}0&\le\frac{1}{\det(I_n+pA)}-\left(\frac{1}{\det(I_n+A)}\right)^p\\&\le\frac{1}{2}\left[\frac{1-\left(\frac{1}{\det(I_n+A)}\right)^{p-1}}{1-\frac{1}{\det(I_n+A)}}\right]\left[\frac{1}{\det(I_n+2A)}-\left(\frac{1}{\det(I_n+A)}\right)^2\right]^{\frac{1}{2}}\\&\le\frac{1}{4}\left[\frac{1-\left(\frac{1}{\det(I_n+A)}\right)^{p-1}}{1-\frac{1}{\det(I_n+A)}}\right].\end{aligned} \tag{3.8}$$

Proof. Take $\Omega=\mathbb{R}^n\times\mathbb{R}^n$, $h(x,y)=\exp(-\langle Cx,x\rangle-\langle Cy,y\rangle)$ and $f(x,y)=\exp(-\langle Ax,x\rangle-\langle Ay,y\rangle)$, $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$. Observe that $0<f(x,y)\le1$ for

$(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$ and by (3.3) for $m=0$, $M=1$ we get

$$\begin{aligned}0 &\le \frac{\int_{\mathbb{R}^n}\exp\left(-\langle (C+pA)x,x\rangle-\langle (C+pA)y,y\rangle\right)dxdy}{\int_{\mathbb{R}^n}\exp\left(-\langle Cx,x\rangle-\langle Cy,y\rangle\right)dxdy}\\
&-\left(\frac{\int_{\mathbb{R}^n}\exp\left(-\langle (C+A)x,x\rangle-\langle (C+A)y,y\rangle\right)dxdy}{\int_{\mathbb{R}^n}\exp\left(-\langle Cx,x\rangle-\langle Cy,y\rangle\right)dxdy}\right)^p\\
&\le \frac{1}{2}\left[\frac{1-\left(\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A)x,x\rangle-\langle (C+A)y,y\rangle)dxdy}{\int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle-\langle Cy,y\rangle)dxdy}\right)^{p-1}}{1-\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A)x,x\rangle-\langle (C+A)y,y\rangle)dxdy}{\int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle-\langle Cy,y\rangle)dxdy}}\right]\\
&\times\left[\frac{\int_{\mathbb{R}^n}\exp\left(-\langle (C+2A)x,x\rangle-\langle (C+2A)y,y\rangle\right)dxdy}{\int_{\mathbb{R}^n}\exp\left(-\langle Cx,x\rangle-\langle Cy,y\rangle\right)dxdy}\right.\\
&\left.-\left(\frac{\int_{\mathbb{R}^n}\exp\left(-\langle (C+A)x,x\rangle-\langle (C+A)y,y\rangle\right)dxdy}{\int_{\mathbb{R}^n}\exp\left(-\langle Cx,x\rangle-\langle Cy,y\rangle\right)dxdy}\right)^2\right]^{1/2}\\
&\le \frac{1}{4}\left[\frac{1-\left(\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A)x,x\rangle-\langle (C+A)y,y\rangle)dxdy}{\int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle-\langle Cy,y\rangle)dxdy}\right)^{p-1}}{1-\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A)x,x\rangle-\langle (C+A)y,y\rangle)dxdy}{\int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle-\langle Cy,y\rangle)dxdy}}\right].\end{aligned}$$

By utilizing the representation (3.6) we get

$$\begin{aligned}0 &\le \frac{K_n(C+pA)}{K_n(C)}-\left(\frac{K_n(C+A)}{K_n(C)}\right)^p\\
&\le \frac{1}{2}\left[\frac{1-\left(\frac{K_n(C+A)}{K_n(C)}\right)^{p-1}}{1-\frac{K_n(C+A)}{K_n(C)}}\right]\left[\frac{K_n(C+2A)}{K_n(C)}-\left(\frac{K_n(C+A)}{K_n(C)}\right)^2\right]^{\frac{1}{2}}\\
&\le \frac{1}{4}\left[\frac{1-\left(\frac{K_n(C+A)}{K_n(C)}\right)^{p-1}}{1-\frac{K_n(C+A)}{K_n(C)}}\right],\end{aligned}$$

which is equivalent to (3.7). □

Remark 4. *If we take $p=2$, then we get*

$$0\le \frac{\det(C)}{\det(C+2A)}-\left(\frac{\det(C)}{\det(C+A)}\right)^2\le\frac{1}{4} \tag{3.9}$$

for positive definite matrices A, C.

In particular, for $C=I_n$, we have

$$0\le \frac{1}{\det(I_n+2A)}-\left(\frac{1}{\det(I_n+A)}\right)^2\le\frac{1}{4}. \tag{3.10}$$

A complex square matrix $H=(h_{ij})$, $i,j=1,...,n$ is said to be Hermitian provided $h_{ij}=\overline{h_{ji}}$ for all i, $j=1,...,n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z)=\sum_{i,j=1}^n a_{ij}z_i\overline{z_j}$ is positive for all $z=(z_1,...,z_n)\in\mathbb{C}^n\setminus\{0\}$.

It is known that, see for instance [12, p. 215], for a positive definite Hermitian matrix H, we have

$$K_n(H) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle \overline{z}, Hz\rangle\right) dxdy = \frac{\pi^n}{\det(H)}, \tag{3.11}$$

where $z = x + iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x, y\rangle$ is understood in the real sense, i.e. $\langle x, y\rangle = \sum_{k=1}^n x_k y_k$.

On making use of a similar argument to the one in Theorem 4 for the representation $K_n(\cdot)$ we can state the same inequalities for positive definite Hermitian matrices H and K.

4. Related Results

We have the following result related to Hölder's inequality:

Theorem 5. *Let p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Assume that A, B are positive definite matrices with $A - (q-1)B > 0$, then*

$$\begin{aligned} 0 &\leq \left[\frac{\det(qB)}{\det(pA)}\right]^{1/2} - \left(\frac{\det(qB)}{\det(A+B)}\right)^{p/2} \\ &\leq \frac{1}{2}\left[\frac{1-\left(\frac{\det(qB)}{\det(A+B)}\right)^{\frac{p-1}{2}}}{1-\left(\frac{\det(qB)}{\det(A+B)}\right)^{1/2}}\right] \\ &\times \left[\left[\frac{\det(qB)}{\det[2A+(2-q)B]}\right]^{1/2} - \frac{\det(qB)}{\det(A+B)}\right]^{1/2} \\ &\leq \frac{1}{4}\left[\frac{1-\left(\frac{\det(qB)}{\det(A+B)}\right)^{\frac{p-1}{2}}}{1-\left(\frac{\det(qB)}{\det(A+B)}\right)^{1/2}}\right]. \end{aligned} \tag{4.1}$$

Proof. If we replace h by g^q and f by $\frac{f}{g^{q-1}}$ in (3.3), then we get

$$\begin{aligned} 0 &\leq \frac{\int_\Omega g^q \left(\frac{f}{g^{q-1}}\right)^p d\mu}{\int_\Omega g^q d\mu} - \left(\frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu}\right)^p \\ &\leq \frac{1}{2}\left[\frac{M^p - \left(\frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu}\right)^p}{M - \frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu}} - \frac{\left(\frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu}\right)^p - m^p}{\frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu} - m}\right] \\ &\times \left[\frac{\int_\Omega g^q \left(\frac{f}{g^{q-1}}\right)^2 d\mu}{\int_\Omega g^q d\mu} - \left(\frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu}\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}\left[\frac{M^p - \left(\frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu}\right)^p}{M - \frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu}} - \frac{\left(\frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu}\right)^p - m^p}{\frac{\int_\Omega g^q \frac{f}{g^{q-1}} d\mu}{\int_\Omega g^q d\mu} - m}\right](M-m), \end{aligned} \tag{4.2}$$

provided that there exists the constants γ, $\Gamma>0$ and such that

$$\gamma \leq \frac{f}{g^{q-1}} \leq \Gamma \ \mu\text{-a.e on } \Omega.$$

The inequality (4.2) is equivalent to

$$\begin{aligned}
0 &\leq \frac{\int_\Omega f^p d\mu}{\int_\Omega g^q d\mu} - \left(\frac{\int_\Omega fg d\mu}{\int_\Omega g^q d\mu}\right)^p \\
&\leq \frac{1}{2}\left[\frac{M^p - \left(\frac{\int_\Omega gf d\mu}{\int_\Omega g^q d\mu}\right)^p}{M - \frac{\int_\Omega gf d\mu}{\int_\Omega g^q d\mu}} - \frac{\left(\frac{\int_\Omega gf d\mu}{\int_\Omega g^q d\mu}\right)^p - m^p}{\frac{\int_\Omega gf d\mu}{\int_\Omega g^q d\mu} - m}\right] \\
&\times \left[\frac{\int_\Omega f^2 g^{2-q} d\mu}{\int_\Omega g^q d\mu} - \left(\frac{\int_\Omega fg d\mu}{\int_\Omega g^q d\mu}\right)^2\right]^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left[\frac{M^p - \left(\frac{\int_\Omega gf d\mu}{\int_\Omega g^q d\mu}\right)^p}{M - \frac{\int_\Omega gf d\mu}{\int_\Omega g^q d\mu}} - \frac{\left(\frac{\int_\Omega gf d\mu}{\int_\Omega g^q d\mu}\right)^p - m^p}{\frac{\int_\Omega gf d\mu}{\int_\Omega g^q d\mu} - m}\right](M-m).
\end{aligned} \tag{4.3}$$

Consider $f(x) = \exp(-\langle Ax, x\rangle)$, $g(x) = \exp(-\langle Bx, x\rangle)$. Then

$$\frac{f(x)}{g^{q-1}(x)} = \frac{\exp(-\langle Ax, x\rangle)}{\exp(-\langle (q-1)Bx, x\rangle)} = \exp(-\langle (A-(q-1)B)x, x\rangle) \leq 1$$

since $A-(q-1)B>0$.

Then by (4.3) we get

$$\begin{aligned}
0 &\leq \frac{\int_{\mathbb{R}^n} \exp(-\langle pAx, x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle qBx, x\rangle)dx} - \left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (A+B)x, x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle qBx, x\rangle)dx}\right)^p \\
&\leq \frac{1}{2}\left[\frac{1-\left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (A+B)x,x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle qBx,x\rangle)dx}\right)^p}{1-\frac{\int_{\mathbb{R}^n} \exp(-\langle (A+B)x,x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle qBx,x\rangle)dx}} - \left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (A+B)x, x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle qBx, x\rangle)dx}\right)^{p-1}\right] \\
&\times \left[\frac{\int_{\mathbb{R}^n} \exp(-\langle [2A+(2-q)B]x, x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle qBx, x\rangle)dx}\right. \\
&\left.-\left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (A+B)x, x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle qBx, x\rangle)dx}\right)^2\right]^{1/2} \\
&\leq \frac{1}{4}\left[\frac{1-\left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (A+B)x,x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle qBx,x\rangle)dx}\right)^p}{1-\frac{\int_{\mathbb{R}^n} \exp(-\langle (A+B)x,x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle qBx,x\rangle)dx}} - \left(\frac{\int_{\mathbb{R}^n} \exp(-\langle (A+B)x, x\rangle)dx}{\int_{\mathbb{R}^n} \exp(-\langle qBx, x\rangle)dx}\right)^{p-1}\right].
\end{aligned}$$

The first equality in (2.1) gives

$$\begin{aligned}
0 &\leq \frac{J_n(pA)}{J_n(qB)} - \left(\frac{J_n(A+B)}{J_n(qB)}\right)^p \\
&\leq \frac{1}{2}\left[\frac{1-\left(\frac{J_n(A+B)}{J_n(qB)}\right)^p}{1-\frac{J_n(A+B)}{J_n(qB)}} - \left(\frac{J_n(A+B)}{J_n(qB)}\right)^{p-1}\right] \\
&\times \left[\frac{J_n(2A+(2-q)B)}{J_n(qB)} - \left(\frac{J_n(A+B)}{J_n(qB)}\right)^2\right]^{1/2} \\
&\leq \frac{1}{4}\left[\frac{1-\left(\frac{J_n(A+B)}{J_n(qB)}\right)^p}{1-\frac{J_n(A+B)}{J_n(qB)}} - \left(\frac{J_n(A+B)}{J_n(qB)}\right)^{p-1}\right].
\end{aligned}$$

□

Remark 5. *If A, B are positive definite matrices with $A-B>0$, then*

$$0 \leq \left[\frac{\det(B)}{\det(A)}\right]^{1/2} - \frac{\det(2B)}{\det(A+B)} \leq \frac{1}{4}.$$

By making use of a similar argument to the one above and utilizing the representation (3.6) we can also state:

Theorem 6. *Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Assume that A, B are positive definite matrices with $A-(q-1)B>0$, then*

$$\begin{aligned}
0 &\leq \frac{\det(qB)}{\det(pA)} - \left(\frac{\det(qB)}{\det(A+B)}\right)^p \\
&\leq \frac{1}{2}\left[\frac{1-\left(\frac{\det(qB)}{\det(A+B)}\right)^{p-1}}{1-\frac{\det(qB)}{\det(A+B)}}\right] \\
&\times \left[\left[\frac{\det(qB)}{\det[2A+(2-q)B]}\right]^{1/2} - \frac{\det(qB)}{\det(A+B)}\right]^{1/2} \\
&\leq \frac{1}{4}\left[\frac{1-\left(\frac{\det(qB)}{\det(A+B)}\right)^{p-1}}{1-\frac{\det(qB)}{\det(A+B)}}\right].
\end{aligned} \tag{4.4}$$

Remark 6. *If A, B are positive definite matrices with $A-B>0$, then*

$$0 \leq \frac{\det(B)}{\det(A)} - \left(\frac{\det(2B)}{\det(A+B)}\right)^2 \leq \frac{1}{4}.$$

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