DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA TWO REVERSES OF JENSEN'S INTEGRAL INEQUALITY

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Abstract. In this paper we prove among others that, if A, C are positive definite matrices and $p \geq 1$, then

$$0 \leq \frac{\det(C)}{\det(C+pA)} - \left(\frac{\det(C)}{\det(C+A)}\right)^p \leq \frac{2^{p-1}-1}{2^{p-1}} \max\left\{1-\frac{\det(C)}{\det(C+A)}, \frac{\det(C)}{\det(C+A)}\right\} \leq \frac{2^{p-1}-1}{2^{p-1}}.$$

and

$$0 \leq \frac{\det(C)}{\det(C+pA)} - \left(\frac{\det(C)}{\det(C+A)}\right)^p \leq p\left(1-\frac{\det(C)}{\det(C+A)}\right)\frac{\det(C)}{\det(C+A)} \leq \frac{1}{4}p.$$

Some results related to reverses of Hölder's inequality are also provided.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$.

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \geq 0$ for μ-a.e. (almost every) $x \in \Omega$, consider the Lebesgue space $L_w(\Omega, \mu) := \{f : \Omega \to \mathbb{R},\ f$ is μ-measurable and $\int_\Omega w(x)|f(x)|\,d\mu(x) < \infty\}$. For simplicity of notation we write everywhere in the sequel $\int_\Omega w d\mu$ instead of $\int_\Omega w(x)\,d\mu(x)$. We also assume that $\int_\Omega w d\mu = 1$.

The following result that provides a reverse of Jensen's integral holds:

Theorem 1 (Dragomir, 2011 [2])**.** *Let $\Phi : [m, M] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on (m, M) and $f : \Omega \to [m, M]$ so that $\Phi \circ f$, $f \in L_w(\Omega, \mu)$, where $w \geq 0$ μ-a.e. on Ω with $\int_\Omega w d\mu = 1$. Then we have the inequality:*

$$0 \leq \int_\Omega w(\Phi \circ f)\,d\mu(x) - \Phi\left(\int_\Omega wf d\mu\right) \leq 2\max\left\{\frac{M - \int_\Omega wf d\mu}{M-m}, \frac{\int_\Omega wf d\mu - m}{M-m}\right\} \tag{1.1}$$

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$$\times\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right]$$
$$\leq\frac{1}{2}\max\left\{M-\int_\Omega wfd\mu,\int_\Omega wfd\mu-m\right\}\left[\Phi'_-(M)-\Phi'_+(m)\right],$$

where Φ'_- is the left and Φ'_+ is the right derivative of the convex function Φ.

Remark 1. *Since, obviously,*

$$\frac{M-\int_\Omega wfd\mu}{M-m},\frac{\int_\Omega wfd\mu-m}{M-m}\leq 1,$$

then we obtain from the first inequality in (1.1) the simpler, however coarser inequality

$$0\leq\int_\Omega w(\Phi\circ f)d\mu(x)-\Phi\left(\int_\Omega wfd\mu\right)\leq 2\left[\frac{\Phi(m)+\Phi(M)}{2}-\Phi\left(\frac{m+M}{2}\right)\right].\tag{1.2}$$

The following reverse of the Jensen's inequality also holds:

Theorem 2 (Dragomir, 2011 [3])**.** *Let $\Phi:[m,M]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on (m,M) and $f:\Omega\to[m,M]$ so that $\Phi\circ f,\ f\in L_w(\Omega,\mu)$, where $w\geq 0$ μ-a.e. on Ω with $\int_\Omega wd\mu=1$. Then we have the inequality:*

$$\begin{aligned}0&\leq\int_\Omega w(\Phi\circ f)d\mu(x)-\Phi\left(\int_\Omega wfd\mu\right)\\&\leq\left(M-\int_\Omega wfd\mu\right)\left(\int_\Omega wfd\mu-m\right)\frac{\Phi'_-(M)-\Phi'_+(m)}{M-m}\\&\leq\frac{1}{4}(M-m)\left[\Phi'_-(M)-\Phi'_+(m)\right].\end{aligned}\tag{1.3}$$

A real square matrix $A=(a_{ij}),\ i,j=1,...,n$ is *symmetric* provided $a_{ij}=a_{ji}$ for all $i,j=1,...,n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x)=\sum_{i,j=1}^n a_{ij}x_ix_j$ is positive for all $x=(x_1,...,x_n)\in\mathbb{R}^n\setminus\{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A>0$, is that all determinants

$$\det(A_k)=\det(a_{ij}),\ i,\ j=1,...,k;\ k=1,...,n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [11, pp. 211-212]

$$J_n(A):=\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle)dx:=\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}\exp(-\langle Ax,x\rangle)dx=\frac{\pi^{n/2}}{[\det(A)]^{1/2}},\tag{1.4}$$

where A is a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.4) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to

Ky Fan ([1, p. 63] or [11, p. 212]), namely

$$\det\left((1-\lambda)A+\lambda B\right)\geq\left[\det(A)\right]^{1-\lambda}\left[\det(B)\right]^{\lambda} \tag{1.5}$$

for any positive definite matrices A, B and $\lambda\in[0,1]$.

By mathematical induction we can get a generalization of (1.5) which was obtained by L. Mirsky in [10], see also [11, p. 212]

$$\det\left(\sum_{j=1}^{m}\lambda_j A_j\right)\geq\prod_{j=1}^{m}\left[\det(A_j)\right]^{\lambda_j},\ m\geq 2, \tag{1.6}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

If we write (1.6) for $A_j=B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^{m}\lambda_j B_j^{-1}\right)\geq\prod_{j=1}^{m}\left[\det\left(B_j^{-1}\right)\right]^{\lambda_j}=\left(\prod_{j=1}^{m}\left[\det(B_j)\right]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^{m}\left[\det(A_j)\right]^{\lambda_j}\geq\det\left[\left(\sum_{j=1}^{m}\lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.7}$$

where $\lambda_j>0$, $j=1,...,m$ with $\sum_{j=1}^{m}\lambda_j=1$ and $A_j>0$, $j=1,...,m$.

Using the representation (1.4) one can also prove the result, see [11, p. 212],

$$\det(A)=\det(A_{1n})\leq\det(A_{1k})\det\left(A_{(k+1)n}\right),\ k=1,...,n; \tag{1.8}$$

where the determinant $\det(A_{rs})$ is defined by

$$\det(A_{rs})=\det(a_{ij}),i,\ j=r,...,s.$$

In particular,

$$\det(A)\leq a_{11}a_{22}...a_{nn}. \tag{1.9}$$

We recall also the Minkowski's type inequality,

$$\left[\det(A+B)\right]^{1/n}\geq\left[\det(A)\right]^{1/n}+\left[\det(B)\right]^{1/n} \tag{1.10}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [11]. For some recent results see [5]-[9].

Motivated by the above results, in this paper we prove among others that, if A, C are positive definite matrices and $p\geq 1$, then

$$\begin{aligned}0&\leq\frac{\det(C)}{\det(C+pA)}-\left(\frac{\det(C)}{\det(C+A)}\right)^{p}\\&\leq\frac{2^{p-1}-1}{2^{p-1}}\max\left\{1-\frac{\det(C)}{\det(C+A)},\frac{\det(C)}{\det(C+A)}\right\}\\&\leq\frac{2^{p-1}-1}{2^{p-1}}.\end{aligned}$$

and

$$0 \leq \frac{\det(C)}{\det(C+pA)} - \left(\frac{\det(C)}{\det(C+A)}\right)^p \leq p\left(1 - \frac{\det(C)}{\det(C+A)}\right)\frac{\det(C)}{\det(C+A)} \leq \frac{1}{4}p.$$

2. Main Results

Our first result is as follows:

Theorem 3. *Assume that A, C are positive definite matrices and $p \geq 1$, then*

$$0 \leq \left(\frac{\det(C)}{\det(C+pA)}\right)^{1/2} - \left(\frac{\det(C)}{\det(C+A)}\right)^{p/2} \leq \frac{2^{p-1}-1}{2^{p-1}} \max\left\{1 - \left(\frac{\det(C)}{\det(C+A)}\right)^{1/2}, \left(\frac{\det(C)}{\det(C+A)}\right)^{1/2}\right\} \leq \frac{2^{p-1}-1}{2^{p-1}}. \tag{2.1}$$

In particular, for $C = I_n$,

$$0 \leq \left(\frac{1}{\det(I_n+pA)}\right)^{1/2} - \left(\frac{1}{\det(I_n+A)}\right)^{p/2} \leq \frac{2^{p-1}-1}{2^{p-1}} \max\left\{1 - \left(\frac{1}{\det(I_n+A)}\right)^{1/2}, \left(\frac{1}{\det(I_n+A)}\right)^{1/2}\right\} \leq \frac{2^{p-1}-1}{2^{p-1}}. \tag{2.2}$$

Proof. If we take $w = h/\int_\Omega h d\mu$ and $\Phi : [m, M] \subset [0, \infty) \to [0, \infty)$, $\Phi(t) = t^p$, $p \geq 1$, then we get by (1.1) that

$$0 \leq \frac{\int_\Omega h f^p d\mu}{\int_\Omega h d\mu} - \left(\frac{\int_\Omega h f d\mu}{\int_\Omega h d\mu}\right)^p \leq 2\max\left\{\frac{M - \frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu}}{M-m}, \frac{\frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu} - m}{M-m}\right\} \times \left[\frac{m^p + M^p}{2} - \left(\frac{m+M}{2}\right)^p\right] \leq 2\left[\frac{m^p + M^p}{2} - \left(\frac{m+M}{2}\right)^p\right]. \tag{2.3}$$

Take $\Omega=\mathbb{R}^n$, $h(x)=\exp(-\langle Cx,x\rangle)$ and $f(x)=\exp(-\langle Ax,x\rangle)$, $x\in\mathbb{R}^n$. Observe that $0<f(x)\le 1$ for $x\in\mathbb{R}^n$ and by (2.3) we get

$$\begin{aligned}0&\le\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+pA)x,x\rangle)\,dx}{\int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle)\,dx}-\left(\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A)x,x\rangle)\,dx}{\int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle)\,dx}\right)^p\\&\le\frac{2^{p-1}-1}{2^{p-1}}\max\left\{1-\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A)x,x\rangle)\,dx}{\int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle)\,dx},\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A)x,x\rangle)\,dx}{\int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle)\,dx}\right\}\\&\le\frac{2^{p-1}-1}{2^{p-1}},\end{aligned}$$

which by representation (1.4) gives

$$\begin{aligned}0&\le\frac{J_n(C+pA)}{J_n(C)}-\left(\frac{J_n(C+A)}{J_n(C)}\right)^p\\&\le\frac{2^{p-1}-1}{2^{p-1}}\max\left\{1-\frac{J_n(C+A)}{J_n(C)},\frac{J_n(C+A)}{J_n(C)}\right\}\le\frac{2^{p-1}-1}{2^{p-1}}.\end{aligned}$$

By using the second identity in (1.4) we derive the desired result (2.1). □

Theorem 4. *Assume that A, C are positive definite matrices and $p\ge 1$, then*

$$\begin{aligned}0&\le\left(\frac{\det(C)}{\det(C+pA)}\right)^{1/2}-\left(\frac{\det(C)}{\det(C+A)}\right)^{p/2}\\&\le p\left(1-\left(\frac{\det(C)}{\det(C+A)}\right)^{1/2}\right)\left(\frac{\det(C)}{\det(C+A)}\right)^{1/2}\le\frac{1}{4}p.\end{aligned}\tag{2.4}$$

In particular, for $C=I_n$,

$$\begin{aligned}0&\le\left(\frac{1}{\det(I_n+pA)}\right)^{1/2}-\left(\frac{1}{\det(I_n+A)}\right)^{p/2}\\&\le p\left(1-\left(\frac{1}{\det(I_n+A)}\right)^{1/2}\right)\left(\frac{1}{\det(I_n+A)}\right)^{1/2}\le\frac{1}{4}p.\end{aligned}\tag{2.5}$$

Proof. If we take $w=h/\int_\Omega hd\mu$ and $\Phi:[m,M]\subset[0,\infty)\to[0,\infty)$, $\Phi(t)=t^p$, $p\ge 1$, then we get by (1.3) that

$$\begin{aligned}0\le 0&\le\frac{\int_\Omega hf^pd\mu}{\int_\Omega hd\mu}-\left(\frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu}\right)^p\\&\le p\frac{M^{p-1}-m^{p-1}}{M-m}\left(M-\frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu}\right)\left(\frac{\int_\Omega hfd\mu}{\int_\Omega hd\mu}-m\right)\\&\le\frac{1}{4}p(M-m)\left(M^{p-1}-m^{p-1}\right).\end{aligned}\tag{2.6}$$

Take $\Omega=\mathbb{R}^n$, $h(x)=\exp(-\langle Cx,x\rangle)$ and $f(x)=\exp(-\langle Ax,x\rangle)$, $x\in\mathbb{R}^n$. Observe that $0<f(x)\le 1$ for $x\in\mathbb{R}^n$ and by (2.6) we get

$$\begin{aligned}0&\le\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+pA)x,x\rangle)\,dx}{\int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle)\,dx}-\left(\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A)x,x\rangle)\,dx}{\int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle)\,dx}\right)^p\\&\le p\left(1-\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A)x,x\rangle)\,dx}{\int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle)\,dx}\right)\frac{\int_{\mathbb{R}^n}\exp(-\langle (C+A)x,x\rangle)\,dx}{\int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle)\,dx}\le\frac{1}{4}p,\end{aligned}$$

which by representation (1.4) gives

$$\begin{aligned}0 &\le \frac{J_n(C+pA)}{J_n(C)} - \left(\frac{J_n(C+A)}{J_n(C)}\right)^p \\ &\le p\left(1-\frac{J_n(C+A)}{J_n(C)}\right)\frac{J_n(C+A)}{J_n(C)} \le \frac{1}{4}p.\end{aligned}$$

By using the second identity in (1.4) we derive the desired result (2.4). □

If we take the square in the representation (1.4), then we get

$$\left(\int_{\mathbb{R}^n} \exp(-\langle x, Ax\rangle)dx\right)^2 = \frac{\pi^n}{\det(A)}.$$

Since

$$\begin{aligned}\left(\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)dx\right)^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle)\exp(-\langle Ay, y\rangle)dxdy \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle))dxdy,\end{aligned}$$

hence

$$K_n(A) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle))dxdy = \frac{\pi^n}{\det(A)} \tag{2.7}$$

for A a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

Utilizing this representation we can get the following results as well:

Theorem 5. *Assume that A, C are positive definite matrices and $p \ge 1$, then*

$$\begin{aligned}0 &\le \frac{\det(C)}{\det(C+pA)} - \left(\frac{\det(C)}{\det(C+A)}\right)^p \\ &\le \frac{2^{p-1}-1}{2^{p-1}}\max\left\{1-\frac{\det(C)}{\det(C+A)}, \frac{\det(C)}{\det(C+A)}\right\} \\ &\le \frac{2^{p-1}-1}{2^{p-1}}.\end{aligned} \tag{2.8}$$

In particular, for $C = I_n$,

$$\begin{aligned}0 &\le \frac{1}{\det(I_n+pA)} - \left(\frac{1}{\det(I_n+A)}\right)^p \\ &\le \frac{2^{p-1}-1}{2^{p-1}}\max\left\{1-\frac{1}{\det(I_n+A)}, \frac{1}{\det(I_n+A)}\right\} \\ &\le \frac{2^{p-1}-1}{2^{p-1}}.\end{aligned} \tag{2.9}$$

Proof. Take $\Omega = \mathbb{R}^n \times \mathbb{R}^n$, $h(x,y) = \exp(-\langle Cx, x\rangle - \langle Cy, y\rangle)$ and $f(x,y) = \exp(-\langle Ax, x\rangle - \langle Ay, y\rangle)$, $x \in \mathbb{R}^n$. Observe that $0 < f(x,y) \le 1$ for $(x,y) \in$

$\mathbb{R}^n \times \mathbb{R}^n$ and by (2.7) we get

$$\begin{aligned}
0 &\leq \frac{\int_{\mathbb{R}^n} \exp\left(-\langle (C+pA)x,x\rangle - \langle (C+pA)y,y\rangle\right) dxdy}{\int_{\mathbb{R}^n} \exp\left(-\langle Cx,x\rangle - \langle Cy,y\rangle\right) dxdy} \\
&- \left(\frac{\int_{\mathbb{R}^n} \exp\left(-\langle (C+A)x,x\rangle - \langle (C+A)y,y\rangle\right) dxdy}{\int_{\mathbb{R}^n} \exp\left(-\langle Cx,x\rangle - \langle Cy,y\rangle\right) dxdy}\right)^p \\
&\leq \frac{2^{p-1}-1}{2^{p-1}} \\
&\times \max\left\{1 - \frac{\int_{\mathbb{R}^n} \exp\left(-\langle (C+A)x,x\rangle - \langle (C+A)y,y\rangle\right) dxdy}{\int_{\mathbb{R}^n} \exp\left(-\langle Cx,x\rangle - \langle Cy,y\rangle\right) dxdy},\right. \\
&\left.\frac{\int_{\mathbb{R}^n} \exp\left(-\langle (C+A)x,x\rangle - \langle (C+A)y,y\rangle\right) dxdy}{\int_{\mathbb{R}^n} \exp\left(-\langle Cx,x\rangle - \langle Cy,y\rangle\right) dxdy}\right\} \\
&\leq \frac{2^{p-1}-1}{2^{p-1}},
\end{aligned}$$

which, by representation (2.7), gives

$$\begin{aligned}
0 &\leq \frac{K_n(C+pA)}{K_n(C)} - \left(\frac{K_n(C+A)}{K_n(C)}\right)^p \\
&\leq \frac{2^{p-1}-1}{2^{p-1}} \max\left\{1 - \frac{J_n(C+A)}{J_n(C)}, \frac{J_n(C+A)}{J_n(C)}\right\} \leq \frac{2^{p-1}-1}{2^{p-1}}.
\end{aligned}$$

By using the second identity in (2.7) we derive the desired result (2.4). □

In a similar fashion we can also prove, by utilizing, representation (2.7):

Theorem 6. *Assume that A, C are positive definite matrices and $p \geq 1$, then*

$$\begin{aligned}
0 &\leq \frac{\det(C)}{\det(C+pA)} - \left(\frac{\det(C)}{\det(C+A)}\right)^p \\
&\leq p\left(1 - \frac{\det(C)}{\det(C+A)}\right)\frac{\det(C)}{\det(C+A)} \leq \frac{1}{4}p.
\end{aligned} \tag{2.10}$$

In particular, for $C = I_n$,

$$\begin{aligned}
0 &\leq \frac{1}{\det(I_n+pA)} - \left(\frac{1}{\det(I_n+A)}\right)^p \\
&\leq p\left(1 - \frac{1}{\det(I_n+A)}\right)\frac{1}{\det(I_n+A)} \leq \frac{1}{4}p.
\end{aligned} \tag{2.11}$$

A complex square matrix $H = (h_{ij})$, $i,j = 1,...,n$ is said to be Hermitian provided $h_{ij} = \overline{h_{ji}}$ for all i, $j = 1,...,n$. A Hermitian matrix is said to be positive definite if the Hermitian form $P(z) = \sum_{i,j=1}^n a_{ij} z_i \overline{z_j}$ is positive for all $z = (z_1,...,z_n) \in \mathbb{C}^n \setminus \{0\}$.

It is known that, see for instance [11, p. 215], for a positive definite Hermitian matrix H, we have

$$K_n(H) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\left(-\langle \overline{z}, Hz\rangle\right) dxdy = \frac{\pi^n}{\det(H)}, \tag{2.12}$$

where $z = x + iy$ and dx and dy denote integration over real n-dimensional space $\mathbb{R}^n$. Here the inner product $\langle x,y\rangle$ is understood in the real sense, i.e. $\langle x,y\rangle = \sum_{k=1}^n x_k y_k$.

On making use of a similar argument to the one in Theorem 5 and Theorem 6 for the representation $K_n(\cdot)$ we can state the same inequalities for positive definite Hermitian matrices H and K.

3. Related Results

We also have the following Hölder's type results:

Theorem 7. *Let p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Assume that A, B are positive definite matrices with $A-(q-1)B>0$, then*

$$0\leq\left[\frac{\det(qB)}{\det(pA)}\right]^{1/2}-\left(\frac{\det(qB)}{\det(A+B)}\right)^{p/2}$$
$$\leq\frac{2^{p-1}-1}{2^{p-1}}\max\left\{1-\left(\frac{\det(qB)}{\det(A+B)}\right)^{1/2},\left(\frac{\det(qB)}{\det(A+B)}\right)^{1/2}\right\}$$
$$\leq\frac{2^{p-1}-1}{2^{p-1}}. \tag{3.1}$$

Proof. If we replace h by g^q and f by $\frac{f}{g^{q-1}}$ in (2.3), then we get

$$0\leq\frac{\int_\Omega g^q\left(\frac{f}{g^{q-1}}\right)^p d\mu}{\int_\Omega g^q d\mu}-\left(\frac{\int_\Omega g^q\frac{f}{g^{q-1}}d\mu}{\int_\Omega g^q d\mu}\right)^p$$
$$\leq\frac{2}{\Gamma-\gamma}\times\left[\frac{\gamma^p+\Gamma^p}{2}-\left(\frac{\gamma+\Gamma}{2}\right)^p\right]$$
$$\times\max\left\{\Gamma-\frac{\int_\Omega g^q\frac{f}{g^{q-1}}d\mu}{\int_\Omega g^q d\mu},\frac{\int_\Omega g^q\frac{f}{g^{q-1}}d\mu}{\int_\Omega g^q d\mu}-\gamma\right\}$$
$$\leq 2\left[\frac{\gamma^p+\Gamma^p}{2}-\left(\frac{\gamma+\Gamma}{2}\right)^p\right], \tag{3.2}$$

provided that there exists the constants γ, $\Gamma>0$ and such that

$$\gamma\leq\frac{f}{g^{q-1}}\leq\Gamma\ \mu\text{-a.e on }\Omega.$$

The inequality (3.2) is equivalent to

$$0\leq\frac{\int_\Omega f^p d\mu}{\int_\Omega g^q d\mu}-\left(\frac{\int_\Omega fg d\mu}{\int_\Omega g^q d\mu}\right)^p$$
$$\leq\frac{2}{\Gamma-\gamma}\times\left[\frac{\gamma^p+\Gamma^p}{2}-\left(\frac{\gamma+\Gamma}{2}\right)^p\right]$$
$$\times\max\left\{\Gamma-\frac{\int_\Omega fg d\mu}{\int_\Omega g^q d\mu},\frac{\int_\Omega fg d\mu}{\int_\Omega g^q d\mu}-\gamma\right\}$$
$$\leq 2\left[\frac{\gamma^p+\Gamma^p}{2}-\left(\frac{\gamma+\Gamma}{2}\right)^p\right]. \tag{3.3}$$

Consider $f(x)=\exp(-\langle Ax,x\rangle)$, $g(x)=\exp(-\langle Bx,x\rangle)$. Then

$$\frac{f(x)}{g^{q-1}(x)}=\frac{\exp(-\langle Ax,x\rangle)}{\exp(-\langle(q-1)Bx,x\rangle)}=\exp(-\langle(A-(q-1)B)x,x\rangle)\leq 1$$

since $A-(q-1)B>0$.

From (3.3) we then get

$$\begin{aligned}
0 &\le \frac{\int_{\mathbb{R}^n} \exp\left(-\langle pAx, x\rangle\right) dx}{\int_{\mathbb{R}^n} \exp\left(-\langle qBx, x\rangle\right) dx} - \left(\frac{\int_{\mathbb{R}^n} \exp\left(-\langle (A+B)x, x\rangle\right) dx}{\int_{\mathbb{R}^n} \exp\left(-\langle qBx, x\rangle\right) dx}\right)^p \\
&\le \frac{2^{p-1}-1}{2^{p-1}} \\
&\times \max\left\{1 - \frac{\int_{\mathbb{R}^n} \exp\left(-\langle (A+B)x, x\rangle\right) dx}{\int_{\mathbb{R}^n} \exp\left(-\langle qBx, x\rangle\right) dx}, \frac{\int_{\mathbb{R}^n} \exp\left(-\langle (A+B)x, x\rangle\right) dx}{\int_{\mathbb{R}^n} \exp\left(-\langle qBx, x\rangle\right) dx}\right\} \\
&\le \frac{2^{p-1}-1}{2^{p-1}},
\end{aligned}$$

which, by representation (1.4), provides

$$\begin{aligned}
0 &\le \frac{J_n(pA)}{J_n(qB)} - \left(\frac{J_n(A+B)}{J_n(qB)}\right)^p \\
&\le \frac{2^{p-1}-1}{2^{p-1}} \max\left\{1 - \frac{J_n(A+B)}{J_n(qB)}, \frac{J_n(A+B)}{J_n(qB)}\right\} \\
&\le \frac{2^{p-1}-1}{2^{p-1}}.
\end{aligned}$$

By using the second identity in (1.4) we derive the desired result (3.1). □

Remark 2. *If $A - B > 0$, then*

$$\begin{aligned}
0 &\le \left[\frac{\det(B)}{\det(A)}\right]^{1/2} - \frac{\det(2B)}{\det(A+B)} \\
&\le \frac{1}{2} \max\left\{1 - \left(\frac{\det(2B)}{\det(A+B)}\right)^{1/2}, \left(\frac{\det(2B)}{\det(A+B)}\right)^{1/2}\right\} \le \frac{1}{2}.
\end{aligned} \tag{3.4}$$

We also have:

Theorem 8. *Let p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Assume that A, B are positive definite matrices with $A - (q-1)B > 0$, then*

$$\begin{aligned}
0 &\le \left[\frac{\det(qB)}{\det(pA)}\right]^{1/2} - \left(\frac{\det(qB)}{\det(A+B)}\right)^{p/2} \\
&\le p\left[1 - \left(\frac{\det(qB)}{\det(A+B)}\right)^{1/2}\right]\left(\frac{\det(qB)}{\det(A+B)}\right)^{1/2} \le \frac{1}{4}p.
\end{aligned} \tag{3.5}$$

The proof goes in a similar way by employing the inequality (2.6).

Remark 3. *If $A - B > 0$, then*

$$\begin{aligned}
0 &\le \left[\frac{\det(B)}{\det(A)}\right]^{1/2} - \frac{\det(2B)}{\det(A+B)} \\
&\le 2\left[1 - \left(\frac{\det(2B)}{\det(A+B)}\right)^{1/2}\right]\left(\frac{\det(2B)}{\det(A+B)}\right)^{1/2} \le \frac{1}{2}.
\end{aligned} \tag{3.6}$$

Finally, by employing the representation (2.7) we can state the following inequalities as well

$$0 \leq \frac{\det(qB)}{\det(pA)} - \left(\frac{\det(qB)}{\det(A+B)}\right)^p \leq \frac{2^{p-1}-1}{2^{p-1}} \max\left\{1 - \frac{\det(qB)}{\det(A+B)}, \frac{\det(qB)}{\det(A+B)}\right\} \leq \frac{2^{p-1}-1}{2^{p-1}} \tag{3.7}$$

and

$$0 \leq \frac{\det(qB)}{\det(pA)} - \left(\frac{\det(qB)}{\det(A+B)}\right)^p \leq p\left(1 - \frac{\det(qB)}{\det(A+B)}\right)\frac{\det(qB)}{\det(A+B)} \leq \frac{1}{4}p, \tag{3.8}$$

for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $A,\ B$ are positive definite matrices with $A - (q-1)B > 0$.

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