p-Schatten norm generalized Canavati fractional Ostrowski, Opial and Grüss type inequalities involving several functions

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Abstract

Using generalized Canavati fractional left and right vectorial Taylor formulae we establish generalized fractional Ostrowski, Opial and Grüss type inequalities for several functions that take values in the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$. The estimates are with respect to all p-Schatten norms, $1 \leq p < \infty$. We finish with applications.

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1 Introduction

The following results inspire our work.

Theorem 1 *(1938, Ostrowski [16]) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_{\infty}^{\sup} := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_{\infty}^{\sup}, \tag{1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

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Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

We mention

Theorem 2 *(1882, Čebyšev [8]) Let $f, g : [a,b] \to \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_\infty([a,b])$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right|$$

$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty . \tag{2}$$

The above integrals are assumed to exist.

The related Grüss type inequalities have many applications to Probability Theory. We presented also ([3], Ch. 8,9) mixed fractional Ostrowski and Grüss-Cebysev type inequalities for several functions, acting to all possible directions. The estimates involve the left and right Caputo fractional derivatives. See also the monographs written by the author [1], Chapters 24-26 and [2], Chapters 2-6.

We are motivated also by S. Dragomir [11] recent work:

An operator $A \in \mathcal{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the p-Schatten norm is finite

$$\|A\|_p := [tr(|A|^p)]^{\frac{1}{p}} < \infty.$$

Assume that $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_q(H)$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, are continuous and B is strongly differentiable on (a,b), then

$$\left\| \int_a^b A(t) B(t) dt - \left(\int_a^b A(s) ds \right) B(u) \right\|_1 \leq$$

$$\sup_{t \in [a,b]} \|B'(t)\|_q \times \begin{cases} \left[\frac{1}{2}(b-a) + \left| u - \frac{a+b}{2} \right| \right] \int_a^b \|A(t)\|_p dt, \\ \left[\frac{(u-a)^{\beta+1} + (b-u)^{\beta+1}}{\beta+1} \right]^{\frac{1}{\beta}} \left(\int_a^b \|A(t)\|_p^\alpha \right)^{\frac{1}{\alpha}}, \\ \text{for } \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \left[\frac{1}{4}(b-a)^2 + \left(u - \frac{a+b}{2} \right)^2 \right] \sup_{t \in [a,b]} \|A(t)\|_p , \end{cases} \tag{3}$$

for all $u \in [a,b]$, an Ostrowski type inequality.

Further inspiration comes from S. Dragomir [12] recent work on Grüss inequalities:

For two continuous functions $A, B : [a,b] \to \mathcal{B}(H)$ we define the noncommutative Cebysev fractional

$$D(A,B) := (b-a)\int_a^b A(t)B(t)\,dt - \int_a^b A(t)\,dt \int_a^b B(t)\,dt.$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, let $A : [a,b] \to \mathcal{B}_p(H)$, $B : [a,b] \to \mathcal{B}_q(H)$ be strongly differentiable functions on the interval (a,b), then

$$\|D(A,B)\|_1 \leq D\left(\int_a^b \|A'(u)\|_p\,du, \int_a^b \|B'(u)\|_q\,du\right) \leq \tag{4}$$

$$\frac{1}{4}(b-a)^2 \int_a^b \|A'(u)\|_p\,du \int_a^b \|B'(u)\|_q\,du.$$

We are also inspired by Z. Opial [15], 1960, famous inequality.

Theorem 3 *Let $x(t) \in C^1([0,h])$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then*

$$\int_0^h |x(t)x'(t)|\,dt \leq \frac{h}{4}\int_0^h (x'(t))^2\,dt. \tag{5}$$

In (5), the constant $\frac{h}{4}$ is the best possible. Inequality (5) holds as equality for the optimal function

$$x(t) = \begin{cases} ct, & 0 \leq t \leq \frac{h}{2}, \\ c(h-t), & \frac{h}{2} \leq t \leq h, \end{cases} \tag{6}$$

where $c > 0$ is an arbitrary constant.

Opial-type inequalities are used a lot in proving uniqueness of solutions to differential equations and also to give upper bounds to their solutions.

For an extensive study about fractional Opial inequalities see the author's monograph [1].

In this article we generalize [3], Ch. 8,9 for several Banach algebra $\mathcal{B}_p(H)$ valued functions, in the sense of developing fractional Ostrowski, Opial and Grüss type inequalities. Now our left and right generalized Canavati fractional derivatives are for Banach space valued functions and our integrals are of Bochner type [13]. Applications finish the article.

2 Background on Vectorial generalized Canavati fractional calculus

All in this section come from [5], pp. 109-115 and [4].

Let $g:[a,b]\to\mathbb{R}$ be a strictly increasing function. such that $g\in C^1([a,b])$, and $g^{-1}\in C^n([g(a),g(b)])$, $n\in\mathbb{N}$, $(X,\|\cdot\|)$ is a Banach space. Let $f\in C^n([a,b],X)$, and call $l:=f\circ g^{-1}:[g(a),g(b)]\to X$. It is clear that $l,l',...,l^{(n)}$ are continuous functions from $[g(a),g(b)]$ into $f([a,b])\subseteq X$.

Let $\nu\geq 1$ such that $[\nu]=n$, $n\in\mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0<\nu<1$, $[\nu]=0$.

I) Let $h\in C([g(a),g(b)],X)$, we define the left Riemann-Liouville Bochner fractional integral as

$$\left(J_\nu^{z_0}h\right)(z):=\frac{1}{\Gamma(\nu)}\int_{z_0}^{z}(z-t)^{\nu-1}h(t)\,dt, \tag{7}$$

for $g(a)\leq z_0\leq z\leq g(b)$, where Γ is the gamma function; $\Gamma(\nu)=\int_0^\infty e^{-t}t^{\nu-1}dt$. We set $J_0^{z_0}h=h$.

Let $\alpha:=\nu-[\nu]$ $(0<\alpha<1)$. We define the subspace $C_{g(x_0)}^{\nu}([g(a),g(b)],X)$ of $C^{[\nu]}([g(a),g(b)],X)$, where $x_0\in[a,b]$ as:

$$C_{g(x_0)}^{\nu}([g(a),g(b)],X)=$$
$$\left\{h\in C^{[\nu]}([g(a),g(b)],X):J_{1-\alpha}^{g(x_0)}h^{([\nu])}\in C^1([g(x_0),g(b)],X)\right\}. \tag{8}$$

So let $h\in C_{g(x_0)}^{\nu}([g(a),g(b)],X)$, we define the left g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(x_0),g(b)]$ as

$$D_{g(x_0)}^{\nu}h:=\left(J_{1-\alpha}^{g(x_0)}h^{([\nu])}\right)'. \tag{9}$$

Clearly, for $h\in C_{g(x_0)}^{\nu}([g(a),g(b)],X)$, there exists

$$\left(D_{g(x_0)}^{\nu}h\right)(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_{g(x_0)}^{z}(z-t)^{-\alpha}h^{([\nu])}(t)\,dt, \tag{10}$$

for all $g(x_0)\leq z\leq g(b)$.

In particular, when $f\circ g^{-1}\in C_{g(x_0)}^{\nu}([g(a),g(b)],X)$, we have that

$$\left(D_{g(x_0)}^{\nu}\left(f\circ g^{-1}\right)\right)(z)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_{g(x_0)}^{z}(z-t)^{-\alpha}\left(f\circ g^{-1}\right)^{([\nu])}(t)\,dt, \tag{11}$$

for all $g(x_0)\leq z\leq g(b)$. We have that $D_{g(x_0)}^{n}\left(f\circ g^{-1}\right)=\left(f\circ g^{-1}\right)^{(n)}$ and $D_{g(x_0)}^{0}\left(f\circ g^{-1}\right)=f\circ g^{-1}$, see [4].

By [4], we have for $f\circ g^{-1}\in C_{g(x_0)}^{\nu}([g(a),g(b)],X)$, where $x_0\in[a,b]$ the following left generalized g-fractional, of Canavati type,X-valued Taylor's formula:

Theorem 4 *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ is fixed.*

(i) If $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!}(g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{12}$$

for all $x_0 \leq x \leq b$.

(ii) If $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{13}$$

for all $x_0 \leq x \leq b$.

II) Let $h \in C([g(a), g(b)], X)$, we define the right Riemann-Liouville Bochner fractional integral as

$$\left(J^{\nu}_{z_0-}h\right)(z) := \frac{1}{\Gamma(\nu)} \int_{z}^{z_0} (t - z)^{\nu-1} h(t)\, dt, \tag{14}$$

for $g(a) \leq z \leq z_0 \leq g(b)$. We set $J^0_{z_0-}h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ as:

$$C^{\nu}_{g(x_0)-}([g(a), g(b)], X) :=$$

$$\left\{h \in C^{[\nu]}([g(a), g(b)], X) : J^{1-\alpha}_{g(x_0)-}h^{([\nu])} \in C^1([g(a), g(x_0)], X)\right\}. \tag{15}$$

So let $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, we define the right g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(a), g(x_0)]$ as

$$D^{\nu}_{g(x_0)-}h := (-1)^{n-1}\left(J^{1-\alpha}_{g(x_0)-}h^{([\nu])}\right)'. \tag{16}$$

Clearly, for $h \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, there exists

$$\left(D^{\nu}_{g(x_0)-}h\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t - z)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{17}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, we have that

$$\left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{z}^{g(x_0)} (t - z)^{-\alpha} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \tag{18}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

We get that

$$\left(D^n_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = (-1)^n \left(f \circ g^{-1}\right)^{(n)}(z) \tag{19}$$

and $\left(D^0_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(z) = \left(f \circ g^{-1}\right)(z)$, all $z \in [g(a), g(b)]$, see [4].

By [4], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ is fixed, the following right generalized g-fractional, of Canavati type, X-valued Taylor's formula:

Theorem 5 *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], X)$, where $x_0 \in [a,b]$ is fixed.*

(i) If $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!}(g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{20}$$

for all $a \leq x \leq x_0$,

(ii) If $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D^{\nu}_{g(x_0)-}\left(f \circ g^{-1}\right)\right)(t)\, dt, \tag{21}$$

all $a \leq x \leq x_0$.

III) Denote by

$$D^{m\nu}_{g(x_0)} = D^{\nu}_{g(x_0)} D^{\nu}_{g(x_0)} \ldots D^{\nu}_{g(x_0)} \quad (m\text{-times}),\ m \in \mathbb{N}. \tag{22}$$

We mention the following modified and generalized left X-valued fractional Taylor's formula of Canavati type:

Theorem 6 *Let $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, strictly increasing: $g^{-1} \in C^1([g(a), g(b)])$. Assume that $\left(D^{i\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right) \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a,b]$, for $i = 0, 1, ..., m$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{(m+1)\nu-1} \left(D^{(m+1)\nu}_{g(x_0)}\left(f \circ g^{-1}\right)\right)(z)\, dz, \tag{23}$$

all $x_0 \leq x \leq b$.

IV) Denote by

$$D_{g(x_0)-}^{m\nu} = D_{g(x_0)-}^{\nu} D_{g(x_0)-}^{\nu} ... D_{g(x_0)-}^{\nu} \quad (m \text{ times}),\ m \in \mathbb{N}. \tag{24}$$

We mention the following modified and generalized right X-valued fractional Taylor's formula of Canavati type:

Theorem 7 *Let $f \in C^1([a,b], X)$, $g \in C^1([a,b])$, strictly increasing: $g^{-1} \in C^1([g(a), g(b)])$. Assume that $\left(D_{g(x_0)-}^{i\nu}(f \circ g^{-1})\right) \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, $0 < \nu < 1$, $x_0 \in [a,b]$, for all $i = 0, 1, ..., m$. Then*

$$f(x) = \frac{1}{\Gamma((m+1)\nu)} \int_{g(x)}^{g(x_0)} (z - g(x))^{(m+1)\nu - 1} \left(D_{g(x_0)-}^{(m+1)\nu}(f \circ g^{-1})\right)(z)\, dz, \tag{25}$$

all $a \le x \le x_0 \le b$.

3 Basic Banach Algebras background

All here come from [17].

We need

Definition 8 *([17], p. 245) A complex algebra is a vector space A over the complex filed $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{26}$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \tag{27}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{28}$$

for all x, y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \le \|x\| \|y\| \quad (x \in A,\ y \in A) \tag{29}$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{30}$$

and

$$\|e\| = 1, \tag{31}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 9 *Commutativity of A is explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (30).

Inequality (29) makes multiplication to be continuous, more precisely left and right continuous, see [17], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [17], p. 247-248, § 10.3.

We also make

Remark 10 *Next we mention about integration of A-valued functions, see [17], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [17], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \ d\mu = \int_Q x f(p) \ d\mu(p) \tag{32}$$

and

$$\left(\int_Q f \ d\mu \right) x = \int_Q f(p) x \ d\mu(p) . \tag{33}$$

The Bochner integrals we will involve in our article follow (32) and (33). Also, let $f \in C([a, b], X)$, where $[a, b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [5], p. 3, f is Bochner integrable.

4 p-Schatten norms background

In this advanced section all come from [11].

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of trace class if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{34}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthornormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the trace of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$tr(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{35}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (35) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 11 *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$tr(A^*) = \overline{tr(A)}; \tag{36}$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then AT, $TA \in \mathcal{B}_1(H)$ and

$$tr(AT) = tr(TA) \quad and \quad |tr(AT)| \le \|A\|_1 \|T\|; \tag{37}$$

(iii) $tr(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|tr\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ then AB, $BA \in \mathcal{B}_1(H)$ and $tr(AB) = tr(BA)$;

(v) $\mathcal{B}_{fin}(H)$ (finite rank operators) is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the p-Schatten norm is finite [19, p. 60-64]

$$\|A\|_p := [tr(|A|^p)]^{\frac{1}{p}} < \infty,$$

$|A|^p$ is an operator notation and not a power.

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{38}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{39}$$

For $p \ge 1$ the functional $\|\cdot\|_p$ is a norm on the $*$-ideal $\mathcal{B}_p(H)$, which is a Banach algebra, and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [19, p. 60-64], for $p \ge 1$,

$$\|A\|_p = \|A^*\|_p, \quad A \in \mathcal{B}_p(H) \tag{40}$$

$$\|AB\|_p \le \|A\|_p \|B\|_p, \quad A, B \in \mathcal{B}_p(H) \tag{41}$$

and

$$\|AB\|_p \le \|A\|_p \|B\|, \quad \|BA\|_p \le \|B\| \|A\|_p, \quad A \in \mathcal{B}_p(H), B \in \mathcal{B}(H). \tag{42}$$

This implies that

$$\|CAB\|_p \leq \|C\| \, \|A\|_p \, \|B\| \, , \;\; A \in \mathcal{B}_p(H) \, , \; B, C \in \mathcal{B}(H) \, . \tag{43}$$

In terms of p-Schatten norm we have the Hölder inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$(|tr(AB)| \leq) \, \|AB\|_1 \leq \|A\|_p \, \|B\|_q \, , \quad A \in \mathcal{B}_p(H) \, , \; B \in \mathcal{B}_q(H) \, . \tag{44}$$

For the theory of trace functionals and their applications the interested reader is referred to [18] and [19].

For some classical trace inequalities see [9], [10] and [14], which are continuations of the work of Bellman [7].

5 Main Results

We start with 1-Schatten norm weighted mixed generalized Canavati fractional Ostrowski type inequalities involving several functions taking values in the Banach algebra $\mathcal{B}_2(H) \subset \mathcal{B}(H)$:

Theorem 12 *Let the $*$-ideal $\mathcal{B}_2(H)$, which $(\mathcal{B}_2(H), \|\cdot\|_2)$ is a Banach algebra; $x_0 \in [a,b] \subset \mathbb{R}$, $\nu \geq 1$, $n = [\nu]$; $f_i \in C^n([a,b], \mathcal{B}_2(H))$, $i = 1, ..., r \in \mathbb{N} - \{1\}$; $g \in C^1([a,b])$, strictly increasing such that $g^{-1} \in C^n([g(a), g(b)])$, with $\left(f_i \circ g^{-1}\right)^{(k)}(g(x_0)) = 0$, $k = 1, ..., n-1$; $i = 1, ..., r$. Assume further that $f_i \circ g^{-1} \in C^{\nu}_{g(x_0)-}([g(a), g(b)], \mathcal{B}_2(H)) \cap C^{\nu}_{g(x_0)}([g(a), g(b)], \mathcal{B}_2(H))$, $i = 1, ..., r$.*

Denote by

$$K(f_1, ..., f_r)(x_0) := \sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) \, dx - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) f_i(x_0) \right]. \tag{45}$$

Then

$$\|K(f_1, ..., f_r)(x_0)\|_1 \leq \frac{1}{\Gamma(\nu+1)} \sum_{i=1}^{r} \left[\left[\left\| \left\| D^{\nu}_{g(x_0)-} \left(f_i \circ g^{-1}\right) \right\|_2 \right\|_{\infty, [g(a), g(x_0)]} (g(x_0) - g(a))^{\nu} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_2 \right) dx \right) \right] + \right. \tag{46}$$

$$\left. \left[\left\| \left\| D^{\nu}_{g(x_0)} \left(f_i \circ g^{-1}\right) \right\|_2 \right\|_{\infty, [g(x_0), g(b)]} (g(b) - g(x_0))^{\nu} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_2 \right) dx \right) \right] \right].$$

Proof. Since $\left(f_i \circ g^{-1}\right)^{(k)}(g(x_0)) = 0$, $k = 1, ..., [\nu] - 1$; $i = 1, ..., r$; we have by Theorem 4 that

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu}\left(f_i \circ g^{-1}\right)\right)(t)\, dt, \quad (47)$$

$\forall\, x \in [x_0, b]$,

and by Theorem 5 that

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu}\left(f_i \circ g^{-1}\right)\right)(t)\, dt, \quad (48)$$

$\forall\, x \in [a, x_0]$, for all $i = 1, ..., r$.

Left multiplying (47) and (48) with $\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)$ we get, respectively,

$$\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu}\left(f_i \circ g^{-1}\right)\right)(t)\, dt, \quad (49)$$

$\forall\, x \in [x_0, b]$,

and

$$\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) = \quad (50)$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)}{\Gamma(\nu)} \int_{g(x)}^{g(x_0)} (t - g(x))^{\nu-1} \left(D_{g(x_0)-}^{\nu}\left(f_i \circ g^{-1}\right)\right)(t)\, dt,$$

$\forall\, x \in [a, x_0]$, for all $i = 1, ..., r$.

Adding (49) and (50) as separate groups, we obtain

$$\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\nu)} \sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D_{g(x_0)}^{\nu}\left(f_i \circ g^{-1}\right)\right)(t)\, dt, \quad (51)$$

$\forall\, x \in [x_0, b]$,

and

$$\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt, \quad (52)$$

$\forall\, x \in [a, x_0]$.

Next, we integrate (51) and (52) with respect to $x \in [a, b]$. We have

$$\sum_{i=1}^{r}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \sum_{i=1}^{r}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)} (g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right], \quad (53)$$

and

$$\sum_{i=1}^{r}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \sum_{i=1}^{r}\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right], \quad (54)$$

Finally, adding (53) and (54) we obtain the useful identity

$$K(f_1, ..., f_r)(x_0) :=$$

$$\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0)\right] =$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right]\right.$$

$$+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)} (g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right]\Bigg]. \tag{55}$$

Therefore, we get that

$$\|K(f_1,...,f_r)(x_0)\|_1 =$$

$$\left\|\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx-\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0)\right]\right\|_1 \leq \frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left\|\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right]\right\|_1\right. \tag{56}$$

$$\left.+\left\|\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)} (g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)dx\right]\right\|_1\right] \leq$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left\|\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)\right\|_1 dx\right]\right. \tag{57}$$

$$\left.+\left[\int_{x_0}^{b}\left\|\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)} (g(x)-t)^{\nu-1}\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\,dt\right)\right\|_1 dx\right]\right] \leq$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} \|f_j(x)\|_2\right)\left(\int_{g(x)}^{g(x_0)} (t-g(x))^{\nu-1}\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\right\|_2 dt\right)dx\right]\right. \tag{58}$$

$$\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} \|f_j(x)\|_2\right)\left(\int_{g(x_0)}^{g(x)} (g(x)-t)^{\nu-1}\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(t)\right\|_2 dt\right)dx\right]\right] =: (*).$$

Hence it holds

$$\|K(f_1,...,f_r)(x_0)\|_1 \leq (*). \tag{59}$$

We have that

$$(*) \leq \frac{1}{\Gamma(\nu+1)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)(g(x_0)-g(x))^{\nu}dx\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)(g(x)-g(x_0))^{\nu}dx\right]\right]\leq \tag{60}$$

$$\frac{1}{\Gamma(\nu+1)}\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{\infty,[g(a),g(x_0)]}\right.\right.$$

$$\left.(g(x_0)-g(a))^{\nu}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right]+ \tag{61}$$

$$\left.\left[\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^{\nu}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right]\right],$$

proving (46). ■

Next comes an L_1 estimate.

Theorem 13 *All as in Theorem 12. Then*

$$\||K(f_1,...,f_r)(x_0)\|_1\leq\frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)(g(x_0)-g(x))^{\nu-1}dx\right]\right. \tag{62}$$

$$\left.+\left[\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)(g(x)-g(x_0))^{\nu-1}dx\right]\right].$$

Proof. We observe that (by (58), (59))

$$(*)\leq\frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)(g(x_0)-g(x))^{\nu-1}dx\right]\right. \tag{63}$$

$$+\left[\left\|\left\|\left(D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)(g(x)-g(x_0))^{\nu-1}dx\right]\Bigg],$$

proving (62). ∎

An L_p estimate follows.

Theorem 14 *All as in Theorem 12. Let now $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\|K(f_1,...,f_r)(x_0)\|_1\leq\frac{1}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{q,[g(a),g(x_0)]}\left(\int_a^{x_0}(g(x_0)-g(x))^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{q,[g(x_0),g(b)]}\left(\int_{x_0}^{b}(g(x)-g(x_0))^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right]\right]. \tag{64}$$

Proof. We have that (by (58), (59))

$$(*)\leq\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)\left(\int_{g(x)}^{g(x_0)}(t-g(x))^{p(\nu-1)}dt\right)^{\frac{1}{p}}\right.\right.$$

$$\left.\left(\int_{g(x)}^{g(x_0)}\left\|\left(D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right)(t)\right\|_2^q dt\right)^{\frac{1}{q}}dx\right]+$$

$$\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-t)^{p(\nu-1)}dt\right)^{\frac{1}{p}}\right.$$

$$\left.\left.\left(\int_{g(x)}^{g(x_0)}\left\|\left(D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right)(t)\right\|_2^q dt\right)^{\frac{1}{q}}dx\right]\right]\leq \tag{65}$$

$$\frac{1}{\Gamma(\nu)}\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)\frac{(g(x_0)-g(x))^{\nu-1+\frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}\left\|\left\|\left(D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{q,[g(a),g(x_0)]}dx\right]\right.$$

$$+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)\frac{(g(x)-g(x_0))^{\nu-1+\frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}}\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)(z)\right\|_2\right\|_{q,[g(x_0),g(b)]}dx\right]\Bigg]$$

$$=\frac{1}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{q,[g(a),g(x_0)]}\left(\int_a^{x_0}(g(x_0)-g(x))^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right. \tag{66}$$

$$\left.+\left\|\left\|\left(D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right)\right\|_2\right\|_{q,[g(x_0),g(b)]}\left(\int_{x_0}^{b}(g(x)-g(x_0))^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right],$$

proving (64). ■

We continue with γ-Schatten norm related Ostrowski fractional inequalities:

Theorem 15 *Let $\gamma\geq 1$, the $*$-ideal $\mathcal{B}_\gamma(H)$, which $\left(\mathcal{B}_\gamma(H),\|\cdot\|_\gamma\right)$ is a Banach algebra; $x_0\in[a,b]\subset\mathbb{R}$, $\nu\geq 1$, $n=[\nu]$; $f_i\in C^n([a,b],\mathcal{B}_\gamma(H))$, $i=1,...,r\in\mathbb{N}-\{1\}$; $g\in C^1([a,b])$, strictly increasing such that $g^{-1}\in C^n([g(a),g(b)])$, with $\left(f_i\circ g^{-1}\right)^{(k)}(g(x_0))=0$, $k=1,...,n-1$; $i=1,...,r$. Assume further that $f_i\circ g^{-1}\in C_{g(x_0)-}^{\nu}([g(a),g(b)],\mathcal{B}_\gamma(H))\cap C_{g(x_0)}^{\nu}([g(a),g(b)],\mathcal{B}_\gamma(H))$, $i=1,...,r$.*

Here $K(f_1,...,f_r)(x_0)$ is as in (45). Then

$$\|K(f_1,...,f_r)(x_0)\|_\gamma\leq\frac{1}{\Gamma(\nu+1)}\sum_{i=1}^{r}\Bigg[\Bigg[\left\|\left\|D_{g(x_0)-}^{\nu}\left(f_i\circ g^{-1}\right)\right\|_\gamma\right\|_{\infty,[g(a),g(x_0)]}$$

$$(g(x_0)-g(a))^{\nu}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_\gamma\right)dx\right)\Bigg]+ \tag{67}$$

$$\left[\left\|\left\|D_{g(x_0)}^{\nu}\left(f_i\circ g^{-1}\right)\right\|_\gamma\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^{\nu}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_\gamma\right)dx\right)\right]\Bigg].$$

Proof. As similar to Theorem 12 is omitted. Use of (41). ■

An L_1 estimate follows:

Theorem 16 *All as in Theorem 15. Then*

$$\Vert |K(f_1,...,f_r)(x_0)\Vert_\gamma \leq \frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\Vert \left\Vert \left(D_{g(x_0)-}^{\nu}\left(f_i \circ g^{-1}\right)\right)\right\Vert_\gamma \right\Vert_{L_1([g(a),g(x_0)])} \int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\Vert f_j(x)\Vert_\gamma\right)(g(x_0)-g(x))^{\nu-1}dx\right]\right. \tag{68}$$

$$\left.+\left[\left\Vert \left\Vert \left(D_{g(x_0)}^{\nu}\left(f_i \circ g^{-1}\right)\right)\right\Vert_\gamma \right\Vert_{L_1([g(x_0),g(b)])} \int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\Vert f_j(x)\Vert_\gamma\right)(g(x)-g(x_0))^{\nu-1}dx\right]\right].$$

Proof. As similar to Theorem 13 is omitted. ■

An L_p estimate follows.

Theorem 17 *All as in Theorem 15. Let now $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\Vert K(f_1,...,f_r)(x_0)\Vert_\gamma \leq \frac{1}{(p(\nu-1)+1)^{\frac{1}{p}}\,\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\Vert \left\Vert \left(D_{g(x_0)-}^{\nu}\left(f_i \circ g^{-1}\right)\right)\right\Vert_\gamma \right\Vert_{q,[g(a),g(x_0)]} \left(\int_a^{x_0}(g(x_0)-g(x))^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\Vert f_j(x)\Vert_\gamma\right)dx\right)\right]\right. \tag{69}$$

$$\left.+\left[\left\Vert \left\Vert \left(D_{g(x_0)}^{\nu}\left(f_i \circ g^{-1}\right)\right)\right\Vert_\gamma \right\Vert_{q,[g(x_0),g(b)]} \left(\int_{x_0}^{b}(g(x)-g(x_0))^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\Vert f_j(x)\Vert_\gamma\right)dx\right)\right]\right].$$

Proof. As similar to Theorem 14 is omitted. ■

When $r=2$ we derive the following p-Schatten norm operator related Ostrowski type Canavati fractional inequalities.

Theorem 18 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, and let the $*$-ideals $\mathcal{B}_p(H)$, $\mathcal{B}_q(H)$, for which $\left(\mathcal{B}_p(H),\Vert\cdot\Vert_p\right)$, $\left(\mathcal{B}_q(H),\Vert\cdot\Vert_q\right)$ are Banach algebras; $x_0\in[a,b]\subset\mathbb{R}$, $\alpha\geq 1$, $n=[\alpha]$; $A_1\in C^n([a,b],\mathcal{B}_p(H))$, $A_2\in C^n([a,b],\mathcal{B}_q(H))$; $g\in C^1([a,b])$, strictly increasing, such that $g^{-1}\in C^n([g(a),g(b)])$, with $\left(A_i\circ g^{-1}\right)^{(k)}(g(x_0))=0$, $k=1,...,n-1$; $i=1,2$. Assume further that $A_1\circ g^{-1}\in C_{g(x_0)-}^{\alpha}([g(a),g(b)],\mathcal{B}_p(H))\cap C_{g(x_0)}^{\alpha}([g(a),g(b)],\mathcal{B}_p(H))$, and $A_2\circ g^{-1}\in C_{g(x_0)-}^{\alpha}([g(a),g(b)],\mathcal{B}_q(H))\cap C_{g(x_0)}^{\alpha}([g(a),g(b)],\mathcal{B}_q(H))$. Then*

1) it holds

$$\Phi\left(A_1,A_2\right)\left(x_0\right):=\int_a^b A_2\left(x\right)A_1\left(x\right)+\int_a^b A_1\left(x\right)A_2\left(x\right)dx-$$

$$\left(\int_a^b A_2\left(x\right)dx\right)A_1\left(x_0\right)-\left(\int_a^b A_1\left(x\right)dx\right)A_2\left(x_0\right)=$$

$$\frac{1}{\Gamma\left(\alpha\right)}\left\{\left[\int_a^{x_0} A_2\left(x\right)\left(\int_{g(x)}^{g(x_0)}\left(z-g\left(x\right)\right)^{\alpha-1}\left(D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right)\left(z\right)dz\right)dx\right]+\right.$$

$$\left[\int_{x_0}^b A_2\left(x\right)\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-z\right)^{\alpha-1}\left(D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right)\left(z\right)dz\right)dx\right]+ \tag{70}$$

$$\left[\int_a^{x_0} A_1\left(x\right)\left(\int_{g(x)}^{g(x_0)}\left(z-g\left(x\right)\right)^{\alpha-1}\left(D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right)\left(z\right)dz\right)dx\right]+$$

$$\left.\left[\int_{x_0}^b A_1\left(x\right)\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-z\right)^{\alpha-1}\left(D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right)\left(z\right)dz\right)dx\right]\right\},$$

2) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, we have that

$$\left\|\Phi\left(A_1,A_2\right)\left(x_0\right)\right\|_1\leq\frac{1}{\Gamma\left(\alpha\right)\left(\gamma\left(\alpha-1\right)+1\right)^{\frac{1}{\gamma}}}$$

$$\left\{\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\left\|A_2\left(x\right)\right\|_q\left(g\left(x_0\right)-g\left(x\right)\right)^{\alpha-\frac{1}{\delta}}dx\right]+\right.$$

$$\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^b\left\|A_2\left(x\right)\right\|_q\left(g\left(x\right)-g\left(x_0\right)\right)^{\alpha-\frac{1}{\delta}}dx\right]+ \tag{71}$$

$$\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\left\|A_1\left(x\right)\right\|_p\left(g\left(x_0\right)-g\left(x\right)\right)^{\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left.\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^b\left\|A_1\left(x\right)\right\|_p\left(g\left(x\right)-g\left(x_0\right)\right)^{\alpha-\frac{1}{\delta}}dx\right]\right\},$$

3) we also obtain

$$\left\|\Phi\left(A_1,A_2\right)\left(x_0\right)\right\|_1\leq\frac{1}{\Gamma\left(\alpha\right)}$$

$$\left\{\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left\|A_2\left(x\right)\right\|_q\left(g\left(x_0\right)-g\left(x\right)\right)^{\alpha-1}dx\right]+\right.$$

$$\left[\left\|\left\|D^{\alpha}_{g(x_0)}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_2(x)\|_q\left(g(x)-g(x_0)\right)^{\alpha-1}dx\right]+$$

$$\left[\left\|\left\|D^{\alpha}_{g(x_0)-}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{L_1([g(a),g(x_0)])}\int_{a}^{x_0}\|A_1(x)\|_p\left(g(x_0)-g(x)\right)^{\alpha-1}dx\right]+$$

$$\left.\left[\left\|\left\|D^{\alpha}_{g(x_0)}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\|A_1(x)\|_p\left(g(x)-g(x_0)\right)^{\alpha-1}dx\right]\right\}, \tag{72}$$

and

4)

$$\|\Phi(A_1,A_2)(x_0)\|_1\leq\frac{1}{\Gamma(\alpha+1)}$$

$$\left\{\left[\left\|\left\|D^{\alpha}_{g(x_0)-}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\|A_2(x)\|_q\left(g(x_0)-g(x)\right)^{\alpha}dx\right]+\right.$$

$$\left[\left\|\left\|D^{\alpha}_{g(x_0)}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_2(x)\|_q\left(g(x)-g(x_0)\right)^{\alpha}dx\right]+$$

$$\left[\left\|\left\|D^{\alpha}_{g(x_0)-}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\|A_1(x)\|_p\left(g(x_0)-g(x)\right)^{\alpha}dx\right]+$$

$$\left.\left[\left\|\left\|D^{\alpha}_{g(x_0)}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_1(x)\|_p\left(g(x)-g(x_0)\right)^{\alpha}dx\right]\right\}. \tag{73}$$

Proof. Here we have that (acting as in the proof of Theorem 12 for $r=2$)

$$\Phi(A_1,A_2)(x_0):=\int_a^b A_2(x)A_1(x)+\int_a^b A_1(x)A_2(x)\,dx-$$

$$\left(\int_a^b A_2(x)\,dx\right)A_1(x_0)-\left(\int_a^b A_1(x)\,dx\right)A_2(x_0)\overset{(55)}{=}$$

$$\frac{1}{\Gamma(\alpha)}\left\{\left[\int_a^{x_0}A_2(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(D^{\alpha}_{g(x_0)-}\left(A_1\circ g^{-1}\right)\right)(z)\,dz\right)dx\right]+\right.$$

$$\left[\int_{x_0}^{b}A_2(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(D^{\alpha}_{g(x_0)}\left(A_1\circ g^{-1}\right)\right)(z)\,dz\right)dx\right]+$$

$$\left[\int_a^{x_0}A_1(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(D^{\alpha}_{g(x_0)-}\left(A_2\circ g^{-1}\right)\right)(z)\,dz\right)dx\right]+$$

$$\left.\left[\int_{x_0}^{b}A_1(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(D^{\alpha}_{g(x_0)}\left(A_2\circ g^{-1}\right)\right)(z)\,dz\right)dx\right]\right\}. \tag{74}$$

Therefore it holds by taking the 1-Schatten norm that

$$\left\|\Phi\left(A_1,A_2\right)\left(x_0\right)\right\|_1=\left\|\int_a^b A_2\left(x\right)A_1\left(x\right)+\int_a^b A_1\left(x\right)A_2\left(x\right)dx-\right.$$

$$\left.\left(\int_a^b A_2\left(x\right)dx\right)A_1\left(x_0\right)-\left(\int_a^b A_1\left(x\right)dx\right)A_2\left(x_0\right)\right\|_1\leq$$

$$\frac{1}{\Gamma\left(\alpha\right)}\left\{\left[\left\|\int_a^{x_0}A_2\left(x\right)\left(\int_{g(x)}^{g(x_0)}\left(z-g\left(x\right)\right)^{\alpha-1}\left(D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right)\left(z\right)dz\right)dx\right\|_1\right]+\right.$$

$$\left[\left\|\int_{x_0}^{b}A_2\left(x\right)\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-z\right)^{\alpha-1}\left(D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right)\left(z\right)dz\right)dx\right\|_1\right]+$$

$$\left[\left\|\int_a^{x_0}A_1\left(x\right)\left(\int_{g(x)}^{g(x_0)}\left(z-g\left(x\right)\right)^{\alpha-1}\left(D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right)\left(z\right)dz\right)dx\right\|_1\right]+$$

$$\left.\left[\left\|\int_{x_0}^{b}A_1\left(x\right)\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-z\right)^{\alpha-1}\left(D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right)\left(z\right)dz\right)dx\right\|_1\right]\right\}\leq \tag{75}$$

$$\frac{1}{\Gamma\left(\alpha\right)}\left\{\left[\int_a^{x_0}\left\|A_2\left(x\right)\left(\int_{g(x)}^{g(x_0)}\left(z-g\left(x\right)\right)^{\alpha-1}\left(D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right)\left(z\right)dz\right)\right\|_1dx\right]+\right.$$

$$\left[\int_{x_0}^{b}\left\|A_2\left(x\right)\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-z\right)^{\alpha-1}\left(D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right)\left(z\right)dz\right)\right\|_1dx\right]+$$

$$\left[\int_a^{x_0}\left\|A_1\left(x\right)\left(\int_{g(x)}^{g(x_0)}\left(z-g\left(x\right)\right)^{\alpha-1}\left(D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right)\left(z\right)dz\right)\right\|_1dx\right]+$$

$$\left.\left[\int_{x_0}^{b}\left\|A_1\left(x\right)\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-z\right)^{\alpha-1}\left(D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right)\left(z\right)dz\right)\right\|_1dx\right]\right\}\leq \tag{76}$$

(by using the p-Schatten norm and Hölder's type inequality (44) for $p,q>1$: $\frac{1}{p}+\frac{1}{q}=1$)

$$\frac{1}{\Gamma\left(\alpha\right)}\left\{\left[\int_a^{x_0}\left\|A_2\left(x\right)\right\|_q\left\|\left(\int_{g(x)}^{g(x_0)}\left(z-g\left(x\right)\right)^{\alpha-1}\left(D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right)\left(z\right)dz\right)\right\|_p dx\right]+\right.$$

$$\left[\int_{x_0}^{b}\left\|A_2\left(x\right)\right\|_q\left\|\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-z\right)^{\alpha-1}\left(D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right)\left(z\right)dz\right)\right\|_p dx\right]+$$

$$\left[\int_a^{x_0} \|A_1(x)\|_p \left\| \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left(D^\alpha_{g(x_0)-}\left(A_2\circ g^{-1}\right)\right)(z)\,dz\right)\right\|_q dx\right]+$$

$$\left[\int_{x_0}^{b} \|A_1(x)\|_p \left\| \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left(D^\alpha_{g(x_0)}\left(A_2\circ g^{-1}\right)\right)(z)\,dz\right)\right\|_q dx\right]\Bigg\} \leq \tag{77}$$

$$\frac{1}{\Gamma(\alpha)}\Bigg\{\left[\int_a^{x_0} \|A_2(x)\|_q \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left\|\left(D^\alpha_{g(x_0)-}\left(A_1\circ g^{-1}\right)\right)(z)\right\|_p dz\right)dx\right]+$$

$$\left[\int_{x_0}^{b} \|A_2(x)\|_q \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left\|\left(D^\alpha_{g(x_0)}\left(A_1\circ g^{-1}\right)\right)(z)\right\|_p dz\right)dx\right]+$$

$$\left[\int_a^{x_0} \|A_1(x)\|_p \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left\|\left(D^\alpha_{g(x_0)-}\left(A_2\circ g^{-1}\right)\right)(z)\right\|_q dz\right)dx\right]+$$

$$\left[\int_{x_0}^{b} \|A_1(x)\|_p \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left\|\left(D^\alpha_{g(x_0)}\left(A_2\circ g^{-1}\right)\right)(z)\right\|_q dz\right)dx\right]\Bigg\}. \tag{78}$$

We have proved, so far, that

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq$$

$$\frac{1}{\Gamma(\alpha)}\Bigg\{\left[\int_a^{x_0} \|A_2(x)\|_q \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left\|\left(D^\alpha_{g(x_0)-}\left(A_1\circ g^{-1}\right)\right)(z)\right\|_p dz\right)dx\right]+$$

$$\left[\int_{x_0}^{b} \|A_2(x)\|_q \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left\|\left(D^\alpha_{g(x_0)}\left(A_1\circ g^{-1}\right)\right)(z)\right\|_p dz\right)dx\right]+$$

$$\left[\int_a^{x_0} \|A_1(x)\|_p \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left\|\left(D^\alpha_{g(x_0)-}\left(A_2\circ g^{-1}\right)\right)(z)\right\|_q dz\right)dx\right]+$$

$$\left[\int_{x_0}^{b} \|A_1(x)\|_p \left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left\|\left(D^\alpha_{g(x_0)}\left(A_2\circ g^{-1}\right)\right)(z)\right\|_q dz\right)dx\right]\Bigg\} =: (\lambda). \tag{79}$$

Let now $\gamma, \delta > 1$ such that $\frac{1}{\gamma}+\frac{1}{\delta}=1$, and we apply the usual Hölder's inequality in (79). Then we have that

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq (\lambda) \leq \frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\Bigg\{\left[\int_a^{x_0} \|A_2(x)\|_q (g(x_0)-g(x))^{\frac{\gamma(\alpha-1)+1}{\gamma}} \left(\int_{g(x)}^{g(x_0)} \left\|\left(D^\alpha_{g(x_0)-}\left(A_1\circ g^{-1}\right)\right)(z)\right\|_p^\delta dz\right)^{\frac{1}{\delta}} dx\right]+$$

$$\left[\int_{x_0}^{b}\left\|A_2(x)\right\|_q\left(g(x)-g(x_0)\right)^{\frac{\gamma(\alpha-1)+1}{\gamma}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right)(z)\right\|_p^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]+$$

$$\left[\int_{a}^{x_0}\left\|A_1(x)\right\|_p\left(g(x_0)-g(x)\right)^{\frac{\gamma(\alpha-1)+1}{\gamma}}\left(\int_{g(x)}^{g(x_0)}\left\|\left(D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right)(z)\right\|_q^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]+ \tag{80}$$

$$\left.\left[\int_{x_0}^{b}\left\|A_1(x)\right\|_p\left(g(x)-g(x_0)\right)^{\frac{\gamma(\alpha-1)+1}{\gamma}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right)(z)\right\|_q^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]\right\}$$

$$\leq\frac{1}{\Gamma(\alpha)\left(\gamma(\alpha-1)+1\right)^{\frac{1}{\gamma}}}$$

$$\left\{\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\delta,[g(a),g(x_0)]}\int_{a}^{x_0}\left\|A_2(x)\right\|_q\left(g(x_0)-g(x)\right)^{\alpha-\frac{1}{\delta}}dx\right]+\right.$$

$$\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\left\|A_2(x)\right\|_q\left(g(x)-g(x_0)\right)^{\alpha-\frac{1}{\delta}}dx\right]+ \tag{81}$$

$$\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\delta,[g(a),g(x_0)]}\int_{a}^{x_0}\left\|A_1(x)\right\|_p\left(g(x_0)-g(x)\right)^{\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left.\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\left\|A_1(x)\right\|_p\left(g(x)-g(x_0)\right)^{\alpha-\frac{1}{\delta}}dx\right]\right\},$$

proving (71).

We also obtain

$$\left\|\Phi(A_1,A_2)(x_0)\right\|_1\leq(\lambda)\leq\frac{1}{\Gamma(\alpha)}$$

$$\left\{\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{L_1([g(a),g(x_0)])}\int_{a}^{x_0}\left\|A_2(x)\right\|_q\left(g(x_0)-g(x)\right)^{\alpha-1}dx\right]+\right.$$

$$\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left\|A_2(x)\right\|_q\left(g(x)-g(x_0)\right)^{\alpha-1}dx\right]+$$

$$\left[\left\|\left\|D_{g(x_0)-}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{L_1([g(a),g(x_0)])}\int_{a}^{x_0}\left\|A_1(x)\right\|_p\left(g(x_0)-g(x)\right)^{\alpha-1}dx\right]+ \tag{82}$$

$$\left.\left[\left\|\left\|D_{g(x_0)}^{\alpha}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left\|A_1(x)\right\|_p\left(g(x)-g(x_0)\right)^{\alpha-1}dx\right]\right\},$$

proving (72).

At last we derive

$$\left\|\Phi\left(A_1,A_2\right)\left(x_0\right)\right\|_1 \leq (\lambda) \leq \frac{1}{\Gamma\left(\alpha+1\right)}$$

$$\left\{\left[\left\|\left\|D^{\alpha}_{g(x_0)-}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\left\|A_2(x)\right\|_q\left(g\left(x_0\right)-g(x)\right)^{\alpha}dx\right]+\right.$$

$$\left[\left\|\left\|D^{\alpha}_{g(x_0)}\left(A_1\circ g^{-1}\right)\right\|_p\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left\|A_2(x)\right\|_q\left(g(x)-g\left(x_0\right)\right)^{\alpha}dx\right]+$$

$$\left[\left\|\left\|D^{\alpha}_{g(x_0)-}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\infty,[g(a),g(x_0)]}\int_a^{x_0}\left\|A_1(x)\right\|_p\left(g\left(x_0\right)-g(x)\right)^{\alpha}dx\right]+$$

$$\left.\left[\left\|\left\|D^{\alpha}_{g(x_0)}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left\|A_1(x)\right\|_p\left(g(x)-g\left(x_0\right)\right)^{\alpha}dx\right]\right\}, \tag{83}$$

proving (73).

The theorem is proved. ■

Next we present p-Schatten left and right generalized Canavati fractional Opial type inequalities:

Theorem 19 *Let the $*$-ideal $\mathcal{B}_2(H)$, which $\left(\mathcal{B}_2(H),\|\cdot\|_2\right)$ is a Banach algebra; $x_0\in[a,b]\subset\mathbb{R}$, $\nu\geq 1$, $n=[\nu]$; $f\in C^n\left([a,b],\mathcal{B}_2(H)\right)$, $g\in C^1([a,b])$, strictly increasing such that $g^{-1}\in C^n\left([g(a),g(b)]\right)$, with $\left(f\circ g^{-1}\right)^{(k)}\left(g\left(x_0\right)\right)=0$, $k=0,1,...,n-1$. Assume further that $f\circ g^{-1}\in C^{\nu}_{g(x_0)}\left([g(a),g(b)],\mathcal{B}_2(H)\right)$.*

Let also $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then

$$\int_{g(x_0)}^{z}\left\|\left(\left(f\circ g^{-1}\right)(w)\right)\left(\left(D^{\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right)(w)\right)\right\|_1 dw\leq \tag{84}$$

$$\frac{2^{-\frac{1}{q}}\left(z-g\left(x_0\right)\right)^{\nu+\frac{1}{p}-\frac{1}{q}}}{\Gamma(\nu)\left[\left(p(\nu-1)+1\right)\left(p(\nu-1)+2\right)\right]^{\frac{1}{p}}}\left(\int_{g(x_0)}^{z}\left\|\left(D^{\nu}_{g(x_0)}\left(f\circ g^{-1}\right)\right)(w)\right\|_2^q dw\right)^{\frac{2}{q}},$$

for all $g\left(x_0\right)\leq z\leq g(b)$.

Proof. Very similar to the proof of Theorem 13 of [6]. Use of (44) for $p=q=2$. ■

A similar result comex next:

Theorem 20 *Let $\gamma\geq 1$, the $*$-ideal $\mathcal{B}_{\gamma}(H)$, which $\left(\mathcal{B}_{\gamma}(H),\|\cdot\|_{\gamma}\right)$ is a Banach algebra; $x_0\in[a,b]\subset\mathbb{R}$, $\nu\geq 1$, $n=[\nu]$; $f\in C^n\left([a,b],\mathcal{B}_{\gamma}(H)\right)$, $g\in C^1([a,b])$, strictly increasing such that $g^{-1}\in C^n\left([g(a),g(b)]\right)$, with $\left(f\circ g^{-1}\right)^{(k)}\left(g\left(x_0\right)\right)=0$, $k=0,1,...,n-1$. Assume further that $f\circ g^{-1}\in C^{\nu}_{g(x_0)}\left([g(a),g(b)],\mathcal{B}_{\gamma}(H)\right)$.*

Let also $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\int_{g(x_0)}^{z} \left\| \left(\left(f \circ g^{-1} \right)(w) \right) \left(\left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right)(w) \right) \right\|_{\gamma} dw \leq \tag{85}$$

$$\frac{2^{-\frac{1}{q}} \left(z - g(x_0) \right)^{\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma(\nu) \left[\left(p(\nu-1)+1 \right) \left(p(\nu-1)+2 \right) \right]^{\frac{1}{p}}} \left(\int_{g(x_0)}^{z} \left\| \left(D_{g(x_0)}^{\nu} \left(f \circ g^{-1} \right) \right)(w) \right\|_{\gamma}^{q} dw \right)^{\frac{2}{q}},$$

for all $g(x_0) \leq z \leq g(b)$.

Proof. Very similar to the proof of Theorem 13 of [6]. Use of (41) for $p = \gamma$.
■

It follows the corresponding right side fractional Opial type inequalities:

Theorem 21 *All as in Theorem 19, however now it is* $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}\left([g(a), g(b)], \mathcal{B}_2(H) \right)$. *Then*

$$\int_{z}^{g(x_0)} \left\| \left(\left(f \circ g^{-1} \right)(w) \right) \left(\left(D_{g(x_0)-}^{\nu} \left(f \circ g^{-1} \right) \right)(w) \right) \right\|_{1} dw \leq$$

$$\frac{2^{-\frac{1}{q}} \left(g(x_0) - z \right)^{\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma(\nu) \left[\left(p(\nu-1)+1 \right) \left(p(\nu-1)+2 \right) \right]^{\frac{1}{p}}} \left(\int_{z}^{g(x_0)} \left\| \left(D_{g(x_0)-}^{\nu} \left(f \circ g^{-1} \right) \right)(t) \right\|_{2}^{q} dt \right)^{\frac{2}{q}}, \tag{86}$$

for all $g(a) \leq z \leq g(x_0)$.

Proof. Based on (20), and as similar to the proof of Theorem 19 is omitted.
■

Next comes another right ride fractional Opial type inequality:

Theorem 22 *All as in Theorem 20, however now it is* $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}\left([g(a), g(b)], \mathcal{B}_{\gamma}(H) \right)$. *Then*

$$\int_{z}^{g(x_0)} \left\| \left(\left(f \circ g^{-1} \right)(w) \right) \left(\left(D_{g(x_0)-}^{\nu} \left(f \circ g^{-1} \right) \right)(w) \right) \right\|_{\gamma} dw \leq$$

$$\frac{2^{-\frac{1}{q}} \left(g(x_0) - z \right)^{\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma(\nu) \left[\left(p(\nu-1)+1 \right) \left(p(\nu-1)+2 \right) \right]^{\frac{1}{p}}} \left(\int_{z}^{g(x_0)} \left\| \left(D_{g(x_0)-}^{\nu} \left(f \circ g^{-1} \right) \right)(t) \right\|_{\gamma}^{q} dt \right)^{\frac{2}{q}}, \tag{87}$$

for all $g(a) \leq z \leq g(x_0)$.

Proof. Based on (20), and as similar to the proof of Theorem 19 is omitted.
■

It follows the modified generalized left $\mathcal{B}_2(H)$-valued fractional Opial inequality:

Theorem 23 *All as in Theorem 6, where $X = \mathcal{B}_2(H)$ and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here we assume that $\frac{1}{(m+1)q} < \nu < 1$. Then*

$$\int_{g(x_0)}^{z} \left\| \left(\left(f \circ g^{-1} \right)(w) \right) \left(\left(D_{g(x_0)}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right)(w) \right) \right\|_1 dw \leq \tag{88}$$

$$\frac{2^{-\frac{1}{q}} \left(z - g(x_0) \right)^{(m+1)\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma\left((m+1)\nu \right) \left[\left(p\left((m+1)\nu - 1 \right) + 1 \right) \left(p\left((m+1)\nu - 1 \right) + 2 \right) \right]^{\frac{1}{p}}}$$

$$\left(\int_{g(x_0)}^{z} \left\| \left(D_{g(x_0)}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right)(t) \right\|_2^q dt \right)^{\frac{2}{q}},$$

for all $g(x_0) \leq z \leq g(b)$.

Proof. As in Theorem 19. ■

Next comes another modified generalized left $\mathcal{B}_\gamma(H)$-valued fractional Opial inequality:

Theorem 24 *All as in Theorem 6, where $X = \mathcal{B}_\gamma(H)$ and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here we assume that $\frac{1}{(m+1)q} < \nu < 1$. Then*

$$\int_{g(x_0)}^{z} \left\| \left(\left(f \circ g^{-1} \right)(w) \right) \left(\left(D_{g(x_0)}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right)(w) \right) \right\|_\gamma dw \leq \tag{89}$$

$$\frac{2^{-\frac{1}{q}} \left(z - g(x_0) \right)^{(m+1)\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma\left((m+1)\nu \right) \left[\left(p\left((m+1)\nu - 1 \right) + 1 \right) \left(p\left((m+1)\nu - 1 \right) + 2 \right) \right]^{\frac{1}{p}}}$$

$$\left(\int_{g(x_0)}^{z} \left\| \left(D_{g(x_0)}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right)(t) \right\|_\gamma^q dt \right)^{\frac{2}{q}},$$

for all $g(x_0) \leq z \leq g(b)$.

Proof. As in Theorem 19. ■

The corresponding modified generalized right $\mathcal{B}_2(H)$-valued fractional Opial inequality comes next:

Theorem 25 *All as in Theorem 7, where $X = \mathcal{B}_2(H)$ and let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Here we assume that $\frac{1}{(m+1)q} < \nu < 1$. Then*

$$\int_{z}^{g(x_0)} \left\| \left(\left(f \circ g^{-1} \right)(w) \right) \left(\left(D_{g(x_0)-}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right)(w) \right) \right\|_1 dw \leq \tag{90}$$

$$\frac{2^{-\frac{1}{q}} \left(g(x_0) - z \right)^{(m+1)\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma\left((m+1)\nu \right) \left[\left(p\left((m+1)\nu - 1 \right) + 1 \right) \left(p\left((m+1)\nu - 1 \right) + 2 \right) \right]^{\frac{1}{p}}}$$

$$\left(\int_z^{g(x_0)} \left\| \left(D_{g(x_0)-}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right) (t) \right\|_2^q dt \right)^{\frac{2}{q}},$$

for all $g(a) \leq z \leq g(x_0)$.

Proof. As in Theorem 19. ∎

The corresponding modified generalized right $\mathcal{B}_\gamma(H)$-valued fractional Opial inequality comes next:

Theorem 26 *All as in Theorem 7, where* $X = \mathcal{B}_\gamma(H)$ *and let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Here we assume that* $\frac{1}{(m+1)q} < \nu < 1$. *Then*

$$\int_z^{g(x_0)} \left\| \left(\left(f \circ g^{-1} \right)(w) \right) \left(\left(D_{g(x_0)-}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right)(w) \right) \right\|_\gamma dw \leq \tag{91}$$

$$\frac{2^{-\frac{1}{q}} \left(g(x_0) - z \right)^{(m+1)\nu + \frac{1}{p} - \frac{1}{q}}}{\Gamma((m+1)\nu) \left[\left(p((m+1)\nu - 1) + 1 \right) \left(p((m+1)\nu - 1) + 2 \right) \right]^{\frac{1}{p}}}$$

$$\left(\int_z^{g(x_0)} \left\| \left(D_{g(x_0)-}^{(m+1)\nu} \left(f \circ g^{-1} \right) \right)(t) \right\|_\gamma^q dt \right)^{\frac{2}{q}},$$

for all $g(a) \leq z \leq g(x_0)$.

Proof. As in Theorem 19. ∎

We make

Remark 27 *(to Theorem 12)*

Case of inequality (46):

Call and assume

$$M_1(f_1, ..., f_r) := \tag{92}$$

$$\max_{i=1,...,r} \left\{ \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)-}^{\nu} \left(f_i \circ g^{-1} \right) \right\|_2 \right\|_{\infty, [g(a), g(x_0)]}, \right.$$

$$\left. \sup_{x_0 \in [a,b]} \left\| \left\| D_{g(x_0)}^{\gamma} \left(f_i \circ g^{-1} \right) \right\|_2 \right\|_{\infty, [g(x_0), g(b)]} \right\} < +\infty.$$

Then

$$\left\| K(f_1, ..., f_r)(x_0) \right\|_1 \leq \text{Right hand side (46)} \leq$$

$$\frac{M_1(f_1, ..., f_r)(g(b) - g(a))^\nu}{\Gamma(\nu + 1)} \sum_{i=1}^r \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^r \left\| f_j(x) \right\|_2 \right) dx \right). \tag{93}$$

We make

Remark 28 *(to Theorem 13)*

Case of inequality (62):

Call and assume

$$M_2(f_1,...,f_r) := \tag{94}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right\|_2\right\|_{L_1([g(a),g(x_0)])},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right\|_2\right\|_{L_1([g(x_0),g(b)])}\right\}<+\infty.$$

Then

$$\|K(f_1,...,f_r)(x_0)\|_1 \leq \textit{Right hand side (62)} \leq$$

$$\frac{M_2(f_1,...,f_r)(g(b)-g(a))^{\nu-1}}{\Gamma(\nu)}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right). \tag{95}$$

We make

Remark 29 *(to Theorem 14)*

Case of inequality (64):

Call and assume $(p,q>1:\frac{1}{p}+\frac{1}{q}=1)$*:*

$$M_3(f_1,...,f_r) := \tag{96}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right\|_2\right\|_{q,([g(a),g(x_0)])},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right\|_2\right\|_{q,([g(x_0),g(b)])}\right\}<+\infty.$$

Then

$$\|K(f_1,...,f_r)(x_0)\|_1 \leq \textit{Right hand side (64)} \leq$$

$$\frac{M_3(f_1,...,f_r)(g(b)-g(a))^{\nu-\frac{1}{q}}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right). \tag{97}$$

We make

Remark 30 *(to Theorem 15)* $(\gamma\geq 1)$

Case of inequality (67):

Call and assume

$$M_1^{\gamma}(f_1,...,f_r) := \tag{98}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right\|_{\gamma}\right\|_{\infty,[g(a),g(x_0)]},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^{\gamma}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right\|_{\gamma}\right\|_{\infty,[g(x_0),g(b)]}\right\}<+\infty.$$

Then

$$\|K(f_1,...,f_r)(x_0)\|_{\gamma}\leq \text{Right hand side (67)} \leq$$

$$\frac{M_1^{\gamma}(f_1,...,f_r)(g(b)-g(a))^{\nu}}{\Gamma(\nu+1)}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_{\gamma}\right)dx\right). \tag{99}$$

We make

Remark 31 *(to Theorem 16) ($\gamma\geq 1$)*
Case of inequality (68):
Call and assume:

$$M_2^{\gamma}(f_1,...,f_r):= \tag{100}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right\|_{\gamma}\right\|_{L_1([g(a),g(x_0)])},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right\|_{\gamma}\right\|_{L_1([g(x_0),g(b)])}\right\}<+\infty.$$

Then

$$\|K(f_1,...,f_r)(x_0)\|_{\gamma}\leq \text{Right hand side (68)} \leq$$

$$\frac{M_2^{\gamma}(f_1,...,f_r)(g(b)-g(a))^{\nu-1}}{\Gamma(\nu)}\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_{\gamma}\right)dx\right). \tag{101}$$

We make

Remark 32 *(to Theorem 17) ($\gamma\geq 1$)*
Case of inequality (69):
Call and assume ($p,q>1:\frac{1}{p}+\frac{1}{q}=1$):

$$M_3^{\gamma}(f_1,...,f_r):= \tag{102}$$

$$\max_{i=1,...,r}\left\{\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(f_i\circ g^{-1}\right)\right\|_{\gamma}\right\|_{q,([g(a),g(x_0)])},\right.$$

$$\left.\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)}\left(f_i\circ g^{-1}\right)\right\|_{\gamma}\right\|_{q,([g(x_0),g(b)])}\right\}<+\infty.$$

Then

$$\left\|K\left(f_{1},...,f_{r}\right)\left(x_{0}\right)\right\|_{\gamma}\leq \text{Right hand side (69)} \leq$$

$$\frac{M_{3}^{\gamma}\left(f_{1},...,f_{r}\right)\left(g\left(b\right)-g\left(a\right)\right)^{\nu-\frac{1}{q}}}{\left(p\left(\nu-1\right)+1\right)^{\frac{1}{p}}\Gamma\left(\nu\right)}\sum_{i=1}^{r}\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_{j}\left(x\right)\right\|_{\gamma}\right)dx\right). \quad (103)$$

Remark 33 *(to Theorem 18)*

i) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, *case of inequality (71):*

Call and assume

$$N_{1}\left(A_{1},A_{2}\right):=$$

$$\max\left\{\sup_{x_{0}\in[a,b]}\left\|\left\|D_{g(x_{0})-}^{\nu}\left(A_{1}\circ g^{-1}\right)\right\|_{p}\right\|_{\delta,[g(a),g(x_{0})]},\sup_{x_{0}\in[a,b]}\left\|\left\|D_{g(x_{0})}^{\nu}\left(A_{1}\circ g^{-1}\right)\right\|_{p}\right\|_{\delta,[g(x_{0}),g(b)]},\right.$$

$$\left.\sup_{x_{0}\in[a,b]}\left\|\left\|D_{g(x_{0})-}^{\nu}\left(A_{2}\circ g^{-1}\right)\right\|_{q}\right\|_{\delta,[g(a),g(x_{0})]},\sup_{x_{0}\in[a,b]}\left\|\left\|D_{g(x_{0})}^{\nu}\left(A_{2}\circ g^{-1}\right)\right\|_{q}\right\|_{\delta,[g(x_{0}),g(b)]}\right\}<+\infty. \quad (104)$$

Then

$$\left\|\Phi\left(A_{1},A_{2}\right)\left(x_{0}\right)\right\|_{1}\leq \text{right hand side (71)} \leq$$

$$\frac{N_{1}\left(A_{1},A_{2}\right)\left(g\left(b\right)-g\left(a\right)\right)^{\alpha-\frac{1}{\delta}}}{\Gamma\left(\alpha\right)\left(\gamma\left(\alpha-1\right)+1\right)^{\frac{1}{\gamma}}}\left[\int_{a}^{b}\left\|A_{1}\left(x\right)\right\|_{p}dx+\int_{a}^{b}\left\|A_{2}\left(x\right)\right\|_{q}dx\right]. \quad (105)$$

ii) case of inequality (72):

Call and assume

$$N_{2}\left(A_{1},A_{2}\right):= \quad (106)$$

$$\max\left\{\sup_{x_{0}\in[a,b]}\left\|\left\|D_{g(x_{0})-}^{\nu}\left(A_{1}\circ g^{-1}\right)\right\|_{p}\right\|_{L_{1}([g(a),g(x_{0})])},\sup_{x_{0}\in[a,b]}\left\|\left\|D_{g(x_{0})}^{\nu}\left(A_{1}\circ g^{-1}\right)\right\|_{p}\right\|_{L_{1}([g(x_{0}),g(b)])},\right.$$

$$\left.\sup_{x_{0}\in[a,b]}\left\|\left\|D_{g(x_{0})-}^{\nu}\left(A_{2}\circ g^{-1}\right)\right\|_{q}\right\|_{L_{1}([g(a),g(x_{0})])},\sup_{x_{0}\in[a,b]}\left\|\left\|D_{g(x_{0})}^{\nu}\left(A_{2}\circ g^{-1}\right)^{-1}\right\|_{q}\right\|_{L_{1}([g(x_{0}),g(b)])}\right\}<+\infty.$$

Then

$$\left\|\Phi\left(A_{1},A_{2}\right)\left(x_{0}\right)\right\|_{1}\leq \text{right hand side (72)} \leq$$

$$\frac{N_{2}\left(A_{1},A_{2}\right)\left(g\left(b\right)-g\left(a\right)\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}\left[\int_{a}^{b}\left\|A_{1}\left(x\right)\right\|_{p}dx+\int_{a}^{b}\left\|A_{2}\left(x\right)\right\|_{q}dx\right]. \quad (107)$$

iii) case of inequality (73):

Call and assume

$$N_{3}\left(A_{1},A_{2}\right):= \quad (108)$$

$$\max\left\{\sup_{x_{0}\in[a,b]}\left\|\left\|D_{g(x_{0})-}^{\nu}\left(A_{1}\circ g^{-1}\right)\right\|_{p}\right\|_{\infty,[g(a),g(x_{0})]},\sup_{x_{0}\in[a,b]}\left\|\left\|D_{g(x_{0})}^{\nu}\left(A_{1}\circ g^{-1}\right)\right\|_{p}\right\|_{\infty,[g(x_{0}),g(b)]},\right.$$

$$\sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)-}\left(A_2\circ g^{-1}\right)\right\|_q\right\|_{\infty,[g(a),g(x_0)]},\ \sup_{x_0\in[a,b]}\left\|\left\|D^{\nu}_{g(x_0)}\left(A_2\circ g^{-1}\right)^{-1}\right\|_q\right\|_{\infty,[g(x_0),g(b)]}\right\}<+\infty.$$

Then

$$\|\Phi(A_1,A_2)(x_0)\|_1 \le \textit{right hand side (73)} \le$$

$$\frac{N_3(A_1,A_2)(g(b)-g(a))^{\alpha}}{\Gamma(\alpha+1)}\left[\int_a^b\|A_1(x)\|_p\,dx+\int_a^b\|A_2(x)\|_q\,dx\right]. \tag{109}$$

We need

Remark 34 *(i) This is regarding Theorems 12-17. Here* $K(f_1,...,f_r)(x_0)$, $x_0\in[a,b]$, *is as in (45). Next we denote and have (case of* $1\le\nu<2$*):*

$$\Delta(f_1,...,f_r) := \int_a^b K(f_1,...,f_r)(x_0)\,dx_0 =$$

$$\sum_{i=1}^{r}\left[(b-a)\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r}f_j(x)\right)f_i(x)\,dx-\left(\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r}f_j(x)\right)dx\right)\left(\int_a^b f_i(x)\,dx\right)\right], \tag{110}$$

(ii) This is regarding Theorem 18. Here $\Phi(A_1,A_2)(x_0)$, $x_0\in[a,b]$, *is as in (70). Next we denote and have (case of* $1\le\alpha<2$*):*

$$\Delta(A_1,A_2) := \int_a^b \Phi(A_1,A_2)(x_0)\,dx_0 =$$

$$(b-a)\left(\int_a^b A_2(x)A_1(x)\,dx+\int_a^b A_1(x)A_2(x)\,dx\right)- \tag{111}$$

$$\left(\int_a^b A_2(x)\,dx\right)\left(\int_a^b A_1(x)\,dx\right)-\left(\int_a^b A_1(x)\,dx\right)\left(\int_a^b A_2(x)\,dx\right).$$

(iii) for $\gamma\ge1$, *it holds*

$$\|\Delta(f_1,...,f_r)\|_{\gamma} \le \int_a^b\|K(f_1,...,f_r)(x)\|_{\gamma}\,dx, \tag{112}$$

and

$$\|\Delta(A_1,A_2)\|_1 \le \int_a^b\|\Phi(A_1,A_2)(x)\|_1\,dx. \tag{113}$$

We give the following set of γ-Schatten norm generalized Canavati type fractional Grüss type inequalities involving several functions over $\mathcal{B}_{\gamma}(H)$, $\gamma\ge1$.

Theorem 35 *All as in Theorem 12, with $1 \leq \nu < 2$ (i.e. $n = 1$). Then*

i)

$$\|\Delta(f_1,...,f_r)\|_1 \leq \frac{M_1(f_1,...,f_r)(g(b)-g(a))^{\nu}(b-a)^2}{\Gamma(\nu+1)} \sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right\|_{\infty,[a,b]}\right), \tag{114}$$

where $M_1(f_1,...,f_r)$ is as in (92),

ii)

$$\|\Delta(f_1,...,f_r)\|_1 \leq \frac{M_2(f_1,...,f_r)(g(b)-g(a))^{\nu-1}(b-a)^2}{\Gamma(\nu)} \sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right\|_{\infty,[a,b]}\right), \tag{115}$$

where $M_2(f_1,...,f_r)$ is as in (94),

iii) when $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, we have

$$\|\Delta(f_1,...,f_r)\|_1 \leq \frac{M_3(f_1,...,f_r)(g(b)-g(a))^{\nu-\frac{1}{q}}(b-a)^2}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} \sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right\|_{\infty,[a,b]}\right), \tag{116}$$

where $M_3(f_1,...,f_r)$ is as in (96).

Proof. By Remarks 34, 27-29 and that

$$\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)dx \leq (b-a)\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right\|_{\infty,[a,b]}.$$

■

We continue with

Theorem 36 *All as in Theorem 15, with $1 \leq \nu < 2$ (i.e. $n = 1$), $\gamma \geq 1$. Then*

i)

$$\|\Delta(f_1,...,f_r)\|_\gamma \leq \frac{M_1^\gamma(f_1,...,f_r)(g(b)-g(a))^{\nu}(b-a)^2}{\Gamma(\nu+1)}$$

$$\sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_{\gamma}\right\|_{\infty,[a,b]}\right), \tag{117}$$

where $M_1^{\gamma}\left(f_1,...,f_r\right)$ *is as in (98),*

ii)

$$\left\|\Delta\left(f_1,...,f_r\right)\right\|_{\gamma}\leq\frac{M_2^{\gamma}\left(f_1,...,f_r\right)\left(g\left(b\right)-g\left(a\right)\right)^{\nu-1}\left(b-a\right)^2}{\Gamma\left(\nu\right)}$$

$$\sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_{\gamma}\right\|_{\infty,[a,b]}\right), \tag{118}$$

where $M_2^{\gamma}\left(f_1,...,f_r\right)$ *is as in (100),*

iii) when $p,q>1:\frac{1}{p}+\frac{1}{q}=1$*, we have*

$$\left\|\Delta\left(f_1,...,f_r\right)\right\|_{\gamma}\leq\frac{M_3^{\gamma}\left(f_1,...,f_r\right)\left(g\left(b\right)-g\left(a\right)\right)^{\nu-\frac{1}{q}}\left(b-a\right)^2}{\left(p\left(\nu-1\right)+1\right)^{\frac{1}{p}}\Gamma\left(\nu\right)}$$

$$\sum_{i=1}^{r}\left(\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_{\gamma}\right\|_{\infty,[a,b]}\right), \tag{119}$$

where $M_3^{\gamma}\left(f_1,...,f_r\right)$ *is as in (102).*

Proof. By Remarks 34, 30-32 and that

$$\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_{\gamma}\right)dx\leq\left(b-a\right)\left\|\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_{\gamma}\right\|_{\infty,[a,b]}.$$

■

Furthermore we have ($r=2$ case of p-Schatten norm Grüss inequalities)

Theorem 37 *All as in Theorem 18, with* $1\leq\alpha<2$ *(i.e.* $[\alpha]=1$*). Then*

i) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$*, we have*

$$\left\|\Delta\left(A_1,A_2\right)\right\|_1\leq\frac{N_1\left(A_1,A_2\right)\left(g\left(b\right)-g\left(a\right)\right)^{\alpha-\frac{1}{\delta}}\left(b-a\right)}{\Gamma\left(\alpha\right)\left(\gamma\left(\alpha-1\right)+1\right)^{\frac{1}{\gamma}}}$$

$$\left[\int_a^b\left\|A_1\left(x\right)\right\|_p dx+\int_a^b\left\|A_2\left(x\right)\right\|_q dx\right], \tag{120}$$

where $N_1(A_1, A_2)$ *is as in (104),*

ii)

$$\|\Delta(A_1, A_2)\|_1 \leq \frac{N_2(A_1, A_2)(g(b) - g(a))^{\alpha-1}(b-a)}{\Gamma(\alpha)}$$

$$\left[\int_a^b \|A_1(x)\|_p dx + \int_a^b \|A_2(x)\|_q dx\right], \tag{121}$$

where $N_2(A_1, A_2)$ *is as in (106),*

and

iii)

$$\|\Delta(A_1, A_2)\|_1 \leq \frac{N_3(A_1, A_2)(g(b) - g(a))^{\alpha}(b-a)}{\Gamma(\alpha+1)}$$

$$\left[\int_a^b \|A_1(x)\|_p dx + \int_a^b \|A_2(x)\|_q dx\right], \tag{122}$$

where $N_3(A_1, A_2)$ *is as in (108).*

Proof. By Remarks 34, 33. ■

6 Applications

We start with applications on Ostrowski type inequalities:

Corollary 38 *(to Theorems 12-14) All as in Theorem 12 for* $g(t) = t$*. Then*

i)

$$\|K(f_1, ..., f_r)(x_0)\|_1 \leq \frac{1}{\Gamma(\nu+1)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-}^{\nu} f_i\right)\right\|_2\right\|_{\infty,[a,x_0]}(x_0-a)^{\nu}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right] + \tag{123}$$

$$\left[\left\|\left\|\left(D_{x_0}^{\nu} f_i\right)\right\|_2\right\|_{\infty,[x_0,b]}(b-x_0)^{\nu}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)dx\right)\right]\right],$$

ii)

$$\|K(f_1, ..., f_r)(x_0)\|_1 \leq \frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-}^{\nu} f_i\right)\right\|_2\right\|_{L_1([a,x_0])}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_2\right)(x_0-x)^{\nu-1}dx\right] +\right.$$

$$\left[\left\|\left\|\left(D_{x_0}^{\nu} f_i\right)\right\|_2\right\|_{L_1([x_0,b])} \int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|f_j(x)\|_2\right) (x-x_0)^{\nu-1} dx\right]\Bigg], \tag{124}$$

iii) when $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*, we have*

$$\|K(f_1, ..., f_r)(x_0)\|_1 \leq \frac{1}{(p(\nu-1)+1)^{\frac{1}{p}} \Gamma(\nu)}$$

$$\sum_{i=1}^{r} \left[\left[\left\|\left\|\left(D_{x_0-}^{\nu} f_i\right)\right\|_2\right\|_{q,[a,x_0]} \left(\int_{a}^{x_0} (x_0-x)^{\nu-\frac{1}{q}} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|f_j(x)\|_2\right) dx\right)\right] + \right.$$

$$\left.\left[\left\|\left\|\left(D_{x_0}^{\nu} f_i\right)\right\|_2\right\|_{q,[x_0,b]} \left(\int_{x_0}^{b} (x-x_0)^{\nu-\frac{1}{q}} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|f_j(x)\|_2\right) dx\right)\right]\right]. \tag{125}$$

It follows:

Corollary 39 *(to Theorems 15-17) All as in Theorem 15 for* $g(t) = t$*,* $\gamma \geq 1$*. Then*

i)

$$\|K(f_1, ..., f_r)(x_0)\|_\gamma \leq \frac{1}{\Gamma(\nu+1)}$$

$$\sum_{i=1}^{r} \left[\left[\left\|\left\|\left(D_{x_0-}^{\nu} f_i\right)\right\|_\gamma\right\|_{\infty,[a,x_0]} (x_0-a)^{\nu} \left(\int_{a}^{x_0} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|f_j(x)\|_\gamma\right) dx\right)\right] + \right.$$

$$\left.\left[\left\|\left\|\left(D_{x_0}^{\nu} f_i\right)\right\|_\gamma\right\|_{\infty,[x_0,b]} (b-x_0)^{\nu} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|f_j(x)\|_\gamma\right) dx\right)\right]\right], \tag{126}$$

ii)

$$\|K(f_1, ..., f_r)(x_0)\|_\gamma \leq \frac{1}{\Gamma(\nu)}$$

$$\sum_{i=1}^{r} \left[\left[\left\|\left\|\left(D_{x_0-}^{\nu} f_i\right)\right\|_\gamma\right\|_{L_1([a,x_0])} \int_{a}^{x_0} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|f_j(x)\|_\gamma\right) (x_0-x)^{\nu-1} dx\right] + \right.$$

$$\left.\left[\left\|\left\|\left(D_{x_0}^{\nu} f_i\right)\right\|_\gamma\right\|_{L_1([x_0,b])} \int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j\neq i}}^{r} \|f_j(x)\|_\gamma\right) (x-x_0)^{\nu-1} dx\right]\right], \tag{127}$$

iii) when $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, we have

$$\|K(f_1,...,f_r)(x_0)\|_\gamma \leq \frac{1}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\|(D_{x_0-}^{\nu}f_i)\|_\gamma\right\|_{q,[a,x_0]}\left(\int_a^{x_0}(x_0-x)^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_\gamma\right)dx\right)\right]+\right.$$

$$\left.\left[\left\|\|(D_{x_0}^{\nu}f_i)\|_\gamma\right\|_{q,[x_0,b]}\left(\int_{x_0}^{b}(x-x_0)^{\nu-\frac{1}{q}}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|_\gamma\right)dx\right)\right]\right]. \quad (128)$$

We continue with

Corollary 40 *(to Theorem 18) All as in Theorem 18, with $g(t)=e^t$. Then*
i) for $\gamma,\delta>1:\frac{1}{\gamma}+\frac{1}{\delta}=1$, we have

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq \frac{1}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\left\{\left[\left\|\|D_{e^{x_0}-}^{\alpha}(A_1\circ\log)\|_p\right\|_{\delta,[e^a,e^{x_0}]}\int_a^{x_0}\|A_2(x)\|_q(e^{x_0}-e^x)^{\alpha-\frac{1}{\delta}}dx\right]+\right.$$

$$\left[\left\|\|D_{e^{x_0}}^{\alpha}(A_1\circ\log)\|_p\right\|_{\delta,[e^{x_0},e^b]}\int_{x_0}^{b}\|A_2(x)\|_q(e^x-e^{x_0})^{\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left[\left\|\|D_{e^{x_0}-}^{\alpha}(A_2\circ\log)\|_q\right\|_{\delta,[e^a,e^{x_0}]}\int_a^{x_0}\|A_1(x)\|_p(e^{x_0}-e^x)^{\alpha-\frac{1}{\delta}}dx\right]+$$

$$\left.\left[\left\|\|D_{e^{x_0}}^{\alpha}(A_2\circ\log)\|_q\right\|_{\delta,[e^{x_0},e^b]}\int_{x_0}^{b}\|A_1(x)\|_p(e^x-e^{x_0})^{\alpha-\frac{1}{\delta}}dx\right]\right\}, \quad (129)$$

ii) it holds

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq \frac{1}{\Gamma(\alpha)}$$

$$\left\{\left[\left\|\|D_{e^{x_0}-}^{\alpha}(A_1\circ\log)\|_p\right\|_{L_1([e^a,e^{x_0}])}\int_a^{x_0}\|A_2(x)\|_q(e^{x_0}-e^x)^{\alpha-1}dx\right]+\right.$$

$$\left[\left\|\|D_{e^{x_0}}^{\alpha}(A_1\circ\log)\|_p\right\|_{L_1([e^{x_0},e^b])}\int_{x_0}^{b}\|A_2(x)\|_q(e^x-e^{x_0})^{\alpha-1}dx\right]+$$

$$\left[\left\|\|D_{e^{x_0}-}^{\alpha}(A_2\circ\log)\|_q\right\|_{L_1([e^a,e^{x_0}])}\int_a^{x_0}\|A_1(x)\|_p(e^{x_0}-e^x)^{\alpha-1}dx\right]+ \quad (130)$$

$$\left[\left\|\left\|D_{e^{x_0}}^{\alpha}\left(A_2\circ\log\right)\right\|_q\right\|_{L_1([e^{x_0},e^b])}\int_{x_0}^{b}\left\|A_1(x)\right\|_p\left(e^x-e^{x_0}\right)^{\alpha-1}dx\right]\Bigg\},$$

and

iii)

$$\left\|\Phi\left(A_1,A_2\right)(x_0)\right\|_1\leq\frac{1}{\Gamma(\alpha+1)}$$

$$\Bigg\{\left[\left\|\left\|D_{e^{x_0}-}^{\alpha}\left(A_1\circ\log\right)\right\|_p\right\|_{\infty,[e^a,e^{x_0}]}\int_a^{x_0}\left\|A_2(x)\right\|_q\left(e^{x_0}-e^x\right)^{\alpha}dx\right]+$$

$$\left[\left\|\left\|D_{e^{x_0}}^{\alpha}\left(A_1\circ\log\right)\right\|_p\right\|_{\infty,[e^{x_0},e^b]}\int_{x_0}^{b}\left\|A_2(x)\right\|_q\left(e^x-e^{x_0}\right)^{\alpha}dx\right]+$$

$$\left[\left\|\left\|D_{e^{x_0}-}^{\alpha}\left(A_2\circ\log\right)\right\|_q\right\|_{\infty,[e^a,e^{x_0}]}\int_a^{x_0}\left\|A_1(x)\right\|_p\left(e^{x_0}-e^x\right)^{\alpha}dx\right]+$$

$$\left[\left\|\left\|D_{e^{x_0}}^{\alpha}\left(A_2\circ\log\right)\right\|_q\right\|_{\infty,[e^{x_0},e^b]}\int_{x_0}^{b}\left\|A_1(x)\right\|_p\left(e^x-e^{x_0}\right)^{\alpha}dx\right]\Bigg\}. \tag{131}$$

We continue with applications on Opial inequalities

Corollary 41 *(to Theorem 19) All as in Theorem 19 with $g(t)=t$. Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{x_0}^{z}\left\|f(w)\left(D_{x_0}^{\nu}f\right)(w)\right\|_1dw\leq$$

$$\frac{2^{-\frac{1}{q}}\left(z-x_0\right)^{\nu+\frac{1}{p}-\frac{1}{q}}}{\Gamma(\nu)\left[\left(p(\nu-1)+1\right)\left(p(\nu-1)+2\right)\right]^{\frac{1}{p}}}\left(\int_{x_0}^{z}\left\|\left(D_{x_0}^{\nu}f\right)(w)\right\|_2^q dw\right)^{\frac{2}{q}}, \tag{132}$$

for all $x_0\leq z\leq b$.

It follows:

Corollary 42 *(to Theorem 20) All as in Theorem 20, $\gamma\geq1$, with $g(t)=e^t$. Let also $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_{e^{x_0}}^{z}\left\|\left((f\circ\log)(w)\right)\left(\left(D_{e^{x_0}}^{\nu}(f\circ\log)\right)(w)\right)\right\|_\gamma dw\leq$$

$$\frac{2^{-\frac{1}{q}}\left(z-e^{x_0}\right)^{\nu+\frac{1}{p}-\frac{1}{q}}}{\Gamma(\nu)\left[\left(p(\nu-1)+1\right)\left(p(\nu-1)+2\right)\right]^{\frac{1}{p}}}\left(\int_{e^{x_0}}^{z}\left\|\left(D_{e^{x_0}}^{\nu}(f\circ\log)\right)(w)\right\|_\gamma^q dw\right)^{\frac{2}{q}}, \tag{133}$$

for all $e^{x_0}\leq z\leq e^b$.

We finish with applications on Grüss inequalities:

Corollary 43 *(to Theorem 35) All as in Theorem 35 with $g(t) = t$ $(1 \leq \nu < 2)$. Then*

i)

$$\|\Delta(f_1,...,f_r)\|_1 \leq \frac{M_1(f_1,...,f_r)(b-a)^{\nu+2}}{\Gamma(\nu+1)} \sum_{i=1}^{r} \left(\left\| \prod_{\substack{j=1 \\ j\neq i}}^{r} \|f_j(x)\|_2 \right\|_{\infty,[a,b]} \right), \tag{134}$$

where $M_1(f_1,...,f_r)$ is as in (92),

ii)

$$\|\Delta(f_1,...,f_r)\|_1 \leq \frac{M_2(f_1,...,f_r)(b-a)^{\nu+1}}{\Gamma(\nu)} \sum_{i=1}^{r} \left(\left\| \prod_{\substack{j=1 \\ j\neq i}}^{r} \|f_j(x)\|_2 \right\|_{\infty,[a,b]} \right), \tag{135}$$

where $M_2(f_1,...,f_r)$ is as in (94),

iii) when $p,q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, we have

$$\|\Delta(f_1,...,f_r)\|_1 \leq \frac{M_3(f_1,...,f_r)(b-a)^{\nu+1+\frac{1}{p}}}{(p(\nu-1)+1)^{\frac{1}{p}}\Gamma(\nu)} \sum_{i=1}^{r} \left(\left\| \prod_{\substack{j=1 \\ j\neq i}}^{r} \|f_j(x)\|_2 \right\|_{\infty,[a,b]} \right), \tag{136}$$

where $M_3(f_1,...,f_r)$ is as in (96).

It follows ($r = 2$ case)

Corollary 44 *(to Theorem 37) All as in Theorem 37, with $[a,b] \subset \mathbb{R}_+ - \{0\}$, and $g(t) = \log t$. Then*

i) for $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, we have

$$\|\Delta(A_1,A_2)\|_1 \leq \frac{N_1(A_1,A_2)\left(\log\frac{b}{a}\right)^{\alpha-\frac{1}{\delta}}(b-a)}{\Gamma(\alpha)(\gamma(\alpha-1)+1)^{\frac{1}{\gamma}}}$$

$$\left[\int_a^b \|A_1(x)\|_p\,dx + \int_a^b \|A_2(x)\|_q\,dx\right], \tag{137}$$

where $N_1(A_1,A_2)$ is as in (104),

ii)

$$\|\Delta(A_1,A_2)\|_1 \leq \frac{N_2(A_1,A_2)\left(\log\frac{b}{a}\right)^{\alpha-1}(b-a)}{\Gamma(\alpha)}$$

$$\left[\int_a^b \|A_1(x)\|_p\,dx + \int_a^b \|A_2(x)\|_q\,dx\right], \tag{138}$$

where $N_2(A_1, A_2)$ *is as in (106),*

and

iii)

$$\|\Delta(A_1, A_2)\|_1 \leq \frac{N_3(A_1, A_2)\left(\log\frac{b}{a}\right)^{\alpha}(b-a)}{\Gamma(\alpha+1)}$$

$$\left[\int_a^b \|A_1(x)\|_p\,dx + \int_a^b \|A_2(x)\|_q\,dx\right], \tag{139}$$

where $N_3(A_1, A_2)$ *is as in (108).*

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