DETERMINANT INEQUALITIES FOR POSITIVE DEFINITE MATRICES VIA ČEBYŠEV'S INTEGRAL INEQUALITY

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Abstract. In this paper we prove among others that, if A, C are positive definite matrices, then

$$\det\left(C+\left(\alpha+\beta\right)A\right)\det\left(C\right)\leq\det\left(C+\alpha A\right)\det\left(C+\beta A\right)$$

for all α, $\beta>0$. If α, $\beta>1$, then also

$$\begin{aligned}0&\leq\left[\det\left(C+\left(\alpha+\beta\right)A\right)\right]^{-1}\left[\det\left(C\right)\right]^{-1}-\left[\det\left(C+\alpha A\right)\right]^{-1}\left[\det\left(C+\beta A\right)\right]^{-1}\\&\leq\alpha\beta\left[\left[\det\left(C+2A\right)\right]^{-1}\left[\det\left(C\right)\right]^{-1}-\left[\det\left(C+A\right)\right]^{-2}\right].\end{aligned}$$

1. Introduction

A real square matrix $A=(a_{ij})$, $i,j=1,...,n$ is *symmetric* provided $a_{ij}=a_{ji}$ for all $i,j=1,...,n$. A real symmetric matrix is said to be *positive definite* provided the quadratic form $Q(x)=\sum_{i,j=1}^{n}a_{ij}x_ix_j$ is positive for all $x=(x_1,...,x_n)\in\mathbb{R}^n\setminus\{0\}$. It is well known that a necessary and sufficient condition for the symmetric matrix A to be positive definite, and we write $A>0$, is that all determinants

$$\det(A_k)=\det\left(a_{ij}\right),\ i,\ j=1,...,k;\ k=1,...,n$$

are positive.

It is know that the following integral representation is valid, see [1, pp. 61-62] or [8, pp. 211-212]

$$\begin{aligned}J_n\left(A\right)&:=\int_{\mathbb{R}^n}\exp(-\left\langle Ax,x\right\rangle)dx:=\int_{-\infty}^{\infty}...\int_{-\infty}^{\infty}\exp(-\left\langle Ax,x\right\rangle)dx\\&=\frac{\pi^{n/2}}{\left[\det\left(A\right)\right]^{1/2}},\end{aligned}\tag{1.1}$$

where A is a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

If we take the square in the representation (1.1), then we get

$$\left(\int_{\mathbb{R}^n}\exp(-\left\langle x,Ax\right\rangle)dx\right)^2=\frac{\pi^n}{\det\left(A\right)}.$$

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Since

$$\left(\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle)dx\right)^2 = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle)\exp(-\langle Ay,y\rangle)dxdy$$
$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle - \langle Ay,y\rangle))dxdy,$$

hence

$$K_n(A) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp(-\langle Ax,x\rangle - \langle Ay,y\rangle))dxdy = \frac{\pi^n}{\det(A)} \tag{1.2}$$

for A a positive definite matrix of order n and $\langle\cdot,\cdot\rangle$ is the usual inner product on $\mathbb{R}^n$.

By utilizing the representation (1.1) and Hölder's integral inequality for multiple integrals one can prove the *logarithmic concavity* of the determinant that is due to Ky Fan ([1, p. 63] or [8, p. 212]), namely

$$\det((1-\lambda)A + \lambda B) \geq [\det(A)]^{1-\lambda}[\det(B)]^{\lambda} \tag{1.3}$$

for any positive definite matrices A, B and $\lambda \in [0,1]$.

By mathematical induction we can get a generalization of (1.3) which was obtained by L. Mirsky in [7], see also [8, p. 212]

$$\det\left(\sum_{j=1}^m \lambda_j A_j\right) \geq \prod_{j=1}^m [\det(A_j)]^{\lambda_j},\ m \geq 2, \tag{1.4}$$

where $\lambda_j > 0$, $j = 1,...,m$ with $\sum_{j=1}^m \lambda_j = 1$ and $A_j > 0$, $j = 1,...,m$.

If we write (1.4) for $A_j = B_j^{-1}$ we get

$$\det\left(\sum_{j=1}^m \lambda_j B_j^{-1}\right) \geq \prod_{j=1}^m \left[\det\left(B_j^{-1}\right)\right]^{\lambda_j} = \left(\prod_{j=1}^m [\det(B_j)]^{\lambda_j}\right)^{-1},$$

which also gives

$$\prod_{j=1}^m [\det(A_j)]^{\lambda_j} \geq \det\left[\left(\sum_{j=1}^m \lambda_j A_j^{-1}\right)^{-1}\right], \tag{1.5}$$

where $\lambda_j > 0$, $j = 1,...,m$ with $\sum_{j=1}^m \lambda_j = 1$ and $A_j > 0$, $j = 1,...,m$.

Using the representation (1.1) one can also prove the result, see [8, p. 212],

$$\det(A) = \det(A_{1n}) \leq \det(A_{1k})\det\left(A_{(k+1)n}\right),\ k = 1,...,n; \tag{1.6}$$

where the determinant $\det(A_{rs})$ is defined by

$$\det(A_{rs}) = \det(a_{ij}),i,\ j = r,...,s.$$

In particular,

$$\det(A) \leq a_{11}a_{22}...a_{nn}. \tag{1.7}$$

We recall also the Minkowski's type inequality,

$$[\det(A+B)]^{1/n} \geq [\det(A)]^{1/n} + [\det(B)]^{1/n} \tag{1.8}$$

for A, B positive definite matrices of order n. For other determinant inequalities see Chapter VIII of the classic book [8]. For some recent results see [2]-[6].

Motivated by the above results, in this paper we prove among others that, if A, C are positive definite matrices, then

$$\det(C+(\alpha+\beta)A)\det(C)\le\det(C+\alpha A)\det(C+\beta A)$$

for all α, $\beta>0$. If α, $\beta>1$, then also

$$\begin{aligned}0&\le[\det(C+(\alpha+\beta)A)]^{-1}[\det(C)]^{-1}\\&-[\det(C+\alpha A)]^{-1}[\det(C+\beta A)]^{-1}\\&\le\alpha\beta\left[[\det(C+2A)]^{-1}[\det(C)]^{-1}-[\det(C+A)]^{-2}\right].\end{aligned}$$

2. Main Results

We have the following result:

Theorem 1. *Assume that A, B are positive definite matrices and α, $\beta>0$, then*

$$\begin{aligned}&[\det(C+(\alpha+\beta)A)]^{-1/2}+[\det(C+(\alpha+\beta)B)]^{-1/2}\\&\ge[\det(C+\alpha A+\beta B)]^{-1/2}+[\det(C+\beta A+\alpha B)]^{-1/2}\end{aligned}\tag{2.1}$$

and

$$\begin{aligned}&[\det(C+(\alpha+\beta)A)]^{-1}+[\det(C+(\alpha+\beta)B)]^{-1}\\&\ge[\det(C+\alpha A+\beta B)]^{-1}+[\det(C+\beta A+\alpha B)]^{-1}.\end{aligned}\tag{2.2}$$

Proof. The function $f(t)=t^p$ is increasing on $[0,\infty)$ for $p>0$. Therefore for α, $\beta>$ we have

$$(t^\alpha-s^\alpha)\left(t^\beta-s^\beta\right)\ge0\text{ for all }t,\ s\ge0$$

namely

$$t^{\alpha+\beta}+s^{\alpha+\beta}\ge t^\alpha s^\beta+t^\beta s^\alpha\text{ for all }t,\ s\ge0.\tag{2.3}$$

Take in (2.3) $t=\exp(-\langle Ax,x\rangle)$ and $s=\exp(-\langle Bx,x\rangle)$ with $x\in\mathbb{R}^n$, then

$$\begin{aligned}&\exp(-\langle[(\alpha+\beta)A]x,x\rangle)+\exp(-\langle[(\alpha+\beta)B]x,x\rangle)\\&\ge\exp(-\langle(\alpha A+\beta B)x,x\rangle)+\exp(-\langle(\beta A+\alpha B)x,x\rangle)\end{aligned}$$

for all $x\in\mathbb{R}^n$.

Now, if we multiply this inequality by $\exp(-\langle Cx,x\rangle)$, then we get

$$\begin{aligned}&\exp(-\langle[C+(\alpha+\beta)A]x,x\rangle)+\exp(-\langle[C+(\alpha+\beta)B]x,x\rangle)\\&\ge\exp(-\langle(C+\alpha A+\beta B)x,x\rangle)+\exp(-\langle(C+\beta A+\alpha B)x,x\rangle)\end{aligned}$$

for all $x\in\mathbb{R}^n$.

If we take the integral over $x\in\mathbb{R}^n$, then we get

$$\begin{aligned}&\int_{\mathbb{R}^n}\exp(-\langle[C+(\alpha+\beta)A]x,x\rangle)dx+\int_{\mathbb{R}^n}\exp(-\langle[C+(\alpha+\beta)B]x,x\rangle)dx\\&\ge\int_{\mathbb{R}^n}\exp(-\langle(C+\alpha A+\beta B)x,x\rangle)dx+\int_{\mathbb{R}^n}\exp(-\langle(C+\beta A+\alpha B)x,x\rangle)dx\end{aligned}$$

and by the first equality in (1.1) we get

$$\begin{aligned}&J_n(C+(\alpha+\beta)A)+J_n(C+(\alpha+\beta)B)\\&\ge J_n(C+\alpha A+\beta B)+J_n(C+\beta A+\alpha B),\end{aligned}$$

which is equivalent to (2.1).

Further, take $t=\exp\left(-\langle Ax,x\rangle-\langle Ay,y\rangle\right)$ and $s=\exp\left(-\langle Bx,x\rangle-\langle By,y\rangle\right)$ with $(x,y)\in\mathbb{R}^n\times\mathbb{R}^n$, then

$$\begin{aligned}&\exp\left(-\langle[(\alpha+\beta)A]x,x\rangle-\langle[(\alpha+\beta)A]y,y\rangle\right)\\&+\exp\left(-\langle[(\alpha+\beta)B]x,x\rangle-\langle[(\alpha+\beta)B]y,y\rangle\right)\\&\geq\exp\left(-\langle(\alpha A+\beta B)x,x\rangle-\langle(\alpha A+\beta B)y,y\rangle\right)\\&+\exp\left(-\langle(\beta A+\alpha B)x,x\rangle-\langle(\beta A+\alpha B)y,y\rangle\right).\end{aligned}$$

By taking the double integral on $\mathbb{R}^n\times\mathbb{R}^n$, then we get

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle[(\alpha+\beta)A]x,x\rangle-\langle[(\alpha+\beta)A]y,y\rangle\right)dxdy\\&+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle[(\alpha+\beta)B]x,x\rangle-\langle[(\alpha+\beta)B]y,y\rangle\right)dxdy\\&\geq\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle(\alpha A+\beta B)x,x\rangle-\langle(\alpha A+\beta B)y,y\rangle\right)dxdy\\&+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle(\beta A+\alpha B)x,x\rangle-\langle(\beta A+\alpha B)y,y\rangle\right)dxdy,\end{aligned}$$

and by first equality in (1.2) we get

$$\begin{aligned}&K_n\left(C+(\alpha+\beta)A\right)+K_n\left(C+(\alpha+\beta)B\right)\\&\geq K_n\left(C+\alpha A+\beta B\right)+K_n\left(C+\beta A+\alpha B\right),\end{aligned}$$

which is equivalent to (2.2). □

Corollary 1. *Assume that* A, B *are positive definite matrices and* $t\in[0,1]$, *then*

$$\begin{aligned}&[\det(C+A)]^{-1/2}+[\det(C+B)]^{-1/2}\\&\geq[\det(C+tA+(1-t)B)]^{-1/2}+[\det(C+(1-t)A+tB)]^{-1/2}\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}&[\det(C+A)]^{-1}+[\det(C+B)]^{-1}\\&\geq[\det(C+tA+(1-t)B)]^{-1}+[\det(C+(1-t)A+tB)]^{-1}.\end{aligned}\tag{2.5}$$

Also

$$\begin{aligned}&\frac{[\det(C+A)]^{-1/2}+[\det(C+B)]^{-1/2}}{2}\\&\geq\int_0^1[\det(C+tA+(1-t)B)]^{-1/2}\,dt\end{aligned}\tag{2.6}$$

and

$$\frac{[\det(C+A)]^{-1}+[\det(C+B)]^{-1}}{2}\geq\int_0^1[\det(C+tA+(1-t)B)]^{-1}\,dt\tag{2.7}$$

Proof. The proof of (2.4) and (2.5) from (2.1) and (2.2) by taking $\alpha=t$ and $\beta=1-t$.

Further, by taking the integral over $t \in [0,1]$ in (2.4) we get

$$\begin{aligned}
&[\det(C+A)]^{-1/2}+[\det(C+B)]^{-1/2} \\
&\geq \int_0^1 [\det(C+tA+(1-t)B)]^{-1/2}\,dt+\int_0^1 [\det(C+(1-t)A+tB)]^{-1/2}\,dt \\
&=2\int_0^1 [\det(C+tA+(1-t)B)]^{-1/2}\,dt,
\end{aligned}$$

which is equivalent to (2.6). □

Remark 1. *If we take* $\alpha=\beta=1$ *in Theorem 1, then we get*

$$\frac{[\det(C+2A)]^{-1/2}+[\det(C+2B)]^{-1/2}}{2}\geq [\det(C+A+B)]^{-1/2} \tag{2.8}$$

and

$$\frac{[\det(C+2A)]^{-1}+[\det(C+2B)]^{-1}}{2}\geq [\det(C+A+B)]^{-1}. \tag{2.9}$$

Also by taking the double integral in Theorem 1, we get

$$\begin{aligned}
&\int_0^1\int_0^1 [\det(C+(\alpha+\beta)A)]^{-1/2}\,d\alpha d\beta \\
&+\int_0^1\int_0^1 [\det(C+(\alpha+\beta)B)]^{-1/2}\,d\alpha d\beta \\
&\geq 2\int_0^1\int_0^1 [\det(C+\alpha A+\beta B)]^{-1/2}\,d\alpha d\beta
\end{aligned} \tag{2.10}$$

and

$$\begin{aligned}
&\int_0^1\int_0^1 [\det(C+(\alpha+\beta)A)]^{-1}\,d\alpha d\beta \\
&+\int_0^1\int_0^1 [\det(C+(\alpha+\beta)B)]^{-1}\,d\alpha d\beta \\
&\geq 2\int_0^1\int_0^1 [\det(C+\alpha A+\beta B)]^{-1}\,d\alpha d\beta.
\end{aligned} \tag{2.11}$$

From a different perspective, we have

Theorem 2. *Assume that* A, B, C, D *are positive definite matrices and* α, $\beta>0$, *then*

$$\begin{aligned}
&[\det(C+(\alpha+\beta)A)]^{-1/2}[\det(D)]^{-1/2} \\
&+[\det(C)]^{-1/2}[\det(D+(\alpha+\beta)B)]^{-1/2} \\
&\geq [\det(C+\alpha A)]^{-1/2}[\det(D+\beta B)]^{-1/2} \\
&+[\det(C+\beta A)]^{-1/2}[\det(D+\alpha B)]^{-1/2}
\end{aligned} \tag{2.12}$$

and

$$[\det(C+(\alpha+\beta)A)]^{-1}[\det(D)]^{-1} + [\det(C)]^{-1}[\det(D+(\alpha+\beta)B)]^{-1} \geq [\det(C+\alpha A)]^{-1}[\det(D+\beta B)]^{-1} + [\det(C+\beta A)]^{-1}[\det(D+\alpha B)]^{-1}. \tag{2.13}$$

Proof. Take in (2.3) $t=\exp(-\langle Ax,x\rangle)$ and $s=\exp(-\langle By,y\rangle)$ with $x,\ y\in\mathbb{R}^n$, then

$$\exp(-\langle(\alpha+\beta)Ax,x\rangle)+\exp(-\langle(\alpha+\beta)By,y\rangle) \geq \exp(-\langle\alpha Ax,x\rangle)\exp(-\langle\beta By,y\rangle)+\exp(-\langle\beta Ax,x\rangle)\exp(-\langle\alpha By,y\rangle)$$

and by multiplication with $\exp(-\langle Cx,x\rangle)$ and $\exp(-\langle Dy,y\rangle)$ where $x,\ y\in\mathbb{R}^n$, we get

$$\exp(-\langle C+(\alpha+\beta)Ax,x\rangle)\exp(-\langle Dy,y\rangle) + \exp(-\langle Cx,x\rangle)\exp(-\langle D+(\alpha+\beta)By,y\rangle) \geq \exp(-\langle(C+\alpha A)x,x\rangle)\exp(-\langle(D+\beta B)y,y\rangle) + \exp(-\langle(C+\beta A)x,x\rangle)\exp(-\langle(D+\alpha B)y,y\rangle).$$

By taking the double integral on $\mathbb{R}^n\times\mathbb{R}^n$, we get

$$\int_{\mathbb{R}^n}\exp(-\langle C+(\alpha+\beta)Ax,x\rangle)\,dx\int_{\mathbb{R}^n}\exp(-\langle Dy,y\rangle)\,dy + \int_{\mathbb{R}^n}\exp(-\langle Cx,x\rangle)\,dx\int_{\mathbb{R}^n}\exp(-\langle D+(\alpha+\beta)By,y\rangle)\,dy \geq \int_{\mathbb{R}^n}\exp(-\langle(C+\alpha A)x,x\rangle)\,dx\int_{\mathbb{R}^n}\exp(-\langle(D+\beta B)y,y\rangle)\,dy + \int_{\mathbb{R}^n}\exp(-\langle(C+\beta A)x,x\rangle)\,dx\int_{\mathbb{R}^n}\exp(-\langle(D+\alpha B)y,y\rangle)\,dy.$$

Utilizing the first equality in (1.1), we get

$$J_n(C+(\alpha+\beta)A)J_n(D)+J_n(C)J_n(D+(\alpha+\beta)B) \geq J_n(C+\alpha A)J_n(D+\beta B)+J_n(C+\beta A)J_n(D+\alpha B),$$

which is equivalent to (2.12).

Take in (2.3) $t=\exp(-\langle Ax,x\rangle-\langle Az,z\rangle)$ and $s=\exp(-\langle By,y\rangle-\langle Bu,u\rangle)$ with $x,\ y,\ z,\ u\in\mathbb{R}^n$, then

$$\exp(-\langle(\alpha+\beta)Ax,x\rangle-\langle(\alpha+\beta)Az,z\rangle) + \exp(-\langle(\alpha+\beta)By,y\rangle-\langle(\alpha+\beta)Bu,u\rangle) \geq \exp(-\langle\alpha Ax,x\rangle-\langle\alpha Az,z\rangle)\exp(-\langle\beta By,y\rangle-\langle\beta Bu,u\rangle) + \exp(-\langle\beta Ax,x\rangle-\langle\beta Az,z\rangle)\exp(-\langle\alpha By,y\rangle-\langle\alpha Bu,u\rangle)$$

and by multiplication with $\exp(-\langle Cx,x\rangle-\langle Cz,z\rangle)$ and $\exp(-\langle Dy,y\rangle-\langle Du,u\rangle)$ where x, y, z, $u\in\mathbb{R}^n$, we get

$$\begin{aligned}&\exp\left(-\left\langle C+(\alpha+\beta)Ax,x\right\rangle-\left\langle C+(\alpha+\beta)Az,z\right\rangle\right)\\&\times\exp\left(-\langle Dy,y\rangle-\langle Du,u\rangle\right)\\&+\exp\left(-\langle Cx,x\rangle-\langle Cz,z\rangle\right)\\&\times\exp\left(-\left\langle D+(\alpha+\beta)By,y\right\rangle-\left\langle D+(\alpha+\beta)Bu,u\right\rangle\right)\\&\geq\exp\left(-\left\langle (C+\alpha A)x,x\right\rangle-\left\langle (C+\alpha A)z,z\right\rangle\right)\\&\times\exp\left(-\left\langle (D+\beta B)y,y\right\rangle-\left\langle (D+\beta B)u,u\right\rangle\right)\\&+\exp\left(-\left\langle (C+\beta A)x,x\right\rangle-\left\langle (C+\beta A)z,z\right\rangle\right)\\&\times\exp\left(-\left\langle (D+\alpha B)y,y\right\rangle-\left\langle (D+\alpha B)u,u\right\rangle\right).\end{aligned}$$

By taking the integral on $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$, we get

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle C+(\alpha+\beta)Ax,x\right\rangle-\left\langle C+(\alpha+\beta)Az,z\right\rangle\right)dxdz\\&\times\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle Dy,y\rangle-\langle Du,u\rangle\right)dydu\\&+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\langle Cx,x\rangle-\langle Cz,z\rangle\right)dxdz\\&\times\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle D+(\alpha+\beta)By,y\right\rangle-\left\langle D+(\alpha+\beta)Bu,u\right\rangle\right)dydu\\&\geq\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle (C+\alpha A)x,x\right\rangle-\left\langle (C+\alpha A)z,z\right\rangle\right)dxdz\\&\times\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle (D+\beta B)y,y\right\rangle-\left\langle (D+\beta B)u,u\right\rangle\right)dydu\\&+\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle (C+\beta A)x,x\right\rangle-\left\langle (C+\beta A)z,z\right\rangle\right)dxdz\\&\times\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left(-\left\langle (D+\alpha B)y,y\right\rangle-\left\langle (D+\alpha B)u,u\right\rangle\right)dydu,\end{aligned}$$

namely, by the representation (1.2),

$$\begin{aligned}&K_n\left(C+(\alpha+\beta)A\right)K_n(D)+K_n(C)\,K_n\left(D+(\alpha+\beta)B\right)\\&\geq K_n(C+\alpha A)\,K_n(D+\beta B)+K_n(C+\beta A)\,K_n(D+\alpha B),\end{aligned}$$

which proves (2.13). □

Remark 2. *If we take $D=C$ and $B=A$, then we get from Theorem 2 that*

$$\det\left(C+(\alpha+\beta)A\right)\det(C)\leq\det(C+\alpha A)\det(C+\beta A)\tag{2.14}$$

for all α, $\beta>0$.

If $t\in[0,1]$, then by (2.14) we get

$$\det(C+A)\det(C)\leq\det(C+tA)\det\left(C+(1-t)A\right).\tag{2.15}$$

If we take $\alpha=\beta=1$, then we get

$$\det(C+2A)\det(C)\leq\left[\det(C+A)\right]^2.\tag{2.16}$$

Corollary 2. *For A, C positive definite matrices, we have*

$$\det(C+A)\det(C)\le\int_0^1\det(C+tA)\det(C+(1-t)A)\,dt. \le\left(\int_0^1\det(C+tA)\,dt\right)^2. \tag{2.17}$$

Proof. We observe that, by taking the integral on (2.15) we get

$$\det(C+A)\det(C)\le\int_0^1\det(C+tA)\det(C+(1-t)A)\,dt. \tag{2.18}$$

By the monotonicity property of the determinant, we observe that the function $f(t):=\det(C+tA)$, $t\in[0,1]$ is monotonic increasing while the function $g(t)=\det(C+(1-t)A)$, $t\in[0,1]$ is monotonic decreasing, and by Čebyšev's inequality for asynchronous functions we have

$$\int_0^1 f(t)\,g(t)\,dt\le\int_0^1 f(t)\,dt\int_0^1 g(t)\,dt,$$

which gives

$$\int_0^1\det(C+tA)\det(C+(1-t)A)\,dt\le\int_0^1\det(C+tA)\,dt\int_0^1\det(C+(1-t)A)\,dt = \left(\int_0^1\det(C+tA)\,dt\right)^2$$

and the inequalities in (2.17) are proved. □

3. Related Results

We also have the reverse inequalities:

Theorem 3. *Assume that A, B are positive definite matrices and α, $\beta\ge1$, then*

$$\begin{aligned}0&\le[\det(C+(\alpha+\beta)A)]^{-1/2}[\det(C)]^{-1/2}\\&-[\det(C+\alpha A)]^{-1/2}[\det(C+\beta A)]^{-1/2}\\&\le\alpha\beta\left[[\det(C+2A)]^{-1/2}[\det(C)]^{-1/2}-[\det(C+A)]^{-1}\right]\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}0&\le[\det(C+(\alpha+\beta)A)]^{-1}[\det(C)]^{-1}\\&-[\det(C+\alpha A)]^{-1}[\det(C+\beta A)]^{-1}\\&\le\alpha\beta\left[[\det(C+2A)]^{-1}[\det(C)]^{-1}-[\det(C+A)]^{-2}\right].\end{aligned} \tag{3.2}$$

Proof. Consider the function $f(t)=t^\alpha$, $\alpha\ge1$ on the interval $[0,1]$. By Lagrange's theorem we get

$$|t^\alpha-s^\alpha|=|f(t)-f(s)|\le\sup_{u\in(0,1)}|f'(u)|\,|t-s|=\alpha|t-s|$$

for all t, $s\in[0,1]$.

Therefore for $\alpha,\ \beta \geq 1$ we derive

$$\begin{aligned} 0 &\leq t^{\alpha+\beta} + s^{\alpha+\beta} - t^{\alpha}s^{\beta} - t^{\beta}s^{\alpha} = \left(t^{\alpha} - s^{\alpha}\right)\left(t^{\beta} - s^{\beta}\right) \\ &= \left|\left(t^{\alpha} - s^{\alpha}\right)\left(t^{\beta} - s^{\beta}\right)\right| \leq \alpha\left|t - s\right|\beta\left|t - s\right| = \alpha\beta\left(t - s\right)^{2} \\ &= \alpha\beta\left(t^{2} - 2ts + s^{2}\right) \end{aligned}$$

for all $t,\ s \in [0,1]$.

If we take $t = \exp\left(-\langle Ax, x\rangle\right)$ and $s = \exp\left(-\langle Ay, y\rangle\right)$ with $x,\ y \in \mathbb{R}^{n}$, then

$$\begin{aligned} &0 \leq \exp\left(-\langle(\alpha+\beta)Ax, x\rangle\right) + \exp\left(-\langle(\alpha+\beta)Ay, y\rangle\right) \\ &- \exp\left(-\langle\alpha Ax, x\rangle\right)\exp\left(-\langle\beta Ay, y\rangle\right) - \exp\left(-\langle\beta Ax, x\rangle\right)\exp\left(-\langle\alpha Ay, y\rangle\right) \\ &\leq \alpha\beta\left(\exp\left(-\langle 2Ax, x\rangle\right) - 2\exp\left(-\langle Ax, x\rangle\right)\exp\left(-\langle Ay, y\rangle\right) + \exp\left(-\langle 2Ay, y\rangle\right)\right) \end{aligned}$$

for all $x,\ y \in \mathbb{R}^{n}$.

If we multiply this inequality by $\exp\left(-\langle Cx, x\rangle\right)$ and $\exp\left(-\langle Cy, y\rangle\right)$ where x, $y \in \mathbb{R}^{n}$, then we get

$$\begin{aligned} &0 \leq \exp\left(-\langle(C+(\alpha+\beta)A)x, x\rangle\right)\exp\left(-\langle Cy, y\rangle\right) \\ &+ \exp\left(-\langle Cx, x\rangle\right)\exp\left(-\langle(C+(\alpha+\beta)A)y, y\rangle\right) \\ &- \exp\left(-\langle(C+\alpha A)x, x\rangle\right)\exp\left(-\langle(C+\beta A)y, y\rangle\right) \\ &- \exp\left(-\langle(C+\beta A)x, x\rangle\right)\exp\left(-\langle(C+\alpha A)y, y\rangle\right) \\ &\leq \alpha\beta\left[\exp\left(-\langle(C+2A)x, x\rangle\right)\exp\left(-\langle Cy, y\rangle\right)\right. \\ &+ \exp\left(-\langle Cx, x\rangle\right)\exp\left(-\langle(C+2A)y, y\rangle\right) \\ &\left.-2\exp\left(-\langle(C+A)x, x\rangle\right)\exp\left(-\langle(C+A)y, y\rangle\right)\right] \end{aligned}$$

and by taking the integral on $\mathbb{R}^{n} \times \mathbb{R}^{n}$ and divide by 2 we obtain

$$\begin{aligned} 0 &\leq J_{n}\left(C+(\alpha+\beta)A\right)J_{n}(C) - J_{n}\left(C+\alpha A\right)J_{n}\left(C+\beta A\right) \\ &\leq \alpha\beta\left[J_{n}\left(C+2A\right)J_{n}(C) - \left[J_{n}\left(C+A\right)\right]^{2}\right], \end{aligned}$$

which is equivalent to (3.1_.

The inequality (3.2) follows in a similar manner and we omit the details. □

Remark 3. *If we take $\beta = \alpha \geq 1$ in Theorem 3, then we obtain*

$$\begin{aligned} 0 &\leq \left[\det\left(C+2\alpha A\right)\right]^{-1/2}\left[\det(C)\right]^{-1/2} - \left[\det\left(C+\alpha A\right)\right]^{-1} \\ &\leq \alpha^{2}\left[\left[\det\left(C+2A\right)\right]^{-1/2}\left[\det(C)\right]^{-1/2} - \left[\det\left(C+A\right)\right]^{-1}\right] \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} 0 &\leq \left[\det\left(C+2\alpha A\right)\right]^{-1}\left[\det(C)\right]^{-1} - \left[\det\left(C+\alpha A\right)\right]^{-2} \\ &\leq \alpha^{2}\left[\left[\det\left(C+2A\right)\right]^{-1}\left[\det(C)\right]^{-1} - \left[\det\left(C+A\right)\right]^{-2}\right]. \end{aligned} \tag{3.4}$$

If $t \in (0,1]$, then $\alpha = \frac{1}{t} \geq 1$ and by (3.3) we get

$$\begin{aligned} 0 &\leq \left[\det\left(C+2\frac{1}{t}A\right)\right]^{-1/2}\left[\det(C)\right]^{-1/2} - \left[\det\left(C+\frac{1}{t}A\right)\right]^{-1} \\ &\leq \frac{1}{t^{2}}\left[\left[\det\left(C+2A\right)\right]^{-1/2}\left[\det(C)\right]^{-1/2} - \left[\det\left(C+A\right)\right]^{-1}\right] \end{aligned}$$

and by taking $B=\frac{1}{t}A$ *we derive*

$$0\le t^{2}\left(\left[\det\left(C+2B\right)\right]^{-1/2}\left[\det\left(C\right)\right]^{-1/2}-\left[\det\left(C+B\right)\right]^{-1}\right)\le\left[\det\left(C+2tB\right)\right]^{-1/2}\left[\det\left(C\right)\right]^{-1/2}-\left[\det\left(C+tB\right)\right]^{-1}\tag{3.5}$$

for B, C *positive definite* $t\in[0,1]$.

Also, by taking the integral over $t\in[0,1]$ *in (3.5), we obtain*

$$0\le\frac{1}{3}\left(\left[\det\left(C+2B\right)\right]^{-1/2}\left[\det\left(C\right)\right]^{-1/2}-\left[\det\left(C+B\right)\right]^{-1}\right)\le\left[\det\left(C\right)\right]^{-1/2}\int_{0}^{1}\left[\det\left(C+2tB\right)\right]^{-1/2}dt-\int_{0}^{1}\left[\det\left(C+tB\right)\right]^{-1}dt.\tag{3.6}$$

Similarly, we have

$$0\le t^{2}\left(\left[\det\left(C+2B\right)\right]^{-1}\left[\det\left(C\right)\right]^{-1}-\left[\det\left(C+B\right)\right]^{-2}\right)\le\left[\det\left(C+2tB\right)\right]^{-1}\left[\det\left(C\right)\right]^{-1}-\left[\det\left(C+tB\right)\right]^{-2}\tag{3.7}$$

and

$$0\le\frac{1}{3}\left(\left[\det\left(C+2B\right)\right]^{-1}\left[\det\left(C\right)\right]^{-1}-\left[\det\left(C+B\right)\right]^{-2}\right)\le\left[\det\left(C\right)\right]^{-1}\int_{0}^{1}\left[\det\left(C+2tB\right)\right]^{-1}dt-\int_{0}^{1}\left[\det\left(C+tB\right)\right]^{-2}dt.\tag{3.8}$$

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