γ- Schatten norm Generalized Canavati Fractional Hilbert-Pachpatte type inequalities for von Neumann-Schatten class $\mathcal{B}_\gamma(H)$ valued functions

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
ganastss@memphis.edu

Abstract

Employing generalized Canavati fractional left and right vectorial Taylor formulae we prove corresponding left and right fractional Hilbert-Pachpatte type inequalities for von Neumann-Schatten class $\mathcal{B}_\gamma(H)$ valued functions. We cover also the sequential fractional case. We finish with applications.

2020 Mathematics Subject Classification : 26A33, 26D10, 26D15.
Keywords and Phrases: generalized Canavati fractional derivative, generalized Canavati fractional inequalities, Hilbert-Pachpatte inequalities, Banach algebra, von Neumann-Schatten class of operators.

1 Introduction

Here all motivation comes from [3].

All fractional terminology comes from section 2 next.

We start with a left generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

Theorem 1 *Let $p, q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$, and $(A, \|\cdot\|)$ is a Banach algebra; and $i = 1, 2$. Let also $x_{0i} \in [a_i, b_i] \subset \mathbb{R}$, $\nu_i \geq 1$, $n_i = [\nu_i]$, $f_i \in C^{n_i}([a_i, b_i], A)$; $g_i \in C^1([a_i, b_i])$, strictly increasing, such that $g_i^{-1} \in C^{n_i}([g_i(a_i), g_i(b_i)])$, with*

$\left(f_i \circ g_i^{-1}\right)^{(k_i)}\left(g_i\left(x_{0i}\right)\right)=0$, $k_i=0,1,...,n_i-1$. *Assume further that* $f_i \circ g_i^{-1} \in C_{g_i(x_{0i})}^{\nu_i}\left(\left[g_i\left(a_i\right), g_i\left(b_i\right)\right], A\right)$. *Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\|\left(f_1 \circ g_1^{-1}\right)\left(z_1\right)\left(f_2 \circ g_2^{-1}\right)\left(z_2\right)\right\| dz_1 dz_2}{\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)} \leq$$

$$\frac{\left(g_1\left(b_1\right)-g_1\left(x_{01}\right)\right)\left(g_2\left(b_2\right)-g_2\left(x_{02}\right)\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)} \tag{1}$$

$$\left\|\left\|D_{g_1(x_{01})}^{\nu_1}\left(f_1 \circ g_1^{-1}\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}\left\|\left\|D_{g_2(x_{02})}^{\nu_2}\left(f_2 \circ g_2^{-1}\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)}.$$

Integrals in this work are of Bochner type [8].

We continue with a right generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

Theorem 2 *All as in Theorem 1, however now it is* $f_i \circ g_i^{-1} \in C_{g_i(x_{0i})-}^{\nu_i}\left(\left[g_i\left(a_i\right), g_i\left(b_i\right)\right], A\right)$, *for* $i=1,2$. *Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\|\left(f_1 \circ g_1^{-1}\right)\left(z_1\right)\left(f_2 \circ g_2^{-1}\right)\left(z_2\right)\right\| dz_1 dz_2}{\left(\frac{(g_1(x_{01})-z_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(g_2(x_{02})-z_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)} \leq$$

$$\frac{\left(g_1\left(x_{01}\right)-g_1\left(a_1\right)\right)\left(g_2\left(x_{02}\right)-g_2\left(a_2\right)\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)} \tag{2}$$

$$\left\|\left\|D_{g_1(x_{01})-}^{\nu_1}\left(f_1 \circ g_1^{-1}\right)\right\|\right\|_{L_q([g_1(a_1),g_1(x_{01})],A)}\left\|\left\|D_{g_2(x_{02})-}^{\nu_2}\left(f_2 \circ g_2^{-1}\right)\right\|\right\|_{L_p([g_2(a_2),g_2(x_{02})],A)}.$$

Next comes a sequential left generalized Canavati fractional Hilbert-Pachpatte type inequality over a Banach algebra.

Theorem 3 *Let* $p,q>1$, *such that* $\frac{1}{p}+\frac{1}{q}=1$, *and* $(A,\|\cdot\|)$ *is a Banach algebra; and* $i=1,2$. *Let also* $f_i \in C^1\left(\left[a_i,b_i\right],A\right)$; $g_i \in C^1\left(\left[a_i,b_i\right]\right)$, *strictly increasing, such that* $g_i^{-1} \in C^1\left(\left[g_i\left(a_i\right), g_i\left(b_i\right)\right]\right)$. *Assume that* $\frac{1}{(m_i+1)q}<\nu_i<1$, $x_{0i} \in \left[a_i,b_i\right]$, *and* $D_{g_i(x_{0i})}^{j_i\nu_i}\left(f_i \circ g_i^{-1}\right) \in C_{g_i(x_{0i})}^{\nu_i}\left(\left[g_i\left(a_i\right), g_i\left(b_i\right)\right], A\right)$, *for* $j_i=0,1,...,m_i \in \mathbb{N}$. *Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\|\left(f_1 \circ g_1^{-1}\right)\left(z_1\right)\left(f_2 \circ g_2^{-1}\right)\left(z_2\right)\right\| dz_1 dz_2}{\left(\frac{(z_1-g_1(x_{01}))^{p((m_1+1)\nu_1-1)+1}}{p(p((m_1+1)\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)}\right)} \leq$$

$$\frac{\left(g_1\left(b_1\right)-g_1\left(x_{01}\right)\right)\left(g_2\left(b_2\right)-g_2\left(x_{02}\right)\right)}{\Gamma\left(\left(m_1+1\right)\nu_1\right)\Gamma\left(\left(m_2+1\right)\nu_2\right)} \tag{3}$$

$$\left\|\left\|D_{g_1(x_{01})}^{(m_1+1)\nu_1}\left(f_1 \circ g_1^{-1}\right)\right\|\right\|_{L_q([g_1(x_{01}),g_1(b_1)],A)}\left\|\left\|D_{g_2(x_{02})}^{(m_2+1)\nu_2}\left(f_2 \circ g_2^{-1}\right)\right\|\right\|_{L_p([g_2(x_{02}),g_2(b_2)],A)}.$$

The right side analog of Theorem 3 follows:

Theorem 4 *Let $p,q>1$, such that $\frac{1}{p}+\frac{1}{q}=1$, and $(A,\|\cdot\|)$ is a Banach algebra; and $i=1,2$. Let also $f_i\in C^1([a_i,b_i],A)$; $g_i\in C^1([a_i,b_i])$, strictly increasing, such that $g_i^{-1}\in C^1([g_i(a_i),g_i(b_i)])$. Assume that $\frac{1}{(m_i+1)q}<\nu_i<1$, $x_{0i}\in[a_i,b_i]$, and $D_{g_i(x_{0i})-}^{j_i\nu_i}\left(f_i\circ g_i^{-1}\right)\in C_{g_i(x_{0i})-}^{\nu_i}([g_i(a_i),g_i(b_i)],A)$, for $j_i=0,1,...,m_i\in\mathbb{N}$. Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})}\int_{g_2(a_2)}^{g_2(x_{02})}\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\left(f_2\circ g_2^{-1}\right)(z_2)\right\|dz_1dz_2}{\left(\frac{(g_1(x_{01})-z_1)^{p((m_1+1)\nu_1-1)+1}}{p(p((m_1+1)\nu_1-1)+1)}+\frac{(g_2(x_{02})-z_2)^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)}\right)}\leq$$

$$\frac{(g_1(x_{01})-g_1(a_1))(g_2(x_{02})-g_2(a_2))}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{4}$$

$$\left\|\left\|D_{g_1(x_{01})-}^{(m_1+1)\nu_1}\left(f_1\circ g_1^{-1}\right)\right\|\right\|_{L_q([g_1(a_1),g_1(x_{01})],A)}\left\|\left\|D_{g_2(x_{02})-}^{(m_2+1)\nu_2}\left(f_2\circ g_2^{-1}\right)\right\|\right\|_{L_p([g_2(a_2),g_2(x_{02})],A)}.$$

Other related inspiration comes from [2].

Let $\gamma\geq1$, in this work we derive γ-Schatten left and right Hilbert-Pachpatte inequalities for Banach algebra $\mathcal{B}_\gamma(H)$ valued functions with respect to their Canavati type generalized left and right fractional derivatives. We cover also the sequential fractional case. We finish with applications.

For $\mathcal{B}_\gamma(H)$ definition see section 4 later.

2 Background on Vectorial generalized Canavati fractional calculus

All in this section come from [2], pp. 109-115 and [1].

Let $g:[a,b]\to\mathbb{R}$ be a strictly increasing function. such that $g\in C^1([a,b])$, and $g^{-1}\in C^n([g(a),g(b)])$, $n\in\mathbb{N}$, $(X,\|\cdot\|)$ is a Banach space. Let $f\in C^n([a,b],X)$, and call $l:=f\circ g^{-1}:[g(a),g(b)]\to X$. It is clear that $l,l',...,l^{(n)}$ are continuous functions from $[g(a),g(b)]$ into $f([a,b])\subseteq X$.

Let $\nu\geq1$ such that $[\nu]=n$, $n\in\mathbb{N}$ as above, where $[\cdot]$ is the integral part of the number.

Clearly when $0<\nu<1$, $[\nu]=0$.

I) Let $h\in C([g(a),g(b)],X)$, we define the left Riemann-Liouville Bochner fractional integral as

$$\left(J_\nu^{z_0}h\right)(z):=\frac{1}{\Gamma(\nu)}\int_{z_0}^{z}(z-t)^{\nu-1}h(t)\,dt, \tag{5}$$

for $g(a)\leq z_0\leq z\leq g(b)$, where Γ is the gamma function; $\Gamma(\nu)=\int_0^\infty e^{-t}t^{\nu-1}dt$. We set $J_0^{z_0}h=h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C^{\nu}_{g(x_0)}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ as:

$$C^{\nu}_{g(x_0)}([g(a), g(b)], X) =$$

$$\left\{ h \in C^{[\nu]}([g(a), g(b)], X) : J^{g(x_0)}_{1-\alpha} h^{([\nu])} \in C^1([g(x_0), g(b)], X) \right\}. \quad (6)$$

So let $h \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, we define the left g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(x_0), g(b)]$ as

$$D^{\nu}_{g(x_0)} h := \left(J^{g(x_0)}_{1-\alpha} h^{([\nu])} \right)'. \quad (7)$$

Clearly, for $h \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, there exists

$$\left(D^{\nu}_{g(x_0)} h \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z-t)^{-\alpha} h^{([\nu])}(t)\, dt, \quad (8)$$

for all $g(x_0) \leq z \leq g(b)$.

In particular, when $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, we have that

$$\left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_{g(x_0)}^{z} (z-t)^{-\alpha} \left(f \circ g^{-1} \right)^{([\nu])}(t)\, dt, \quad (9)$$

for all $g(x_0) \leq z \leq g(b)$. We have that $D^{n}_{g(x_0)} \left(f \circ g^{-1} \right) = \left(f \circ g^{-1} \right)^{(n)}$ and $D^{0}_{g(x_0)} \left(f \circ g^{-1} \right) = f \circ g^{-1}$, see [1].

By [1], we have for $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ the following left generalized g-fractional, of Canavati type,X-valued Taylor's formula:

Theorem 5 *Let $f \circ g^{-1} \in C^{\nu}_{g(x_0)}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed.*

(i) If $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1} \right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t)\, dt, \quad (10)$$

for all $x_0 \leq x \leq b$.

(ii) If $0 < \nu < 1$, we get

$$f(x) = \frac{1}{\Gamma(\nu)} \int_{g(x_0)}^{g(x)} (g(x) - t)^{\nu-1} \left(D^{\nu}_{g(x_0)} \left(f \circ g^{-1} \right) \right)(t)\, dt, \quad (11)$$

for all $x_0 \leq x \leq b$.

II) Let $h \in C([g(a), g(b)], X)$, we define the right Riemann-Liouville Bochner fractional integral as

$$\left(J_{z_0-}^{\nu} h\right)(z) := \frac{1}{\Gamma(\nu)} \int_z^{z_0} (t-z)^{\nu-1} h(t)\, dt, \tag{12}$$

for $g(a) \leq z \leq z_0 \leq g(b)$. We set $J_{z_0-}^0 h = h$.

Let $\alpha := \nu - [\nu]$ $(0 < \alpha < 1)$. We define the subspace $C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$ of $C^{[\nu]}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ as:

$$C_{g(x_0)-}^{\nu}([g(a), g(b)], X) :=$$

$$\left\{h \in C^{[\nu]}([g(a), g(b)], X) : J_{g(x_0)-}^{1-\alpha} h^{([\nu])} \in C^1([g(a), g(x_0)], X)\right\}. \tag{13}$$

So let $h \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, we define the right g-generalized X-valued fractional derivative of h of order ν, of Canavati type, over $[g(a), g(x_0)]$ as

$$D_{g(x_0)-}^{\nu} h := (-1)^{n-1} \left(J_{g(x_0)-}^{1-\alpha} h^{([\nu])}\right)'. \tag{14}$$

Clearly, for $h \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, there exists

$$\left(D_{g(x_0)-}^{\nu} h\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t-z)^{-\alpha} h^{([\nu])}(t)\, dt, \tag{15}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

In particular, when $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, we have that

$$\left(D_{g(x_0)-}^{\nu}\left(f \circ g^{-1}\right)\right)(z) = \frac{(-1)^{n-1}}{\Gamma(1-\alpha)} \frac{d}{dz} \int_z^{g(x_0)} (t-z)^{-\alpha} \left(f \circ g^{-1}\right)^{([\nu])}(t)\, dt, \tag{16}$$

for all $g(a) \leq z \leq g(x_0) \leq g(b)$.

We get that

$$\left(D_{g(x_0)-}^{n}\left(f \circ g^{-1}\right)\right)(z) = (-1)^n \left(f \circ g^{-1}\right)^{(n)}(z) \tag{17}$$

and $\left(D_{g(x_0)-}^{0}\left(f \circ g^{-1}\right)\right)(z) = \left(f \circ g^{-1}\right)(z)$, all $z \in [g(a), g(b)]$, see [1].

By [1], we have for $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed, the following right generalized g-fractional, of Canavati type, X-valued Taylor's formula:

Theorem 6 *Let $f \circ g^{-1} \in C_{g(x_0)-}^{\nu}([g(a), g(b)], X)$, where $x_0 \in [a, b]$ is fixed.*

(i) If $\nu \geq 1$, then

$$f(x) - f(x_0) = \sum_{k=1}^{[\nu]-1} \frac{\left(f \circ g^{-1}\right)^{(k)}(g(x_0))}{k!} (g(x) - g(x_0))^k +$$

$$\frac{1}{\Gamma(\nu)}\int_{g(x)}^{g(x_0)}(t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f\circ g^{-1}\right)\right)(t)\,dt, \tag{18}$$

for all $a\leq x\leq x_0$,

(ii) If $0<\nu<1$, *we get*

$$f(x)=\frac{1}{\Gamma(\nu)}\int_{g(x)}^{g(x_0)}(t-g(x))^{\nu-1}\left(D_{g(x_0)-}^{\nu}\left(f\circ g^{-1}\right)\right)(t)\,dt, \tag{19}$$

all $a\leq x\leq x_0$.

III) Denote by

$$D_{g(x_0)}^{m\nu}=D_{g(x_0)}^{\nu}D_{g(x_0)}^{\nu}...D_{g(x_0)}^{\nu}\quad (m\text{-times}),\ m\in\mathbb{N}. \tag{20}$$

We mention the following modified and generalized left X-valued fractional Taylor's formula of Canavati type:

Theorem 7 *Let* $f\in C^1([a,b],X)$, $g\in C^1([a,b])$, *strictly increasing:* $g^{-1}\in C^1([g(a),g(b)])$. *Assume that* $\left(D_{g(x_0)}^{i\nu}\left(f\circ g^{-1}\right)\right)\in C_{g(x_0)}^{\nu}([g(a),g(b)],X)$, $0<\nu<1$, $x_0\in[a,b]$, *for* $i=0,1,...,m$. *Then*

$$f(x)=\frac{1}{\Gamma((m+1)\nu)}\int_{g(x_0)}^{g(x)}(g(x)-z)^{(m+1)\nu-1}\left(D_{g(x_0)}^{(m+1)\nu}\left(f\circ g^{-1}\right)\right)(z)\,dz, \tag{21}$$

all $x_0\leq x\leq b$.

IV) Denote by

$$D_{g(x_0)-}^{m\nu}=D_{g(x_0)-}^{\nu}D_{g(x_0)-}^{\nu}...D_{g(x_0)-}^{\nu}\quad (m\text{ times}),\ m\in\mathbb{N}. \tag{22}$$

We mention the following modified and generalized right X-valued fractional Taylor's formula of Canavati type:

Theorem 8 *Let* $f\in C^1([a,b],X)$, $g\in C^1([a,b])$, *strictly increasing:* $g^{-1}\in C^1([g(a),g(b)])$. *Assume that* $\left(D_{g(x_0)-}^{i\nu}\left(f\circ g^{-1}\right)\right)\in C_{g(x_0)-}^{\nu}([g(a),g(b)],X)$, $0<\nu<1$, $x_0\in[a,b]$, *for all* $i=0,1,...,m$. *Then*

$$f(x)=\frac{1}{\Gamma((m+1)\nu)}\int_{g(x)}^{g(x_0)}(z-g(x))^{(m+1)\nu-1}\left(D_{g(x_0)-}^{(m+1)\nu}\left(f\circ g^{-1}\right)\right)(z)\,dz, \tag{23}$$

all $a\leq x\leq x_0\leq b$.

3 Basic Banach Algebras background

All here come from [10].

We need

Definition 9 *([10], p. 245) A complex algebra is a vector space A over the complex filed $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{24}$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \tag{25}$$

and

$$\alpha(xy) = (\alpha x)y = x(\alpha y), \tag{26}$$

for all x, y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A,\ y \in A) \tag{27}$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{28}$$

and

$$\|e\| = 1, \tag{29}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 10 *Commutativity of A will be explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (28).

Inequality (27) makes multiplication to be continuous, more precisely left and right continuous, see [10], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [10], p. 247-248, § 10.3.

We also make

Remark 11 *Next we mention about integration of A-valued functions, see [10], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [10], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \ d\mu = \int_Q x f(p) \ d\mu(p) \tag{30}$$

and

$$\left(\int_Q f \ d\mu \right) x = \int_Q f(p) \, x \ d\mu(p) \, . \tag{31}$$

The Bochner integrals we will involve in our article follow (30) and (31). Also, let $f \in C([a,b], X)$, where $[a,b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [2], p. 3, f is Bochner integrable.

4 p-Schatten norms background

In this advanced section all come from [7].

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of trace class if

$$\|A\|_1 := \sum_{i \in I} \langle |A| \, e_i, e_i \rangle < \infty. \tag{32}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthornormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the trace of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$tr(A) := \sum_{i \in I} \langle A e_i, e_i \rangle \, , \tag{33}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (33) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 12 *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$tr(A^*) = \overline{tr(A)}; \tag{34}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$*, then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$tr(AT) = tr(TA) \quad and \quad |tr(AT)| \leq \|A\|_1 \|T\|; \tag{35}$$

(iii) $tr(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|tr\| = 1$*;*
(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $tr(AB) = tr(BA)$*;*
(v) $\mathcal{B}_{fin}(H)$ *(finite rank operators) is a dense subspace of* $\mathcal{B}_1(H)$*.*

An operator $A \in \mathcal{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the p-Schatten norm is finite [12, p. 60-64]

$$\|A\|_p := [tr(|A|^p)]^{\frac{1}{p}} < \infty,$$

$|A|^p$ is an operator notation and not a power.

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{36}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{37}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a norm on the $*$-ideal $\mathcal{B}_p(H)$, which is a Banach algebra, and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [12, p. 60-64], for $p \geq 1$,

$$\|A\|_p = \|A^*\|_p, \quad A \in \mathcal{B}_p(H) \tag{38}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \quad A, B \in \mathcal{B}_p(H) \tag{39}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \quad \|BA\|_p \leq \|B\| \|A\|_p, \quad A \in \mathcal{B}_p(H), B \in \mathcal{B}(H). \tag{40}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \quad A \in \mathcal{B}_p(H), B, C \in \mathcal{B}(H). \tag{41}$$

In terms of p-Schatten norm we have the Hölder inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$:

$$(|tr(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \quad A \in \mathcal{B}_p(H), B \in \mathcal{B}_q(H). \tag{42}$$

For the theory of trace functionals and their applications the interested reader is referred to [11] and [12].

For some classical trace inequalities see [5], [6] and [9], which are continuations of the work of Bellman [4].

5 Main Results

We start with a 1-2-Schatten norms left generalized Canavati fractional Hilbert-Pachpatte type inequality over $\mathcal{B}_2(H)$.

Theorem 13 *Let $p, q > 1$, such that $\frac{1}{p}+\frac{1}{q}=1$, and $\left(\mathcal{B}_2(H), \|\cdot\|_2\right)$ is the $*$-ideal; $i=1,2$. Let also $x_{0i} \in [a_i, b_i] \subset \mathbb{R}$, $\nu_i \geq 1$, $n_i=[\nu_i]$, $f_i \in C^{n_i}([a_i, b_i], \mathcal{B}_2(H))$; $g_i \in C^1([a_i, b_i])$, strictly increasing, such that $g_i^{-1} \in C^{n_i}([g_i(a_i), g_i(b_i)])$, with $\left(f_i \circ g_i^{-1}\right)^{(k_i)}(g_i(x_{0i}))=0$, $k_i=0,1,...,n_i-1$. Assume further that $f_i \circ g_i^{-1} \in C^{\nu_i}_{g_i(x_{0i})}([g_i(a_i), g_i(b_i)], \mathcal{B}_2(H))$. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\|\left(f_1 \circ g_1^{-1}\right)(z_1)\left(f_2 \circ g_2^{-1}\right)(z_2)\right\|_1 dz_1 dz_2}{\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)} \leq$$

$$\frac{(g_1(b_1)-g_1(x_{01}))(g_2(b_2)-g_2(x_{02}))}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{43}$$

$$\left\|\left\|D^{\nu_1}_{g_1(x_{01})}\left(f_1 \circ g_1^{-1}\right)\right\|_2\right\|_{L_q([g_1(x_{01}), g_1(b_1)], \mathcal{B}_2(H))}\left\|\left\|D^{\nu_2}_{g_2(x_{02})}\left(f_2 \circ g_2^{-1}\right)\right\|_2\right\|_{L_p([g_2(x_{02}), g_2(b_2)], \mathcal{B}_2(H))}.$$

Proof. By (10) and assumptions we get that

$$\left(f_i \circ g_i^{-1}\right)(z_i)=\frac{1}{\Gamma(\nu_i)} \int_{g_i(x_{0i})}^{z_i}(z_i-t_i)^{\nu_i-1}\left(D^{\nu_i}_{g_i(x_{0i})}\left(f_i \circ g_i^{-1}\right)\right)(t_i)\, dt_i, \tag{44}$$

for all $g_i(x_{0i}) \leq z_i \leq g_i(b_i)$; $i=1,2$.

By Hölder's inequality we obtain

$$\left\|\left(f_1 \circ g_1^{-1}\right)(z_1)\right\|_2 \leq \frac{1}{\Gamma(\nu_1)} \int_{g_1(x_{01})}^{z_1}(z_1-t_1)^{\nu_1-1}\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1 \circ g_1^{-1}\right)\right)(t_1)\right\|_2 dt_1 \leq$$

$$\frac{1}{\Gamma(\nu_1)}\left(\int_{g_1(x_{01})}^{z_1}(z_1-t_1)^{p(\nu_1-1)} dt_1\right)^{\frac{1}{p}}\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1 \circ g_1^{-1}\right)\right)(t_1)\right\|_2^q dt_1\right)^{\frac{1}{q}}=$$

$$\frac{1}{\Gamma(\nu_1)} \frac{(z_1-g_1(x_{01}))^{\frac{p(\nu_1-1)+1}{p}}}{(p(\nu_1-1)+1)^{\frac{1}{p}}}\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1 \circ g_1^{-1}\right)\right)(t_1)\right\|_2^q dt_1\right)^{\frac{1}{q}}. \tag{45}$$

That is

$$\left\|\left(f_1 \circ g_1^{-1}\right)(z_1)\right\|_2 \leq \frac{1}{\Gamma(\nu_1)} \frac{(z_1-g_1(x_{01}))^{\frac{p(\nu_1-1)+1}{p}}}{(p(\nu_1-1)+1)^{\frac{1}{p}}}$$

$$\left(\int_{g_1(x_{01})}^{z_1}\left\|\left(D^{\nu_1}_{g_1(x_{01})}\left(f_1 \circ g_1^{-1}\right)\right)(t_1)\right\|_2^q dt_1\right)^{\frac{1}{q}}, \tag{46}$$

for all $g_1(x_{01}) \leq z_1 \leq g_1(b_1)$.

Similarly, we prove that

$$\left\|\left(f_2 \circ g_2^{-1}\right)(z_2)\right\|_2 \leq \frac{1}{\Gamma(\nu_2)} \frac{(z_2 - g_2(x_{02}))^{\frac{q(\nu_2-1)+1}{q}}}{(q(\nu_2-1)+1)^{\frac{1}{q}}}$$

$$\left(\int_{g_2(x_{02})}^{z_2} \left\|\left(D_{g_2(x_{02})}^{\nu_2}\left(f_2 \circ g_2^{-1}\right)\right)(t_2)\right\|_2^p dt_2\right)^{\frac{1}{p}}, \tag{47}$$

for all $g_2(x_{02}) \leq z_2 \leq g_2(b_2)$.

Therefore we have

$$\left\|\left(f_1 \circ g_1^{-1}\right)(z_1)\right\|_2 \leq \frac{1}{\Gamma(\nu_1)} \frac{(z_1 - g_1(x_{01}))^{\frac{p(\nu_1-1)+1}{p}}}{(p(\nu_1-1)+1)^{\frac{1}{p}}}$$

$$\left\|\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1 \circ g_1^{-1}\right)\right)\right\|_2\right\|_{q,[g_1(x_{01}),g_1(b_1)]}, \tag{48}$$

for all $g_1(x_{01}) \leq z_1 \leq g_1(b_1)$;

and

$$\left\|\left(f_2 \circ g_2^{-1}\right)(z_2)\right\|_2 \leq \frac{1}{\Gamma(\nu_2)} \frac{(z_2 - g_2(x_{02}))^{\frac{q(\nu_2-1)+1}{q}}}{(q(\nu_2-1)+1)^{\frac{1}{q}}}$$

$$\left\|\left\|\left(D_{g_2(x_{02})}^{\nu_2}\left(f_2 \circ g_2^{-1}\right)\right)\right\|_2\right\|_{p,[g_2(x_{02}),g_2(b_2)]}, \tag{49}$$

for all $g_2(x_{02}) \leq z_2 \leq g_2(b_2)$.

Hence we get that

$$\left\|\left(f_1 \circ g_1^{-1}\right)(z_1)\right\|_2 \left\|\left(f_2 \circ g_2^{-1}\right)(z_2)\right\|_2 \leq \frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)(p(\nu_1-1)+1)^{\frac{1}{p}}(q(\nu_2-1)+1)^{\frac{1}{q}}}$$

$$(z_1 - g_1(x_{01}))^{\frac{p(\nu_1-1)+1}{p}} (z_2 - g_2(x_{02}))^{\frac{q(\nu_2-1)+1}{q}} \tag{50}$$

$$\left\|\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1 \circ g_1^{-1}\right)\right)\right\|_2\right\|_{q,[g_1(x_{01}),g_1(b_1)]} \left\|\left\|\left(D_{g_2(x_{02})}^{\nu_2}\left(f_2 \circ g_2^{-1}\right)\right)\right\|_2\right\|_{p,[g_2(x_{02}),g_2(b_2)]} \leq$$

(using Young's inequality for $a, b \geq 0$, $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$)

$$\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \left(\frac{(z_1 - g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)} + \frac{(z_2 - g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)$$

$$\left\|\left\|\left(D_{g_1(x_{01})}^{\nu_1}\left(f_1 \circ g_1^{-1}\right)\right)\right\|_2\right\|_{L_q([g_1(x_{01}),g_1(b_1)],\mathcal{B}_2(H))} \left\|\left\|\left(D_{g_2(x_{02})}^{\nu_2}\left(f_2 \circ g_2^{-1}\right)\right)\right\|_2\right\|_{L_p([g_2(x_{02}),g_2(b_2)],\mathcal{B}_2(H))},$$

$\forall (z_1, z_2) \in [g_1(x_{01}), g_1(b_1)] \times [g_2(x_{02}), g_2(b_2)]$.

So far we have

$$\frac{\left\|\left(f_1 \circ g_1^{-1}\right)(z_1)\left(f_2 \circ g_2^{-1}\right)(z_2)\right\|_1}{\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)} + \frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)} \overset{(42)}{\leq} \tag{51}$$

$$\frac{\left\| \left(f_1 \circ g_1^{-1} \right) (z_1) \right\|_2 \left\| \left(f_2 \circ g_2^{-1} \right) (z_2) \right\|_2}{\left(\frac{(z_1 - g_1(x_{01}))^{p(\nu_1 - 1)+1}}{p(p(\nu_1-1)+1)} + \frac{(z_2 - g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)} \right)} \leq \tag{52}$$

$$\frac{1}{\Gamma(\nu_1)\Gamma(\nu_2)} \left\| \left\| \left(D_{g_1(x_{01})}^{\nu_1} \left(f_1 \circ g_1^{-1} \right) \right) \right\|_2 \right\|_{L_q([g_1(x_{01}), g_1(b_1)], \mathcal{B}_2(H))}$$

$$\left\| \left\| \left(D_{g_2(x_{02})}^{\nu_2} \left(f_2 \circ g_2^{-1} \right) \right) \right\|_2 \right\|_{L_p([g_2(x_{02}), g_2(b_2)], \mathcal{B}_2(H))},$$

$\forall\ (z_1, z_2) \in [g_1(x_{01}), g_1(b_1)] \times [g_2(x_{02}), g_2(b_2)]$.

The denominators in (51), (52) can be zero only when both $z_1 = g_1(x_{01})$ and $z_2 = g_2(x_{02})$.

Therefore we obtain (43), by integrating (51), (52) over $[g_1(x_{01}), g_1(b_1)] \times [g_2(x_{02}), g_2(b_2)]$. ■

We continue with the corresponding right generalized Canavati fractional Hilbert-Pachpatte type inequality over $\mathcal{B}_2(H)$.

Theorem 14 *All as in Theorem 13, however now it is* $f_i \circ g_i^{-1} \in C_{g_i(x_{0i})-}^{\nu_i}([g_i(a_i), g_i(b_i)], \mathcal{B}_2(H))$, *for* $i = 1, 2$. *Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1} \right) (z_1) \left(f_2 \circ g_2^{-1} \right) (z_2) \right\|_1 dz_1 dz_2}{\left(\frac{(g_1(x_{01}) - z_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)} + \frac{(g_2(x_{02}) - z_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)} \right)} \leq$$

$$\frac{(g_1(x_{01}) - g_1(a_1))(g_2(x_{02}) - g_2(a_2))}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{53}$$

$$\left\| \left\| D_{g_1(x_{01})-}^{\nu_1} \left(f_1 \circ g_1^{-1} \right) \right\|_2 \right\|_{L_q([g_1(a_1), g_1(x_{01})], \mathcal{B}_2(H))} \left\| \left\| D_{g_2(x_{02})-}^{\nu_2} \left(f_2 \circ g_2^{-1} \right) \right\|_2 \right\|_{L_p([g_2(a_2), g_2(x_{02})], \mathcal{B}_2(H))}.$$

Proof. Similar to Theorem 13, by using now (18). ■

Next comes a sequential analogous left generalized Canavati fractional Hilbert-Pachpatte type inequality over $\mathcal{B}_2(H)$.

Theorem 15 *Let* $p, q > 1$, *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *and* $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is the* $*$*-ideal;* $i = 1, 2$. *Let also* $f_i \in C^1([a_i, b_i], \mathcal{B}_2(H))$; $g_i \in C^1([a_i, b_i])$, *strictly increasing, such that* $g_i^{-1} \in C^1([g_i(a_i), g_i(b_i)])$. *Assume that* $\frac{1}{(m_i+1)q} < \nu_i < 1$, $x_{0i} \in [a_i, b_i]$, *and* $D_{g_i(x_{0i})}^{j_i \nu_i} \left(f_i \circ g_i^{-1} \right) \in C_{g_i(x_{0i})}^{\nu_i}([g_i(a_i), g_i(b_i)], \mathcal{B}_2(H))$, *for* $j_i = 0, 1, ..., m_i \in \mathbb{N}$. *Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1} \right) (z_1) \left(f_2 \circ g_2^{-1} \right) (z_2) \right\|_1 dz_1 dz_2}{\left(\frac{(z_1 - g_1(x_{01}))^{p((m_1+1)\nu_1-1)+1}}{p(p((m_1+1)\nu_1-1)+1)} + \frac{(z_2 - g_2(x_{02}))^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)} \right)} \leq$$

$$\frac{(g_1(b_1) - g_1(x_{01}))(g_2(b_2) - g_2(x_{02}))}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{54}$$

$$\left\| \left\| D_{g_1(x_{01})}^{(m_1+1)\nu_1} \left(f_1 \circ g_1^{-1} \right) \right\|_2 \right\|_{L_q([g_1(x_{01}), g_1(b_1)], \mathcal{B}_2(H))} \left\| \left\| D_{g_2(x_{02})}^{(m_2+1)\nu_2} \left(f_2 \circ g_2^{-1} \right) \right\|_2 \right\|_{L_p([g_2(x_{02}), g_2(b_2)], \mathcal{B}_2(H))}.$$

Proof. Using (21), as similar to Theorem 13 the proof is omitted. ■

The right side analog of Theorem 15 follows:

Theorem 16 *Let $p,q>1$, such that $\frac{1}{p}+\frac{1}{q}=1$, and $\left(\mathcal{B}_2(H), \|\cdot\|_2\right)$ is the $*$-ideal; $i=1,2$. Let also $f_i \in C^1\left([a_i,b_i], \mathcal{B}_2(H)\right)$; $g_i \in C^1\left([a_i,b_i]\right)$, strictly increasing, such that $g_i^{-1} \in C^1\left([g_i(a_i), g_i(b_i)]\right)$. Assume that $\frac{1}{(m_i+1)q} < \nu_i < 1$, $x_{0i} \in [a_i,b_i]$, and $D^{j_i \nu_i}_{g_i(x_{0i})-}\left(f_i \circ g_i^{-1}\right) \in C^{\nu_i}_{g_i(x_{0i})-}\left([g_i(a_i), g_i(b_i)], \mathcal{B}_2(H)\right)$, for $j_i = 0,1,...,m_i \in \mathbb{N}$. Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\|_1 dz_1 dz_2}{\left(\frac{(g_1(x_{01})-z_1)^{p((m_1+1)\nu_1-1)+1}}{p(p((m_1+1)\nu_1-1)+1)} + \frac{(g_2(x_{02})-z_2)^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)} \right)} \le$$

$$\frac{(g_1(x_{01})-g_1(a_1))(g_2(x_{02})-g_2(a_2))}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{55}$$

$$\left\| \left\| D^{(m_1+1)\nu_1}_{g_1(x_{01})-}\left(f_1 \circ g_1^{-1}\right) \right\|_2 \right\|_{L_q([g_1(a_1), g_1(x_{01})], \mathcal{B}_2(H))} \left\| \left\| D^{(m_2+1)\nu_2}_{g_2(x_{02})-}\left(f_2 \circ g_2^{-1}\right) \right\|_2 \right\|_{L_p([g_2(a_2), g_2(x_{02})], \mathcal{B}_2(H))}.$$

Proof. Using (23), as similar to Theorem 13 is omitted. ■

We continue with a γ-Schatten norm left generalized Canavati fractional Hilbert-Pachpatte type inequality over $\mathcal{B}_\gamma(H)$, $\gamma \ge 1$.

Theorem 17 *Let $\gamma \ge 1$, $p,q>1$, such that $\frac{1}{p}+\frac{1}{q}=1$, and $\left(\mathcal{B}_\gamma(H), \|\cdot\|_\gamma\right)$ is the $*$-ideal; $i=1,2$. Let also $x_{0i} \in [a_i,b_i] \subset \mathbb{R}$, $\nu_i \ge 1$, $n_i = [\nu_i]$, $f_i \in C^{n_i}\left([a_i,b_i], \mathcal{B}_\gamma(H)\right)$; $g_i \in C^1\left([a_i,b_i]\right)$, strictly increasing, such that $g_i^{-1} \in C^{n_i}\left([g_i(a_i), g_i(b_i)]\right)$, with $\left(f_i \circ g_i^{-1}\right)^{(k_i)}(g_i(x_{0i})) = 0$, $k_i = 0,1,...,n_i-1$. Assume further that $f_i \circ g_i^{-1} \in C^{\nu_i}_{g_i(x_{0i})}\left([g_i(a_i), g_i(b_i)], \mathcal{B}_\gamma(H)\right)$. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\|_\gamma dz_1 dz_2}{\left(\frac{(z_1-g_1(x_{01}))^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)} + \frac{(z_2-g_2(x_{02}))^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)} \right)} \le$$

$$\frac{(g_1(b_1)-g_1(x_{01}))(g_2(b_2)-g_2(x_{02}))}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{56}$$

$$\left\| \left\| D^{\nu_1}_{g_1(x_{01})}\left(f_1 \circ g_1^{-1}\right) \right\|_\gamma \right\|_{L_q([g_1(x_{01}), g_1(b_1)], \mathcal{B}_\gamma(H))} \left\| \left\| D^{\nu_2}_{g_2(x_{02})}\left(f_2 \circ g_2^{-1}\right) \right\|_\gamma \right\|_{L_p([g_2(x_{02}), g_2(b_2)], \mathcal{B}_\gamma(H))}.$$

Proof. Similar to Theorem 13, by using norm (39). ■

We continue with the corresponding right generalized Canavati fractional Hilbert-Pachpatte type inequality over $\mathcal{B}_\gamma(H)$.

Theorem 18 *All as in Theorem 17, however now it is $f_i \circ g_i^{-1} \in C^{\nu_i}_{g_i(x_{0i})-}\left([g_i(a_i), g_i(b_i)], \mathcal{B}_\gamma(H)\right)$, for $i=1,2$. Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})} \int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\|_\gamma dz_1 dz_2}{\left(\frac{(g_1(x_{01})-z_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)} + \frac{(g_2(x_{02})-z_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)} \right)} \le$$

$$\frac{\left(g_1\left(x_{01}\right)-g_1\left(a_1\right)\right)\left(g_2\left(x_{02}\right)-g_2\left(a_2\right)\right)}{\Gamma\left(\nu_1\right)\Gamma\left(\nu_2\right)} \tag{57}$$

$$\left\|\left\|D^{\nu_1}_{g_1(x_{01})-}\left(f_1\circ g_1^{-1}\right)\right\|_\gamma\right\|_{L_q([g_1(a_1),g_1(x_{01})],\mathcal{B}_\gamma(H))}\left\|\left\|D^{\nu_2}_{g_2(x_{02})-}\left(f_2\circ g_2^{-1}\right)\right\|_\gamma\right\|_{L_p([g_2(a_2),g_2(x_{02})],\mathcal{B}_\gamma(H))}.$$

Proof. Similar to Theorem 13, by using now (39). ■

Next comes a sequential analogous left generalized Canavati fractional Hilbert-Pachpatte type inequality over $\mathcal{B}_\gamma(H)$.

Theorem 19 *Let $\gamma\geq 1$, $p,q>1$, such that $\frac{1}{p}+\frac{1}{q}=1$, and $\left(\mathcal{B}_\gamma(H),\|\cdot\|_\gamma\right)$ is the $*$-ideal; $i=1,2$. Let also $f_i\in C^1([a_i,b_i],\mathcal{B}_\gamma(H))$; $g_i\in C^1([a_i,b_i])$, strictly increasing, such that $g_i^{-1}\in C^1([g_i(a_i),g_i(b_i)])$. Assume that $\frac{1}{(m_i+1)q}<\nu_i<1$, $x_{0i}\in[a_i,b_i]$, and $D^{j_i\nu_i}_{g_i(x_{0i})}\left(f_i\circ g_i^{-1}\right)\in C^{\nu_i}_{g_i(x_{0i})}([g_i(a_i),g_i(b_i)],\mathcal{B}_\gamma(H))$, for $j_i=0,1,...,m_i\in\mathbb{N}$. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)}\int_{g_2(x_{02})}^{g_2(b_2)}\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\left(f_2\circ g_2^{-1}\right)(z_2)\right\|_\gamma dz_1dz_2}{\left(\frac{(z_1-g_1(x_{01}))^{p((m_1+1)\nu_1-1)+1}}{p(p((m_1+1)\nu_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)}\right)}\leq$$

$$\frac{\left(g_1\left(b_1\right)-g_1\left(x_{01}\right)\right)\left(g_2\left(b_2\right)-g_2\left(x_{02}\right)\right)}{\Gamma\left(\left(m_1+1\right)\nu_1\right)\Gamma\left(\left(m_2+1\right)\nu_2\right)} \tag{58}$$

$$\left\|\left\|D^{(m_1+1)\nu_1}_{g_1(x_{01})}\left(f_1\circ g_1^{-1}\right)\right\|_\gamma\right\|_{L_q([g_1(x_{01}),g_1(b_1)],\mathcal{B}_\gamma(H))}\left\|\left\|D^{(m_2+1)\nu_2}_{g_2(x_{02})}\left(f_2\circ g_2^{-1}\right)\right\|_\gamma\right\|_{L_p([g_2(x_{02}),g_2(b_2)],\mathcal{B}_\gamma(H))}.$$

Proof. Using (21), as similar to Theorem 13 the proof is omitted. ■

The right side analog of Theorem 19 follows:

Theorem 20 *Let $\gamma\geq 1$, $p,q>1$, such that $\frac{1}{p}+\frac{1}{q}=1$, and $\left(\mathcal{B}_\gamma(H),\|\cdot\|_\gamma\right)$ is the $*$-ideal; $i=1,2$. Let also $f_i\in C^1([a_i,b_i],\mathcal{B}_\gamma(H))$; $g_i\in C^1([a_i,b_i])$, strictly increasing, such that $g_i^{-1}\in C^1([g_i(a_i),g_i(b_i)])$. Assume that $\frac{1}{(m_i+1)q}<\nu_i<1$, $x_{0i}\in[a_i,b_i]$, and $D^{j_i\nu_i}_{g_i(x_{0i})-}\left(f_i\circ g_i^{-1}\right)\in C^{\nu_i}_{g_i(x_{0i})-}([g_i(a_i),g_i(b_i)],\mathcal{B}_\gamma(H))$, for $j_i=0,1,...,m_i\in\mathbb{N}$. Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})}\int_{g_2(a_2)}^{g_2(x_{02})}\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\left(f_2\circ g_2^{-1}\right)(z_2)\right\|_\gamma dz_1dz_2}{\left(\frac{(g_1(x_{01})-z_1)^{p((m_1+1)\nu_1-1)+1}}{p(p((m_1+1)\nu_1-1)+1)}+\frac{(g_2(x_{02})-z_2)^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)}\right)}\leq$$

$$\frac{\left(g_1\left(x_{01}\right)-g_1\left(a_1\right)\right)\left(g_2\left(x_{02}\right)-g_2\left(a_2\right)\right)}{\Gamma\left(\left(m_1+1\right)\nu_1\right)\Gamma\left(\left(m_2+1\right)\nu_2\right)} \tag{59}$$

$$\left\|\left\|D^{(m_1+1)\nu_1}_{g_1(x_{01})-}\left(f_1\circ g_1^{-1}\right)\right\|_\gamma\right\|_{L_q([g_1(a_1),g_1(x_{01})],\mathcal{B}_\gamma(H))}\left\|\left\|D^{(m_2+1)\nu_2}_{g_2(x_{02})-}\left(f_2\circ g_2^{-1}\right)\right\|_\gamma\right\|_{L_p([g_2(a_2),g_2(x_{02})],\mathcal{B}_\gamma(H))}.$$

Proof. Using (23), as similar to Theorem 13 is omitted. ■

6 Applications

We give the following γ-Schatten Hilbert-Pachpatte fractional inequalities:

Corollary 21 *(to Theorem 13) All as in Theorem 13 for $g_1(t) = g_2(t) = t$. Then*

$$\int_{x_{01}}^{b_1}\int_{x_{02}}^{b_2} \frac{\left\| f_1(z_1) f_2(z_2) \right\|_1 dz_1 dz_2}{\left(\frac{(z_1 - x_{01})^{p(\nu_1 - 1)+1}}{p(p(\nu_1-1)+1)} + \frac{(z_2 - x_{02})^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)} \right)} \leq \tag{60}$$

$$\frac{(b_1 - x_{01})(b_2 - x_{02})}{\Gamma(\nu_1)\Gamma(\nu_2)} \left\| \left\| D_{x_{01}}^{\nu_1} f_1 \right\|_2 \right\|_{L_q([x_{01}, b_1], \mathcal{B}_2(H))} \left\| \left\| D_{x_{02}}^{\nu_2} f_2 \right\|_2 \right\|_{L_p([x_{02}, b_2], \mathcal{B}_2(H))}.$$

We continue with

Corollary 22 *(to Theorem 15) All as in Theorem 15 for $g_1(t) = g_2(t) = t$. Then*

$$\int_{x_{01}}^{b_1}\int_{x_{02}}^{b_2} \frac{\left\| f_1(z_1) f_2(z_2) \right\|_1 dz_1 dz_2}{\left(\frac{(z_1 - x_{01})^{p((m_1+1)\nu_1 - 1)+1}}{p(p((m_1+1)\nu_1-1)+1)} + \frac{(z_2 - x_{02})^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)} \right)} \leq \tag{61}$$

$$\frac{(b_1 - x_{01})(b_2 - x_{02})}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)}$$

$$\left\| \left\| D_{x_{01}}^{(m_1+1)\nu_1} f_1 \right\|_2 \right\|_{L_q([x_{01}, b_1], \mathcal{B}_2(H))} \left\| \left\| D_{x_{02}}^{(m_2+1)\nu_2} f_2 \right\|_2 \right\|_{L_p([x_{02}, b_2], \mathcal{B}_2(H))}.$$

Next we present

Corollary 23 *(to Theorem 18) All as in Theorem 18 for $g_1(t) = g_2(t) = e^t$. Then*

$$\int_{e^{a_1}}^{e^{x_{01}}}\int_{e^{a_2}}^{e^{x_{02}}} \frac{\left\| (f_1 \circ \log)(z_1)(f_2 \circ \log)(z_2) \right\|_\gamma dz_1 dz_2}{\left(\frac{(e^{x_{01}} - z_1)^{p(\nu_1 - 1)+1}}{p(p(\nu_1-1)+1)} + \frac{(e^{x_{02}} - z_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)} \right)} \leq$$

$$\frac{(e^{x_{01}} - e^{a_1})(e^{x_{02}} - e^{a_2})}{\Gamma(\nu_1)\Gamma(\nu_2)} \tag{62}$$

$$\left\| \left\| D_{e^{x_{01}}-}^{\nu_1} (f_1 \circ \log) \right\|_\gamma \right\|_{L_q([e^{a_1}, e^{x_{01}}], \mathcal{B}_\gamma(H))} \left\| \left\| D_{e^{x_{02}}-}^{\nu_2} (f_2 \circ \log) \right\|_\gamma \right\|_{L_p([e^{a_2}, e^{x_{02}}], \mathcal{B}_\gamma(H))}.$$

We finish with

Corollary 24 *(to Theorem 20) All as in Theorem 20, with $g_1(t) = g_2(t) = \log t$, where $[a_i, b_i] \subset \mathbb{R}_* - \{0\}$, $i = 1, 2$. Then*

$$\int_{\log a_1}^{\log x_{01}}\int_{\log a_2}^{\log x_{02}} \frac{\left\| (f_1 \circ e^t)(z_1)(f_2 \circ e^t)(z_2) \right\|_\gamma dz_1 dz_2}{\left(\frac{(\log x_{01} - z_1)^{p((m_1+1)\nu_1 - 1)+1}}{p(p((m_1+1)\nu_1-1)+1)} + \frac{(\log x_{02} - z_2)^{q((m_2+1)\nu_2-1)+1}}{q(q((m_2+1)\nu_2-1)+1)} \right)} \leq$$

$$\frac{\left(\log \frac{x_{01}}{a_1}\right)\left(\log \frac{x_{02}}{a_2}\right)}{\Gamma((m_1+1)\nu_1)\Gamma((m_2+1)\nu_2)} \tag{63}$$

$$\left\| \left\| D_{\log x_{01}-}^{(m_1+1)\nu_1} (f_1 \circ e^t) \right\|_\gamma \right\|_{L_q([\log a_1, \log x_{01}], \mathcal{B}_\gamma(H))} \left\| \left\| D_{\log x_{02}-}^{(m_2+1)\nu_2} (f_2 \circ e^t) \right\|_\gamma \right\|_{L_p([\log a_2, \log x_{02}], \mathcal{B}_\gamma(H))}.$$

References

[1] G.A. Anastassiou, *Strong mixed and generalized fractional calculus for Banach space valued functions*, Mat. Vesnik, 69(3) (2017), 176-191.

[2] G.A. Anastassiou, *Intelligent Computations: Abstract Fractional Calculus, Inequalities, Approximations*, Springer, Heidelberg, New York, 2018.

[3] G.A. Anastassiou, *Generalized Canavati Fractional Hilbert-Pachpatte type inequalities for Banach algebra valued functions*, Journal of Computational Analysis and Applications, accepted, 2021.

[4] R. Bellman, *Some inequalities for positive definite matrices*, in E.F. Beckenbach (Ed.), General Inequalities 2, Proceedings of the 2nd International Conference on General Inequalities, Birkhauser, Basel, 1980, 89-90.

[5] D. Chang, *A matrix trace inequality for products of Hermitian matrices*, J. Math. Anal. Appl., 237 (1999), 721-725.

[6] I.D. Coop, *On matrix trace inequalities and related topics for products of Hermitian matrix*, J. Math. Anal. Appl. 188 (1994), 999-1001.

[7] S.S. Dragomir, *p-Schatten norm inequalities of Ostrowski's type*, RGMIA Res. Rep. Coll. 24 (2021), Art. 108, 19 pp.

[8] J. Mikusinski, *The Bochner integral*, Academic Press, New York, 1978.

[9] H. Neudecker, *A matrix trace inequality*, J. Math. Anal. Appl., 166 (1992), 302-303.

[10] W. Rudin, *Functional Analysis*, Second Edition, McGraw-Hill, Inc., New York, 1991.

[11] B. Simon, *Trace ideals and Their Applications*, Cambridge University Press, Cambridge, 1979.

[12] V.A. Zagrebvov, *Gibbs Semigroups*, Operator Theory: Advances and Applications, Volume 273, Birkhauser, 2019.