γ-Schatten norm Generalized Ostrowski, Opial and Hilbert-Pachpatte type inequalities for von Neumann-Schatten class $\mathcal{B}_\gamma(H)$ valued functions with integer vectorial derivatives

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Abstract

Using a generalized vectorial Taylor formula involving ordinary vector derivatives we establish mixed Ostrowski, Opial and Hilbert-Pachpatte type inequalities for several von Neumann-Schatten class $\mathcal{B}_\gamma(H)$ valued functions. The estimates are with respect to all norms $\|\cdot\|_p$, $1 \le p \le \infty$. We finish with applications.

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1 Introduction

Our main motivation is [3].

We mention a uniform mixed generalized Ostrowski type inequality for several functions that are Banach algebra valued.

Theorem 1 *([3]) Let $n \in \mathbb{N}$ and $f_i \in C^n([a,b], A)$, $i = 1, ..., r \in \mathbb{N} - \{1\}$; where $[a,b] \subset \mathbb{R}$ and $(A, \|\cdot\|)$ is a Banach algebra. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We assume that $\left(f_i \circ g^{-1}\right)^{(j)}(g(x_0)) = 0$, $j = 1, ..., n-1$; $i = 1, ..., r$; where $x_0 \in [a,b]$ be fixed. Denote by*

$$E(f_1, ..., f_r)(x_0) :=$$

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$$\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx-\left(\int_a^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)dx\right) f_i(x_0)\right]. \quad (1)$$

Then

1)

$$E(f_1,...,f_r)(x_0)=\frac{1}{(n-1)!}$$

$$\sum_{i=1}^{r}\left[(-1)^n\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]+\right. \quad (2)$$

$$\left.\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right],$$

and

2)

$$\|E(f_1,...,f_r)(x_0)\|\leq\frac{1}{n!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(a),g(x_0)]}(g(x_0)-g(a))^n\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]+\right.\right. \quad (3)$$

$$\left.\left.\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^n\left(\int_{x_0}^b\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right]\right\}.$$

We also mention a left generalized Opial type inequality for ordinary vector valued derivatives:

Theorem 2 *([3]) Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$*, and* $n\in\mathbb{N}$*,* $f\in C^n([a,b],A)$*; where* $[a,b]\subset\mathbb{R}$ *and* $(A,\|\cdot\|)$ *is a Banach algebra. Let* $g\in C^1([a,b])$*, strictly increasing, such that* $g^{-1}\in C^n([g(a),g(b)])$*. We assume that* $\left(f\circ g^{-1}\right)^{(j)}(g(x_0))=0$*,* $j=0,1,...,n-1$*; where* $x_0\in[a,b]$ *be fixed. Then*

$$\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)(z)\left(f\circ g^{-1}\right)^{(n)}(z)\right\|dz\leq$$

$$\frac{(g(x)-g(x_0))^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}(n-1)!\left[(p(n-1)+1)(p(n-1)+2)\right]^{\frac{1}{p}}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(f\circ g^{-1}\right)^{(n)}(z)\right\|^q dz\right)^{\frac{2}{q}}, \quad (4)$$

for all $x_0 \leq x \leq b$.

We also mention a left generalized Hilbert-Pachpatte inequality for ordinary vector valued derivatives.

Theorem 3 *([3]) Let* $i = 1, 2$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $n_i \in \mathbb{N}$, $f_i \in C^{n_i}([a_i, b_i], A)$; *where* $[a_i, b_i] \subset \mathbb{R}$ *and* $(A, \|\cdot\|)$ *is a Banach algebra. Let* $g_i \in C^1([a_i, b_i])$, *strictly increasing, such that* $g_i^{-1} \in C^{n_i}([g_i(a_i), g_i(b_i)])$. *We assume that* $\left(f_i \circ g_i^{-1}\right)^{(j_i)}(g_i(x_{0i})) = 0$, $j_i = 0, 1, ..., n_i - 1$; *where* $x_{0i} \in [a_i, b_i]$ *be fixed. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)} \int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\| \left(f_1 \circ g_1^{-1}\right)(z_1) \left(f_2 \circ g_2^{-1}\right)(z_2) \right\| dz_1 dz_2}{\left(\frac{(z_1 - g_1(x_{01}))^{p(n_1-1)+1}}{p(p(n_1-1)+1)} + \frac{(z_2 - g_2(x_{02}))^{q(n_2-1)+1}}{q(q(n_2-1)+1)} \right)} \leq$$

$$\frac{(g_1(b_1) - g_1(x_{01}))(g_2(b_2) - g_2(x_{02}))}{(n_1-1)!(n_2-1)!} \tag{5}$$

$$\left\| \left\| \left(f_1 \circ g_1^{-1}\right)^{(n_1)} \right\| \right\|_{L_q([g_1(x_{01}), g_1(b_1)], A)} \left\| \left\| \left(f_2 \circ g_2^{-1}\right)^{(n_2)} \right\| \right\|_{L_p([g_2(x_{02}), g_2(b_2)], A)}.$$

We are motivated also by S. Dragomir [7] recent work:

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H.

An operator $A \in \mathcal{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the p-Schatten norm is finite

$$\|A\|_p := \left[tr\left(|A|^p\right)\right]^{\frac{1}{p}} < \infty.$$

Assume that $A : [a, b] \to \mathcal{B}_p(H)$, $B : [a, b] \to \mathcal{B}_q(H)$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, are continuous and B is strongly differentiable on (a, b), then

$$\left\| \int_a^b A(t) B(t) dt - \left(\int_a^b A(s) ds \right) B(u) \right\|_1 \leq$$

$$\sup_{t \in [a,b]} \|B'(t)\|_q \times \begin{cases} \left[\frac{1}{2}(b-a) + \left|u - \frac{a+b}{2}\right|\right] \int_a^b \|A(t)\|_p dt, \\ \\ \left[\frac{(u-a)^{\beta+1} + (b-u)^{\beta+1}}{\beta+1}\right]^{\frac{1}{\beta}} \left(\int_a^b \|A(t)\|_p^\alpha\right)^{\frac{1}{\alpha}}, \\ \text{for } \alpha, \beta > 1 \text{ with } \frac{1}{\alpha} + \frac{1}{\beta} = 1 \\ \\ \left[\frac{1}{4}(b-a)^2 + \left(u - \frac{a+b}{2}\right)^2\right] \sup_{t \in [a,b]} \|A(t)\|_p, \end{cases} \tag{6}$$

for all $u \in [a, b]$, an Ostrowski type inequality.

Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

We presented ([1], Ch. 8,9) mixed fractional Ostrowski inequalities for several functions for various norms.

In this article we generalize [1], Ch. 8,9 for several $\mathcal{B}_p(H)$ valued functions by using ordinary vector valued derivatives and our integrals here are of Bochner type [8].

Opial-type inequalities are used a lot in proving uniqueness of solutions to differential equations and also to give upper bounds to their solutions.

In this work we also derive Opial type inequalities for $\mathcal{B}_p(H)$ valued functions with respect to ordinary vector valued derivatives.

Additionally we include in this article related to $\mathcal{B}_p(H)$ Hilbert-Pachpatte type inequalities, [10]. We finish with selective applications to Ostrowski, Opial and Hilbert-Pachpatte inequalities.

2 Background

We use the following generalized vector Taylor's formula:

Theorem 4 *([2], p. 97) Let $n \in \mathbb{N}$ and $f \in C^n([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ is a Banach space. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. Let any $x, y \in [a,b]$. Then*

$$f(x) = f(y) + \sum_{i=1}^{n-1} \frac{(g(x) - g(y))^i}{i!} \left(f \circ g^{-1}\right)^{(i)}(g(y)) \tag{7}$$

$$+\frac{1}{(n-1)!} \int_{g(y)}^{g(x)} (g(x) - z)^{n-1} \left(f \circ g^{-1}\right)^{(n)}(z)\, dz.$$

The derivatives here are defined similarly to the numerical ones, see [12], pp. 83-86.

The above integral is of Bochner type [8], and so are the integrals in this work. By [2], p. 3, if $f \in C([a,b], X)$ then f is Bochner integrable.

3 About basic Banach Algebras

All here come from [11].

We need

Definition 5 *([11], p. 245) A complex algebra is a vector space A over the complex field $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x(yz) = (xy)z, \tag{8}$$

$$(x+y)z = xz + yz, \quad x(y+z) = xy + xz, \tag{9}$$

and

$$\alpha(xy) = (\alpha x) y = x(\alpha y), \tag{10}$$

for all x, y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A,\ y \in A) \tag{11}$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{12}$$

and

$$\|e\| = 1, \tag{13}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

We make

Remark 6 *Commutativity of A will be explicited stated when needed.*

There exists at most one $e \in A$ that satisfies (12).

Inequality (11) makes multiplication to be continuous, more precisely left and right continuous, see [11], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [11], p. 247-248, § 10.3.

We also make

Remark 7 *Next we mention about integration of A-valued functions, see [11], p. 259, § 10.22:*

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [11], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \ d\mu = \int_Q x f(p) \ d\mu(p) \tag{14}$$

and

$$\left(\int_Q f \ d\mu \right) x = \int_Q f(p) x \ d\mu(p). \tag{15}$$

The Bochner integrals we will involve in our article follow (14) and (15).

4 p-Schatten norms background

In this advanced section all come from [7].

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on H. If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is of trace class if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i \rangle < \infty. \tag{16}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthornormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the trace of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$tr(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle, \tag{17}$$

where $\{e_i\}_{i \in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (17) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 8 *We have:*

(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and

$$tr(A^*) = \overline{tr(A)}; \tag{18}$$

(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then AT, $TA \in \mathcal{B}_1(H)$ and

$$tr(AT) = tr(TA) \quad and \quad |tr(AT)| \leq \|A\|_1 \|T\|; \tag{19}$$

(iii) $tr(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|tr\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ then AB, $BA \in \mathcal{B}_1(H)$ and $tr(AB) = tr(BA)$;

(v) $\mathcal{B}_{fin}(H)$ (finite rank operators) is a dense subspace of $\mathcal{B}_1(H)$.

An operator $A \in \mathcal{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \leq p < \infty$ if the p-Schatten norm is finite [14, p. 60-64]

$$\|A\|_p := [tr(|A|^p)]^{\frac{1}{p}} < \infty,$$

$|A|^p$ is an operator notation and not a power.

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{20}$$

and

$$\|A\|_1 \geq \|A\|_p \geq \|A\|_q \geq \|A\|. \tag{21}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a norm on the $*$-ideal $\mathcal{B}_p(H)$, which is a Banach algebra, and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [14, p. 60-64], for $p \geq 1$,

$$\|A\|_p = \|A^*\|_p, \quad A \in \mathcal{B}_p(H) \tag{22}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \quad A, B \in \mathcal{B}_p(H) \tag{23}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \quad \|BA\|_p \leq \|B\| \|A\|_p, \quad A \in \mathcal{B}_p(H), \, B \in \mathcal{B}(H). \tag{24}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \quad A \in \mathcal{B}_p(H), \, B, C \in \mathcal{B}(H). \tag{25}$$

In terms of p-Schatten norm we have the Hölder inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1:$

$$(|tr(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \quad A \in \mathcal{B}_p(H), \, B \in \mathcal{B}_q(H). \tag{26}$$

For the theory of trace functionals and their applications the interested reader is referred to [13] and [14].

For some classical trace inequalities see [5], [6] and [9], which are continuations of the work of Bellman [4].

5 Main Results

We start with 1-2-Schatten norms mixed generalized Ostrowski type inequalities for several functions that are Banach algebra $\mathcal{B}_2(H) \subset \mathcal{B}(H)$ valued. A uniform estimate follows.

Theorem 9 *Let $n \in \mathbb{N}$ and $f_i \in C^n([a,b], \mathcal{B}_2(H))$, $i = 1, ..., r \in \mathbb{N} - \{1\}$; where $[a,b] \subset \mathbb{R}$ and $\mathcal{B}_2(H)$ is a $*$-ideal, which $(\mathcal{B}_2(H), \|\cdot\|_2)$ is a Banach algebra. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We assume that $\left(f_i \circ g^{-1}\right)^{(j)}(g(x_0)) = 0$, $j = 1, ..., n-1$; $i = 1, ..., r$; where $x_0 \in [a,b]$ be fixed. Denote by*

$$E(f_1, ..., f_r)(x_0) :=$$

$$\sum_{i=1}^{r} \left[\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) \, dx - \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) f_i(x_0) \right]. \tag{27}$$

Then

$$\|E(f_1, ..., f_r)(x_0)\|_1 \leq \frac{1}{n!}$$

$$\left\{ \sum_{i=1}^{r} \left[\left[\left\| \left\| \left(f_i \circ g^{-1} \right)^{(n)} \right\|_2 \right\|_{\infty, [g(a), g(x_0)]} (g(x_0) - g(a))^n \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \| f_j(x) \|_2 \right) dx \right) \right] + \right. \right. \tag{28}$$

$$\left. \left. \left[\left\| \left\| \left(f_i \circ g^{-1} \right)^{(n)} \right\|_2 \right\|_{\infty, [g(x_0), g(b)]} (g(b) - g(x_0))^n \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \| f_j(x) \|_2 \right) dx \right) \right] \right] \right\}.$$

Proof. Let $x_0 \in [a, b]$ such that $\left(f_i \circ g^{-1} \right)^{(j)} (g(x_0)) = 0$, $j = 1, ..., n-1$; $i = 1, ..., r$. Let $x \in [a, x_0]$, then by Theorem 4 we have

$$f_i(x) - f_i(x_0) = \frac{1}{(n-1)!} \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(f_i \circ g^{-1} \right)^{(n)} (z) \, dz \tag{29}$$

$$= \frac{(-1)^n}{(n-1)!} \int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(f_i \circ g^{-1} \right)^{(n)} (z) \, dz,$$

for $i = 1, ..., r$.

And for $x \in [x_0, b]$, then again by Theorem 4 we get

$$f_i(x) - f_i(x_0) = \frac{1}{(n-1)!} \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(f_i \circ g^{-1} \right)^{(n)} (z) \, dz, \tag{30}$$

for $i = 1, ..., r$.

We multiply (29) by $\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right)$ to get:

$$\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) - \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x_0) =$$

$$\frac{\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) (-1)^n}{(n-1)!} \int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(f_i \circ g^{-1} \right)^{(n)} (z) \, dz, \tag{31}$$

$\forall \, x \in [a, x_0]$; for $i = 1, ..., r$.

Similarly, we get (by (30))

$$\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) - \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x_0) =$$

$$\frac{\left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right)}{(n-1)!} \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(f_i \circ g^{-1} \right)^{(n)} (z) \, dz, \tag{32}$$

$\forall\ x \in [x_0, b]$; for $i = 1, ..., r$.

Adding (31) and (32) as separate groups, we obtain

$$\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{(-1)^n}{(n-1)!}\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\int_{g(x)}^{g(x_0)} (z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz, \tag{33}$$

$\forall\ x \in [a, x_0]$,

and

$$\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{(n-1)!}\sum_{i=1}^{r}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\int_{g(x_0)}^{g(x)} (g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz, \tag{34}$$

$\forall\ x \in [x_0, b]$.

Next, we integrate (33) and (34) with respect to $x \in [a, b]$. We have

$$\sum_{i=1}^{r}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \sum_{i=1}^{r}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) dx\right) f_i(x_0) = \tag{35}$$

$$\frac{(-1)^n}{(n-1)!}\sum_{i=1}^{r}\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)} (z-g(x))^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right) dx\right],$$

and

$$\sum_{i=1}^{r}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \sum_{i=1}^{r}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) dx\right) f_i(x_0) = \tag{36}$$

$$\frac{1}{(n-1)!}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)} (g(x)-z)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}(z)\,dz\right) dx\right].$$

Finally, adding (35) and (36) we obtain the useful identity

$$E\left(f_1,...,f_r\right)\left(x_0\right)=$$

$$\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)f_i\left(x\right)dx-\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)dx\right)f_i\left(x_0\right)\right]=\frac{1}{(n-1)!}$$

$$\sum_{i=1}^{r}\left[(-1)^n\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)\left(\int_{g(x)}^{g(x_0)}\left(z-g\left(x\right)\right)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}\left(z\right)dz\right)dx\right]+\right.$$

$$\left.\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-z\right)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}\left(z\right)dz\right)dx\right]\right],\qquad(37)$$

see also (2).

Therefore, we get that

$$\left\|E\left(f_1,...,f_r\right)\left(x_0\right)\right\|_1=$$

$$\left\|\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)f_i\left(x\right)dx-\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)dx\right)f_i\left(x_0\right)\right]\right\|_1\leq\frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left\|\left[\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)\left(\int_{g(x)}^{g(x_0)}\left(z-g\left(x\right)\right)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}\left(z\right)dz\right)dx\right]\right\|_1\right.\right.$$

$$\left.\left.+\left\|\left[\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-z\right)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}\left(z\right)dz\right)dx\right]\right\|_1\right]\right\}\leq$$

$$\frac{1}{(n-1)!}\left\{\sum_{i=1}^{r}\left[\left[\int_a^{x_0}\left\|\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)\left(\int_{g(x)}^{g(x_0)}\left(z-g\left(x\right)\right)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}\left(z\right)dz\right)\right\|_1dx\right]\right.\right.$$

$$\left.\left.+\left[\int_{x_0}^b\left\|\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j\left(x\right)\right)\left(\int_{g(x_0)}^{g(x)}\left(g\left(x\right)-z\right)^{n-1}\left(f_i\circ g^{-1}\right)^{(n)}\left(z\right)dz\right)\right\|_1dx\right]\right]\right\}\overset{(20)}{\leq}\qquad(38)$$

$$\frac{1}{(n-1)!}\left\{\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|_2 dz\right)dx\right]\right.\right.$$

$$\left.\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left\|\left(f_i\circ g^{-1}\right)^{(n)}(z)\right\|_2 dz\right)dx\right]\right]\right\}=:(\xi). \tag{39}$$

Hence it holds

$$\left\|E(f_1,...,f_r)(x_0)\right\|_1\leq(\xi). \tag{40}$$

We have that

$$(\xi)\leq\frac{1}{n!}\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|_2\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)(g(x_0)-g(x))^{n}dx\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|_2\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)(g(x)-g(x_0))^{n}dx\right]\right]\right\}\leq \tag{41}$$

$$\frac{1}{n!}\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|_2\right\|_{\infty,[g(a),g(x_0)]}(g(x_0)-g(a))^{n}\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)dx\right)\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|_2\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^{n}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)dx\right)\right]\right]\right\}, \tag{42}$$

proving (28). ■

Next comes an L_1 estimate.

Theorem 10 *All as in Theorem 9. Then*

$$\left\|E(f_1,...,f_r)(x_0)\right\|_1\leq\frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|_2\right\|_{L_1([g(a),g(x_0)])}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\left\|f_j(x)\right\|_2\right)(g(x_0)-g(x))^{n-1}dx\right]\right.\right.$$

$$+\left[\left\|\left\|\left(f_i \circ g^{-1}\right)^{(n)}\right\|_2\right\|_{L_1([g(x_0),g(b)])} \int_{x_0}^{b} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|f_j(x)\|_2\right) (g(x)-g(x_0))^{n-1}\, dx\right]\Bigg]\Bigg\}. \tag{43}$$

Proof. By (39), (40), we get that

$$\|E(f_1,...,f_r)(x_0)\|_1 \leq (\xi) \leq \frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i \circ g^{-1}\right)^{(n)}\right\|_2\right\|_{L_1([g(a),g(x_0)])} \int_{a}^{x_0} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|f_j(x)\|_2\right) (g(x_0)-g(x))^{n-1}\, dx\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i \circ g^{-1}\right)^{(n)}\right\|_2\right\|_{L_1([g(x_0),g(b)])} \int_{x_0}^{b} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|f_j(x)\|_2\right) (g(x)-g(x_0))^{n-1}\, dx\right]\right]\right\}, \tag{44}$$

proving (43). ■

An L_p estimate follows.

Theorem 11 *All as in Theorem 9, and let $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$. Then*

$$\|E(f_1,...,f_r)(x_0)\|_1 \leq \frac{1}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(f_i \circ g^{-1}\right)^{(n)}\right\|_2\right\|_{L_q([g(a),g(x_0)])} \left(\int_{a}^{x_0} (g(x_0)-g(x))^{n-\frac{1}{q}} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|f_j(x)\|_2\right) dx\right)\right.$$

$$\left.+\left\|\left\|\left(f_i \circ g^{-1}\right)^{(n)}\right\|_2\right\|_{L_q([g(x_0),g(b)])} \left(\int_{x_0}^{b} (g(x)-g(x_0))^{n-\frac{1}{q}} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|f_j(x)\|_2\right) dx\right)\right]. \tag{45}$$

Proof. By (39), (40), we get that

$$\|E(f_1,...,f_r)(x_0)\|_1 \leq (\xi) \leq \frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|f_j(x)\|_2\right) \left(\int_{g(x)}^{g(x_0)} (z-g(x))^{p(n-1)}\, dz\right)^{\frac{1}{p}}\right.\right.\right.$$

$$\left(\int_{g(x)}^{g(x_0)} \left\| \left(f_i \circ g^{-1}\right)^{(n)}(z) \right\|_2^q dz \right)^{\frac{1}{q}} dx \Bigg] +$$

$$\Bigg[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_2 \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{p(n-1)} dz \right)^{\frac{1}{p}} \tag{46}$$

$$\left(\int_{g(x_0)}^{g(x)} \left\| \left(f_i \circ g^{-1}\right)^{(n)}(z) \right\|_2^q dz \right)^{\frac{1}{q}} dx \Bigg] \Bigg] \Bigg\} = \frac{1}{(n-1)!}$$

$$\Bigg\{ \sum_{i=1}^{r} \Bigg[\Bigg[\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_2 \right) \frac{(g(x_0) - g(x))^{\frac{p(n-1)|+1}{p}}}{(p(n-1)+1)^{\frac{1}{p}}} \left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\|_2 \right\|_{L_q([g(a), g(x_0)])} dx \Bigg]$$

$$+ \Bigg[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_2 \right) \frac{(g(x) - g(x_0))^{\frac{p(n-1)|+1}{p}}}{(p(n-1)+1)^{\frac{1}{p}}} \left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\|_2 \right\|_{L_q([g(x_0), g(b)])} dx \Bigg] \Bigg] \Bigg\}$$

$$= \frac{1}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}$$

$$\Bigg\{ \sum_{i=1}^{r} \Bigg[\left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\|_2 \right\|_{L_q([g(a), g(x_0)])} \left(\int_a^{x_0} (g(x_0) - g(x))^{n - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_2 \right) dx \right)$$

$$+ \left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\|_2 \right\|_{L_q([g(x_0), g(b)])} \left(\int_{x_0}^{b} (g(x) - g(x_0))^{n - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_2 \right) dx \right) \Bigg] \Bigg\}, \tag{47}$$

proving (45). ■

We continue with mixed generalized Ostrowski type inequalities for several functions that are Banach algebra $\mathcal{B}_\gamma(H) \subset \mathcal{B}(H)$, $\gamma \geq 1$, valued. A uniform estimate follows.

Theorem 12 *Let $\gamma \geq 1$, $n \in \mathbb{N}$ and $f_i \in C^n([a,b], \mathcal{B}_\gamma(H))$, $i = 1, ..., r \in \mathbb{N} - \{1\}$; where $[a,b] \subset \mathbb{R}$ and $\mathcal{B}_\gamma(H)$ is a $*$-ideal, which $\left(\mathcal{B}_\gamma(H), \|\cdot\|_\gamma\right)$ is a Banach algebra. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We assume that $\left(f_i \circ g^{-1}\right)^{(j)}(g(x_0)) = 0$, $j = 1, ..., n-1$; $i = 1, ..., r$; where $x_0 \in [a,b]$ be fixed. Denote again by*

$$E(f_1, ..., f_r)(x_0) :=$$

$$\sum_{i=1}^{r}\left[\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)f_i(x)\,dx-\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)dx\right)f_i(x_0)\right]. \tag{48}$$

Then

$$\|E(f_1,...,f_r)(x_0)\|_\gamma \leq \frac{1}{n!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|_\gamma\right\|_{\infty,[g(a),g(x_0)]}(g(x_0)-g(a))^n\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_\gamma\right)dx\right)\right]+\right.$$

$$\left.\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|_\gamma\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^n\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_\gamma\right)dx\right)\right]\right]\right\}. \tag{49}$$

Proof. As similar to Theorem 9 is omitted, use of (23). ■

An L_1 estimate follows:

Theorem 13 *All as in Theorem 12. Then*

$$\|E(f_1,...,f_r)(x_0)\|_\gamma \leq \frac{1}{(n-1)!}$$

$$\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|_\gamma\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_\gamma\right)(g(x_0)-g(x))^{n-1}dx\right]\right.\right.$$

$$\left.\left.+\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|_\gamma\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_\gamma\right)(g(x)-g(x_0))^{n-1}dx\right]\right]\right\}. \tag{50}$$

Proof. As simlar to Theorem 10 is omitted. ■

An L_p estimate follows.

Theorem 14 *All as in Theorem 12, and let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\|E(f_1,...,f_r)(x_0)\|_\gamma \leq \frac{1}{(n-1)!\,(p(n-1)+1)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(f_i\circ g^{-1}\right)^{(n)}\right\|_\gamma\right\|_{L_q([g(a),g(x_0)])}\left(\int_a^{x_0}(g(x_0)-g(x))^{n-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|_\gamma\right)dx\right)\right.$$

$$+\left\| \left\| \left(f_i \circ g^{-1}\right)^{(n)} \right\|_{\gamma} \right\|_{L_q([g(x_0),g(b)])} \left(\int_{x_0}^{b} (g(x) - g(x_0))^{n-\frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\|_{\gamma} \right) dx \right) \Bigg] . \tag{51}$$

Proof. As similar to Theorem 11 is omitted. ■

When $r = 2$ we derive the following p-Schatten norm operator related Ostrowski type inequalities.

Theorem 15 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and let the $*$-ideals $\mathcal{B}_p(H)$, $\mathcal{B}_q(H)$, for which $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$, $\left(\mathcal{B}_q(H), \|\cdot\|_q\right)$ are Banach algebras; $n \in \mathbb{N}$, $x_0 \in [a,b] \subset \mathbb{R}$, $A_1 \in C^n([a,b], \mathcal{B}_p(H))$, $A_2 \in C^n([a,b], \mathcal{B}_q(H))$; $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$, with $\left(A_i \circ g^{-1}\right)^{(k)}(g(x_0)) = 0$, $k = 1, ..., n-1$; $i = 1, 2$. Then*

1)

$$\Phi(A_1, A_2)(x_0) := \int_a^b A_2(x) A_1(x) + \int_a^b A_1(x) A_2(x) dx -$$

$$\left(\int_a^b A_2(x) dx \right) A_1(x_0) - \left(\int_a^b A_1(x) dx \right) A_2(x_0) =$$

$$\frac{1}{(n-1)!} \Bigg\{ (-1)^n \left[\int_a^{x_0} A_2(x) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(A_1 \circ g^{-1}\right)^{(n)}(z) dz \right) dx \right] +$$

$$\left[\int_{x_0}^{b} A_2(x) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(A_1 \circ g^{-1}\right)^{(n)}(z) dz \right) dx \right] +$$

$$(-1)^n \left[\int_a^{x_0} A_1(x) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{n-1} \left(A_2 \circ g^{-1}\right)^{(n)}(z) dz \right) dx \right] + \tag{52}$$

$$\left[\int_{x_0}^{b} A_1(x) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(A_2 \circ g^{-1}\right)^{(n)}(z) dz \right) dx \right] \Bigg\} .$$

2) for $\gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1$, we obtain

$$\|\Phi(A_1, A_2)(x_0)\|_1 \leq \frac{1}{(n-1)! (\gamma(n-1)+1)^{\frac{1}{\gamma}}}$$

$$\Bigg\{ \left[\left\| \left\| \left(A_1 \circ g^{-1}\right)^{(n)} \right\|_p \right\|_{\delta,[g(a),g(x_0)]} \int_a^{x_0} \|A_2(x)\|_q (g(x_0) - g(x))^{n-\frac{1}{\delta}} dx \right] +$$

$$\left[\left\| \left\| \left(A_1 \circ g^{-1}\right)^{(n)} \right\|_p \right\|_{\delta,[g(x_0),g(b)]} \int_{x_0}^{b} \|A_2(x)\|_q (g(x) - g(x_0))^{n-\frac{1}{\delta}} dx \right] + \tag{53}$$

$$\left[\left\|\left\|\left(A_2 \circ g^{-1}\right)^{(n)}\right\|_q\right\|_{\delta,[g(a),g(x_0)]} \int_a^{x_0} \|A_1(x)\|_p \left(g(x_0) - g(x)\right)^{n-\frac{1}{\delta}} dx\right] +$$

$$\left.\left[\left\|\left\|\left(A_2 \circ g^{-1}\right)^{(n)}\right\|_q\right\|_{\delta,[g(x_0),g(b)]} \int_{x_0}^{b} \|A_1(x)\|_p \left(g(x) - g(x_0)\right)^{n-\frac{1}{\delta}} dx\right]\right\},$$

3)

$$\|\Phi(A_1, A_2)(x_0)\|_1 \leq \frac{1}{(n-1)!}$$

$$\left\{\left[\left\|\left\|\left(A_1 \circ g^{-1}\right)^{(n)}\right\|_p\right\|_{L_1([g(a),g(x_0)])} \int_a^{x_0} \|A_2(x)\|_q \left(g(x_0) - g(x)\right)^{n-1} dx\right] +\right.$$

$$\left[\left\|\left\|\left(A_1 \circ g^{-1}\right)^{(n)}\right\|_p\right\|_{L_1([g(x_0),g(b)])} \int_{x_0}^{b} \|A_2(x)\|_q \left(g(x) - g(x_0)\right)^{n-1} dx\right] + \quad (54)$$

$$\left[\left\|\left\|\left(A_2 \circ g^{-1}\right)^{(n)}\right\|_q\right\|_{L_1([g(a),g(x_0)])} \int_a^{x_0} \|A_1(x)\|_p \left(g(x_0) - g(x)\right)^{n-1} dx\right] +$$

$$\left.\left[\left\|\left\|\left(A_2 \circ g^{-1}\right)^{(n)}\right\|_q\right\|_{L_1([g(x_0),g(b)])} \int_{x_0}^{b} \|A_1(x)\|_p \left(g(x) - g(x_0)\right)^{n-1} dx\right]\right\},$$

and

4)

$$\|\Phi(A_1, A_2)(x_0)\|_1 \leq \frac{1}{n!}$$

$$\left\{\left[\left\|\left\|\left(A_1 \circ g^{-1}\right)^{(n)}\right\|_p\right\|_{\infty,[g(a),g(x_0)]} \int_a^{x_0} \|A_2(x)\|_q \left(g(x_0) - g(x)\right)^{n} dx\right] +\right.$$

$$\left[\left\|\left\|\left(A_1 \circ g^{-1}\right)^{(n)}\right\|_p\right\|_{\infty,[g(x_0),g(b)]} \int_{x_0}^{b} \|A_2(x)\|_q \left(g(x) - g(x_0)\right)^{n} dx\right] + \quad (55)$$

$$\left[\left\|\left\|\left(A_2 \circ g^{-1}\right)^{(n)}\right\|_q\right\|_{\infty,[g(a),g(x_0)]} \int_a^{x_0} \|A_1(x)\|_p \left(g(x_0) - g(x)\right)^{n} dx\right] +$$

$$\left.\left[\left\|\left\|\left(A_2 \circ g^{-1}\right)^{(n)}\right\|_q\right\|_{\infty,[g(x_0),g(b)]} \int_{x_0}^{b} \|A_1(x)\|_p \left(g(x) - g(x_0)\right)^{n} dx\right]\right\}.$$

Proof. Writing (37) for $r = 2$ and for A_1, A_2, we derive

$$\Phi(A_1, A_2)(x_0) = \int_a^b A_2(x) A_1(x) + \int_a^b A_1(x) A_2(x) dx -$$

$$\left(\int_a^b A_2(x) dx\right) A_1(x_0) - \left(\int_a^b A_1(x) dx\right) A_2(x_0) =$$

$$\frac{1}{(n-1)!}\left\{(-1)^n\left[\int_a^{x_0} A_2(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(A_1\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]+\right.$$
$$\left[\int_{x_0}^{b} A_2(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(A_1\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]+$$
$$(-1)^n\left[\int_a^{x_0} A_1(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(A_2\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]+ \tag{56}$$
$$\left.\left[\int_{x_0}^{b} A_1(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(A_2\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right\},$$

proving (52).

Hence it holds

$$\|\Phi(A_1,A_2)(x_0)\|_1=\left\|\int_a^b A_2(x)A_1(x)+\int_a^b A_1(x)A_2(x)\,dx-\right.$$
$$\left.\left(\int_a^b A_2(x)\,dx\right)A_1(x_0)-\left(\int_a^b A_1(x)\,dx\right)A_2(x_0)\right\|_1=$$
$$\frac{1}{(n-1)!}\left\|\left\{(-1)^n\left[\int_a^{x_0} A_2(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(A_1\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]+\right.\right.$$
$$\left[\int_{x_0}^{b} A_2(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(A_1\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]+$$
$$(-1)^n\left[\int_a^{x_0} A_1(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(A_2\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]+ \tag{57}$$
$$\left.\left.\left[\int_{x_0}^{b} A_1(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(A_2\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right\}\right\|_1\leq$$
$$\frac{1}{(n-1)!}\left\{\left\|\left[\int_a^{x_0} A_2(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(A_1\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right\|_1+\right.$$
$$\left\|\left[\int_{x_0}^{b} A_2(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(A_1\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right\|_1+$$
$$\left\|\left[\int_a^{x_0} A_1(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(A_2\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right\|_1+ \tag{58}$$
$$\left.\left\|\left[\int_{x_0}^{b} A_1(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(A_2\circ g^{-1}\right)^{(n)}(z)\,dz\right)dx\right]\right\|_1\right\}\leq$$

$$\frac{1}{(n-1)!}\left\{\left[\int_a^{x_0}\left\|A_2(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(A_1\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|_1 dx\right]+\right.$$
$$\left[\int_{x_0}^{b}\left\|A_2(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(A_1\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|_1 dx\right]+$$
$$\left[\int_a^{x_0}\left\|A_1(x)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(A_2\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|_1 dx\right]+ \tag{59}$$
$$\left.\left[\int_{x_0}^{b}\left\|A_1(x)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(A_2\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|_1 dx\right]\right\}\leq$$

(by using the p-Schatten norm and Hölder's type inequality (26) for $p,q>1$: $\frac{1}{p}+\frac{1}{q}=1$)

$$\frac{1}{(n-1)!}\left\{\left[\int_a^{x_0}\|A_2(x)\|_q\left\|\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(A_1\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|_p dx\right]+\right.$$
$$\left[\int_{x_0}^{b}\|A_2(x)\|_q\left\|\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(A_1\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|_p dx\right]+$$
$$\left[\int_a^{x_0}\|A_1(x)\|_p\left\|\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left(A_2\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|_q dx\right]+ \tag{60}$$
$$\left.\left[\int_{x_0}^{b}\|A_1(x)\|_p\left\|\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left(A_2\circ g^{-1}\right)^{(n)}(z)\,dz\right)\right\|_q dx\right]\right\}\leq$$
$$\frac{1}{(n-1)!}\left\{\left[\int_a^{x_0}\|A_2(x)\|_q\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left\|\left(A_1\circ g^{-1}\right)^{(n)}(z)\right\|_p dz\right)dx\right]+\right.$$
$$\left[\int_{x_0}^{b}\|A_2(x)\|_q\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left\|\left(A_1\circ g^{-1}\right)^{(n)}(z)\right\|_p dz\right)dx\right]+$$
$$\left[\int_a^{x_0}\|A_1(x)\|_p\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left\|\left(A_2\circ g^{-1}\right)^{(n)}(z)\right\|_q dz\right)dx\right]+ \tag{61}$$
$$\left.\left[\int_{x_0}^{b}\|A_1(x)\|_p\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left\|\left(A_2\circ g^{-1}\right)^{(n)}(z)\right\|_q dz\right)dx\right]\right\}.$$

So far we have proved that

$$\|\Phi(A_1,A_2)(x_0)\|_1\leq$$

$$\frac{1}{(n-1)!}\left\{\left[\int_a^{x_0}\|A_2(x)\|_q\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left\|\left(A_1\circ g^{-1}\right)^{(n)}(z)\right\|_p dz\right)dx\right]+\right.$$
$$\left[\int_{x_0}^{b}\|A_2(x)\|_q\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left\|\left(A_1\circ g^{-1}\right)^{(n)}(z)\right\|_p dz\right)dx\right]+$$
$$\left[\int_a^{x_0}\|A_1(x)\|_p\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{n-1}\left\|\left(A_2\circ g^{-1}\right)^{(n)}(z)\right\|_q dz\right)dx\right]+ \quad (62)$$
$$\left.\left[\int_{x_0}^{b}\|A_1(x)\|_p\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{n-1}\left\|\left(A_2\circ g^{-1}\right)^{(n)}(z)\right\|_q dz\right)dx\right]\right\}=:(\lambda).$$

Let now $\gamma,\delta>1$ such that $\frac{1}{\gamma}+\frac{1}{\delta}=1$, and we apply the usual Hölder's inequality in (62). Then, we have that

$$\|\Phi(A_1,A_2)(x_0)\|_1\le(\lambda)\le\frac{1}{(n-1)!\,(\gamma(n-1)+1)^{\frac{1}{\gamma}}}$$
$$\left\{\left[\int_a^{x_0}\|A_2(x)\|_q(g(x_0)-g(x))^{\frac{\gamma(n-1)+1}{\gamma}}\left(\int_{g(x)}^{g(x_0)}\left\|\left(A_1\circ g^{-1}\right)^{(n)}(z)\right\|_p^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]+\right.$$
$$\left[\int_{x_0}^{b}\|A_2(x)\|_q(g(x)-g(x_0))^{\frac{\gamma(n-1)+1}{\gamma}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(A_1\circ g^{-1}\right)^{(n)}(z)\right\|_p^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]+ \quad (63)$$
$$\left[\int_a^{x_0}\|A_1(x)\|_p(g(x_0)-g(x))^{\frac{\gamma(n-1)+1}{\gamma}}\left(\int_{g(x)}^{g(x_0)}\left\|\left(A_2\circ g^{-1}\right)^{(n)}(z)\right\|_q^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]+$$
$$\left.\left[\int_{x_0}^{b}\|A_1(x)\|_p(g(x)-g(x_0))^{\frac{\gamma(n-1)+1}{\gamma}}\left(\int_{g(x_0)}^{g(x)}\left\|\left(A_2\circ g^{-1}\right)^{(n)}(z)\right\|_q^{\delta}dz\right)^{\frac{1}{\delta}}dx\right]\right\}$$
$$\le\frac{1}{(n-1)!\,(\gamma(n-1)+1)^{\frac{1}{\gamma}}}$$
$$\left\{\left[\left\|\left\|\left(A_1\circ g^{-1}\right)^{(n)}\right\|_p\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\|A_2(x)\|_q(g(x_0)-g(x))^{n-\frac{1}{\delta}}dx\right]+\right.$$
$$\left[\left\|\left\|\left(A_1\circ g^{-1}\right)^{(n)}\right\|_p\right\|_{\delta,[g(x_0),g(b)]}\int_{x_0}^{b}\|A_2(x)\|_q(g(x)-g(x_0))^{n-\frac{1}{\delta}}dx\right]+ \quad (64)$$
$$\left[\left\|\left\|\left(A_2\circ g^{-1}\right)^{(n)}\right\|_q\right\|_{\delta,[g(a),g(x_0)]}\int_a^{x_0}\|A_1(x)\|_p(g(x_0)-g(x))^{n-\frac{1}{\delta}}dx\right]+$$

$$\left[\left\|\left\|\left(A_2 \circ g^{-1}\right)^{(n)}\right\|_q\right\|_{\delta,[g(x_0),g(b)]} \int_{x_0}^{b} \|A_1(x)\|_p \left(g(x)-g(x_0)\right)^{n-\frac{1}{\delta}} dx\right]\Bigg\},$$

proving (53).

We also obtain

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq (\lambda) \leq \frac{1}{(n-1)!}$$

$$\Bigg\{\left[\left\|\left\|\left(A_1 \circ g^{-1}\right)^{(n)}\right\|_p\right\|_{\infty,[g(a),g(x_0)]} \int_{a}^{x_0} \|A_2(x)\|_q \left(g(x_0)-g(x)\right)^{n-1} dx\right]+$$

$$\left[\left\|\left\|\left(A_1 \circ g^{-1}\right)^{(n)}\right\|_p\right\|_{L_1([g(x_0),g(b)])} \int_{x_0}^{b} \|A_2(x)\|_q \left(g(x)-g(x_0)\right)^{n-1} dx\right]+ \quad (65)$$

$$\left[\left\|\left\|\left(A_2 \circ g^{-1}\right)^{(n)}\right\|_q\right\|_{L_1([g(a),g(x_0)])} \int_{a}^{x_0} \|A_1(x)\|_p \left(g(x_0)-g(x)\right)^{n-1} dx\right]+$$

$$\left[\left\|\left\|\left(A_2 \circ g^{-1}\right)^{(n)}\right\|_q\right\|_{L_1([g(x_0),g(b)])} \int_{x_0}^{b} \|A_1(x)\|_p \left(g(x)-g(x_0)\right)^{n-1} dx\right]\Bigg\},$$

proving (54).

At last we derive

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq (\lambda) \leq \frac{1}{n!}$$

$$\Bigg\{\left[\left\|\left\|\left(A_1 \circ g^{-1}\right)^{(n)}\right\|_p\right\|_{\infty,[g(a),g(x_0)]} \int_{a}^{x_0} \|A_2(x)\|_q \left(g(x_0)-g(x)\right)^{n} dx\right]+$$

$$\left[\left\|\left\|\left(A_1 \circ g^{-1}\right)^{(n)}\right\|_p\right\|_{\infty,[g(x_0),g(b)]} \int_{x_0}^{b} \|A_2(x)\|_q \left(g(x)-g(x_0)\right)^{n} dx\right]+ \quad (66)$$

$$\left[\left\|\left\|\left(A_2 \circ g^{-1}\right)^{(n)}\right\|_q\right\|_{\infty,[g(a),g(x_0)]} \int_{a}^{x_0} \|A_1(x)\|_p \left(g(x_0)-g(x)\right)^{n} dx\right]+$$

$$\left[\left\|\left\|\left(A_2 \circ g^{-1}\right)^{(n)}\right\|_q\right\|_{\infty,[g(x_0),g(b)]} \int_{x_0}^{b} \|A_1(x)\|_p \left(g(x)-g(x_0)\right)^{n} dx\right]\Bigg\},$$

proving (55).

The theorem is proved. ■

Next we present a left generalized Opial type inequality involving 1-2-Schatten norms.

Theorem 16 *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $n \in \mathbb{N}$, $f \in C^n([a,b], \mathcal{B}_2(H))$; where $[a,b] \subset \mathbb{R}$ and $\mathcal{B}_2(H)$ is the $*$-ideal. Let $g \in C^1([a,b])$, strictly increasing, such that $g^{-1} \in C^n([g(a), g(b)])$. We assume that $\left(f \circ g^{-1}\right)^{(j)}(g(x_0)) = 0$, $j = 0, 1, ..., n-1$; where $x_0 \in [a,b]$ be fixed. Then*

$$\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1}\right)(z) \left(f \circ g^{-1}\right)^{(n)}(z) \right\|_1 dz \leq$$

$$\frac{(g(x) - g(x_0))^{n + \frac{1}{p} - \frac{1}{q}}}{2^{\frac{1}{q}} (n-1)! \left[(p(n-1)+1)(p(n-1)+2)\right]^{\frac{1}{p}}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|_2^q dz \right)^{\frac{2}{q}}, \tag{67}$$

for all $x_0 \leq x \leq b$.

Proof. Let $x_0 \in [a,b]$ such that $\left(f \circ g^{-1}\right)^{(j)}(g(x_0)) = 0$, $j = 0, 1, ..., n-1$. For $x \in [x_0, b]$ by Theorem 4 we have

$$\left(f \circ g^{-1}\right)(g(z)) = \frac{1}{(n-1)!} \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left(f \circ g^{-1}\right)^{(n)}(z)\, dz. \tag{68}$$

By Hölder's inequality we obtain

$$\left\| \left(f \circ g^{-1}\right)(g(x)) \right\|_2 \leq \frac{1}{(n-1)!} \int_{g(x_0)}^{g(x)} (g(x) - z)^{n-1} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|_2 dz \leq \tag{69}$$

$$\frac{1}{(n-1)!} \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{p(n-1)} dt \right)^{\frac{1}{p}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|_2^q dz \right)^{\frac{1}{q}} =$$

$$\frac{1}{(n-1)!} \frac{(g(x) - g(x_0))^{\frac{p(n-1)+1}{p}}}{(p(n-1)+1)^{\frac{1}{p}}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|_2^q dz \right)^{\frac{1}{q}}.$$

Call

$$\varphi(g(x)) := \int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|_2^q dz, \tag{70}$$

$\varphi(g(x_0)) = 0$.

Thus

$$\frac{d\varphi(g(x))}{dg(x)} = \left\| \left(f \circ g^{-1}\right)^{(n)}(g(x)) \right\|_2^q \geq 0, \tag{71}$$

and

$$\left(\frac{d\varphi(g(x))}{dg(x)} \right)^{\frac{1}{q}} = \left\| \left(f \circ g^{-1}\right)^{(n)}(g(x)) \right\|_2 \geq 0, \tag{72}$$

$\forall\ g(x) \in [g(x_0), g(b)]$.

Consequently, we get

$$\left\| \left(f \circ g^{-1} \right) (g(w)) \right\|_2 \left\| \left(f \circ g^{-1} \right)^{(n)} (g(w)) \right\|_2 \leq$$

$$\frac{(g(w) - g(x_0))^{\frac{p(n-1)+1}{p}}}{(n-1)! (p(n-1)+1)^{\frac{1}{p}}} \left(\varphi(g(w)) \frac{d\varphi(g(w))}{dg(w)} \right)^{\frac{1}{q}}, \tag{73}$$

$\forall\ g(w) \in [g(x_0), g(b)]$.

Then we observe that

$$\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1} \right) (g(w)) \left(f \circ g^{-1} \right)^{(n)} (g(w)) \right\|_1 dg(w) \overset{(26)}{\leq}$$

$$\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1} \right) (g(w)) \right\|_2 \left\| \left(f \circ g^{-1} \right)^{(n)} (g(w)) \right\|_2 dg(w) \leq$$

$$\frac{1}{(n-1)! (p(n-1)+1)^{\frac{1}{p}}}$$

$$\int_{g(x_0)}^{g(x)} (g(w) - g(x_0))^{\frac{p(n-1)+1}{p}} \left(\varphi(g(w)) \frac{d\varphi(g(w))}{dg(w)} \right)^{\frac{1}{q}} dg(w) \leq \tag{74}$$

$$\frac{1}{(n-1)! (p(n-1)+1)^{\frac{1}{p}}}$$

$$\left(\int_{g(x_0)}^{g(x)} (g(w) - g(x_0))^{p(n-1)+1} dg(w) \right)^{\frac{1}{p}} \left(\int_{g(x_0)}^{g(x)} \varphi(g(w)) \frac{d\varphi(g(w))}{dg(w)} dg(w) \right)^{\frac{1}{q}} =$$

$$\frac{1}{(n-1)! (p(n-1)+1)^{\frac{1}{p}} (p(n-1)+2)^{\frac{1}{p}}}$$

$$(g(x) - g(x_0))^{\frac{p(n-1)+2}{p}} \left(\int_{g(x_0)}^{g(x)} \varphi(g(w))\ d\varphi(g(w)) \right)^{\frac{1}{q}} = \tag{75}$$

$$\frac{(g(x) - g(x_0))^{n+\frac{1}{p}-\frac{1}{q}}}{(n-1)! (p(n-1)+1)^{\frac{1}{p}} (p(n-1)+2)^{\frac{1}{p}}} \left(\frac{\varphi^2(g(x))}{2} \right)^{\frac{1}{q}} =$$

$$\frac{(g(x) - g(x_0))^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}} (n-1)! ((p(n-1)+1)(p(n-1)+2))^{\frac{1}{p}}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1} \right)^{(n)} (z) \right\|_2^q dz \right)^{\frac{2}{q}}, \tag{76}$$

for all $g(x_0) \leq g(x) \leq g(b)$, proving (67). ■

The corresponding $\mathcal{B}_2(H)$ right generalized Opial type inequality follows:

Theorem 17 *All as in Theorem 16. Then*

$$\int_{g(x)}^{g(x_0)} \left\| \left(f \circ g^{-1}\right)(z) \left(f \circ g^{-1}\right)^{(n)}(z) \right\|_1 dz \leq$$

$$\frac{\left(g\left(x_0\right)-g\left(x\right)\right)^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}\left(n-1\right)!\left(\left(p\left(n-1\right)+1\right)\left(p\left(n-1\right)+2\right)\right)^{\frac{1}{p}}} \left(\int_{g(x)}^{g(x_0)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|_2^q dz \right)^{\frac{2}{q}}, \tag{77}$$

for all $a \leq x \leq x_0$.

Proof. As similar to Theorem 16 is omitted. ■

A $\mathcal{B}_\gamma(H)$, $\gamma \geq 1$, left Opial inequality follows:

Theorem 18 *Let* $\gamma \geq 1$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, *and* $n \in \mathbb{N}$, $f \in C^n([a,b], \mathcal{B}_\gamma(H))$; *where* $[a,b] \subset \mathbb{R}$ *and* $\mathcal{B}_\gamma(H)$ *is the* $*$*-ideal. Let* $g \in C^1([a,b])$, *strictly increasing, such that* $g^{-1} \in C^n([g(a), g(b)])$. *We assume that* $\left(f \circ g^{-1}\right)^{(j)}(g(x_0)) = 0$, $j = 0, 1, ..., n-1$; *where* $x_0 \in [a,b]$ *be fixed. Then*

$$\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1}\right)(z) \left(f \circ g^{-1}\right)^{(n)}(z) \right\|_\gamma dz \leq$$

$$\frac{\left(g\left(x\right)-g\left(x_0\right)\right)^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}\left(n-1\right)!\left[\left(p\left(n-1\right)+1\right)\left(p\left(n-1\right)+2\right)\right]^{\frac{1}{p}}} \left(\int_{g(x_0)}^{g(x)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|_\gamma^q dz \right)^{\frac{2}{q}}, \tag{78}$$

for all $x_0 \leq x \leq b$.

Proof. As similar to Theorem 16 is omitted. Use of (23). ■

A $\mathcal{B}_\gamma(H)$, $\gamma \geq 1$, right Opial inequality follows:

Theorem 19 *All as in Theorem 18. Then*

$$\int_{g(x)}^{g(x_0)} \left\| \left(f \circ g^{-1}\right)(z) \left(f \circ g^{-1}\right)^{(n)}(z) \right\|_\gamma dz \leq$$

$$\frac{\left(g\left(x_0\right)-g\left(x\right)\right)^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}\left(n-1\right)!\left(\left(p\left(n-1\right)+1\right)\left(p\left(n-1\right)+2\right)\right)^{\frac{1}{p}}} \left(\int_{g(x)}^{g(x_0)} \left\| \left(f \circ g^{-1}\right)^{(n)}(z) \right\|_\gamma^q dz \right)^{\frac{2}{q}}, \tag{79}$$

for all $a \leq x \leq x_0$.

Proof. As similar to Theorem 16 is omitted. ■

Next we present a $\mathcal{B}_2(H)$ left generalized Hilbert-Pachpatte inequality for ordinary derivatives.

Theorem 20 *Let* $i=1,2;\, p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *and* $n_i\in\mathbb{N}$, $f_i\in C^{n_i}([a_i,b_i],\mathcal{B}_2(H))$*; where* $[a_i,b_i]\subset\mathbb{R}$ *and* $\mathcal{B}_2(H)$ *is the* $*$*-ideal. Let* $g_i\in C^1([a_i,b_i])$*, strictly increasing, such that* $g_i^{-1}\in C^{n_i}([g_i(a_i),g_i(b_i)])$*. We assume that* $\left(f_i\circ g_i^{-1}\right)^{(j_i)}(g_i(x_{0i}))=0$*,* $j_i=0,1,\dots,n_i-1$*; where* $x_{0i}\in[a_i,b_i]$ *be fixed. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)}\int_{g_2(x_{02})}^{g_2(b_2)}\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\left(f_2\circ g_2^{-1}\right)(z_2)\right\|_1 dz_1dz_2}{\left(\frac{(z_1-g_1(x_{01}))^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right)}\leq$$

$$\frac{(g_1(b_1)-g_1(x_{01}))(g_2(b_2)-g_2(x_{02}))}{(n_1-1)!\,(n_2-1)!}\tag{80}$$

$$\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|_2\right\|_{L_q([g_1(x_{01}),g_1(b_1)],\mathcal{B}_2(H))}\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|_2\right\|_{L_p([g_2(x_{02}),g_2(b_2)],\mathcal{B}_2(H))}.$$

Proof. Let $i=1,2$; $x_0\in[a_i,b_i]$, such that $\left(f_i\circ g_i^{-1}\right)^{(j_i)}(g_i(x_{0i}))=0$, $j_i=0,1,\dots,n_i-1$.

For $x_i\in[x_{0i},b_i]$ by Theorem 4 we have

$$\left(f_i\circ g_i^{-1}\right)(g_i(x_i))=\frac{1}{(n_i-1)!}\int_{g_i(x_{0i})}^{g_i(x_i)}(g_i(x_i)-z_i)^{n_i-1}\left(f_i\circ g_i^{-1}\right)^{(n_i)}(z_i)\,dz_i.\tag{81}$$

As in (69) we have

$$\left\|\left(f_1\circ g_1^{-1}\right)(g_1(x_1))\right\|_2\leq\frac{1}{(n_1-1)!}\frac{(g_1(x_1)-g_1(x_{01}))^{\frac{p(n_1-1)+1}{p}}}{(p(n_1-1)+1)^{\frac{1}{p}}}$$

$$\left(\int_{g_1(x_{01})}^{g_1(x_1)}\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}(z)\right\|_2^q dz\right)^{\frac{1}{q}}\leq$$

$$\frac{1}{(n_1-1)!}\frac{(g_1(x_1)-g_1(x_{01}))^{\frac{p(n_1-1)+1}{p}}}{(p(n_1-1)+1)^{\frac{1}{p}}}\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|_2\right\|_{L_q([g_1(x_{01}),g_1(b_1)])},\tag{82}$$

for all $x_1\in[x_{01},b_1]$.

Similarly, we obtain that

$$\left\|\left(f_2\circ g_2^{-1}\right)(g_2(x_2))\right\|_2\leq\frac{1}{(n_2-1)!}\frac{(g_2(x_2)-g_2(x_{02}))^{\frac{q(n_2-1)+1}{q}}}{(q(n_2-1)+1)^{\frac{1}{q}}}$$

$$\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|_2\right\|_{L_p([g_2(x_{02}),g_2(b_2)])},\tag{83}$$

for all $x_2\in[x_{02},b_2]$.

By (82) and (83) we get

$$\left\|\left(f_1\circ g_1^{-1}\right)(g_1(x_1))\left(f_2\circ g_2^{-1}\right)(g_2(x_2))\right\|_1\overset{(26)}{\leq}$$

$$\left\|\left(f_1\circ g_1^{-1}\right)(g_1(x_1))\right\|_2\left\|\left(f_2\circ g_2^{-1}\right)(g_2(x_2))\right\|_2\leq\frac{1}{(n_1-1)!\,(n_2-1)!}$$

$$\frac{(g_1(x_1)-g_1(x_{01}))^{\frac{p(n_1-1)+1}{p}}}{(p(n_1-1)+1)^{\frac{1}{p}}}\frac{(g_2(x_2)-g_2(x_{02}))^{\frac{q(n_2-1)+1}{q}}}{(q(n_2-1)+1)^{\frac{1}{q}}} \tag{84}$$

$$\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|_2\right\|_{L_q([g_1(x_{01}),g_1(b_1)])}\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|_2\right\|_{L_p([g_2(x_{02}),g_2(b_2)])}\leq$$

(using Young's inequality for $a,b\geq 0$, $a^{\frac{1}{p}}b^{\frac{1}{q}}\leq\frac{a}{p}+\frac{b}{q}$)

$$\frac{1}{(n_1-1)!\,(n_2-1)!}\left(\frac{(g_1(x_1)-g_1(x_{01}))^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(g_2(x_2)-g_2(x_{02}))^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right) \tag{85}$$

$$\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|_2\right\|_{L_q([g_1(x_{01}),g_1(b_1)])}\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|_2\right\|_{L_p([g_2(x_{02}),g_2(b_2)])},$$

$\forall\ (x_1,x_2)\in[x_{01},b_1]\times[x_{02},b_2]$.

So far we have

$$\frac{\left\|\left(f_1\circ g_1^{-1}\right)(g_1(x_1))\left(f_2\circ g_2^{-1}\right)(g_2(x_2))\right\|_1}{\left(\frac{(g_1(x_1)-g_1(x_{01}))^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(g_2(x_2)-g_2(x_{02}))^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right)}\leq \tag{86}$$

$$\frac{1}{(n_1-1)!\,(n_2-1)!}\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|_2\right\|_{L_q([g_1(x_{01}),g_1(b_1)],\mathcal{B}_2(H))}$$

$$\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|_2\right\|_{L_p([g_2(x_{02}),g_2(b_2)],\mathcal{B}_2(H))},$$

$\forall\ (x_1,x_2)\in[x_{01},b_1]\times[x_{02},b_2]$.

The denominator in (86) can be zero, only when both $g_1(x_1)=g_1(x_{01})$ and $g_2(x_2)=g_2(x_{02})$.

Therefore we obtain (80), by integrating (86) over $[g_1(x_{01}),g_1(b_1)]\times[g_2(x_{02}),g_2(b_2)]$.

■

It follows the $\mathcal{B}_2(H)$ right generalized Hilbert-Pachpate inequality for ordinary derivatives.

Theorem 21 *All as in Theorem 20. Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})}\int_{g_2(a_2)}^{g_2(x_{02})}\frac{\left\|\left(f_1\circ g_1^{-1}\right)(z_1)\left(f_2\circ g_2^{-1}\right)(z_2)\right\|_1 dz_1dz_2}{\left(\frac{(g_1(x_{01})-z_1)^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(g_2(x_{02})-z_2)^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right)}\leq$$

$$\frac{(g_1(x_{01})-g_1(a_1))(g_2(x_{02})-g_2(a_2))}{(n_1-1)!\,(n_2-1)!} \tag{87}$$

$$\left\|\left\|\left(f_1\circ g_1^{-1}\right)^{(n_1)}\right\|_2\right\|_{L_q([g_1(a_1),g_1(x_{01})],\mathcal{B}_2(H))}\left\|\left\|\left(f_2\circ g_2^{-1}\right)^{(n_2)}\right\|_2\right\|_{L_p([g_2(a_2),g_2(x_{02})],\mathcal{B}_2(H))}.$$

Proof. As similar to Theorem 20 is omitted. ■

Next we present a $\mathcal{B}_{\gamma}(H)$, $\gamma \geq 1$, left generalized Hilbert-Pachpatte inequality for ordinary derivatives.

Theorem 22 *Let $\gamma \geq 1$, $i = 1,2$; $p,q > 1 : \frac{1}{p}+\frac{1}{q} = 1$, and $n_i \in \mathbb{N}$, $f_i \in C^{n_i}([a_i,b_i], \mathcal{B}_{\gamma}(H))$; where $[a_i,b_i] \subset \mathbb{R}$ and $\mathcal{B}_{\gamma}(H)$ is the $*$-ideal. Let $g_i \in C^1([a_i,b_i])$, strictly increasing, such that $g_i^{-1} \in C^{n_i}([g_i(a_i), g_i(b_i)])$. We assume that $\left(f_i \circ g_i^{-1}\right)^{(j_i)}(g_i(x_{0i})) = 0$, $j_i = 0,1,...,n_i-1$; where $x_{0i} \in [a_i,b_i]$ be fixed. Then*

$$\int_{g_1(x_{01})}^{g_1(b_1)}\int_{g_2(x_{02})}^{g_2(b_2)} \frac{\left\|\left(f_1 \circ g_1^{-1}\right)(z_1)\left(f_2 \circ g_2^{-1}\right)(z_2)\right\|_{\gamma} dz_1 dz_2}{\left(\frac{(z_1-g_1(x_{01}))^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(z_2-g_2(x_{02}))^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right)} \leq$$

$$\frac{(g_1(b_1)-g_1(x_{01}))(g_2(b_2)-g_2(x_{02}))}{(n_1-1)!(n_2-1)!} \tag{88}$$

$$\left\|\left\|\left(f_1 \circ g_1^{-1}\right)^{(n_1)}\right\|_{\gamma}\right\|_{L_q([g_1(x_{01}),g_1(b_1)],\mathcal{B}_{\gamma}(H))}\left\|\left\|\left(f_2 \circ g_2^{-1}\right)^{(n_2)}\right\|_{\gamma}\right\|_{L_p([g_2(x_{02}),g_2(b_2)],\mathcal{B}_{\gamma}(H))}.$$

Proof. As similar to Theorem 20 is omitted. Use of (23). ■

It follows the $\mathcal{B}_{\gamma}(H)$, $\gamma \geq 1$, right generalized Hilbert-Pachpate inequality for ordinary derivatives.

Theorem 23 *All as in Theorem 22. Then*

$$\int_{g_1(a_1)}^{g_1(x_{01})}\int_{g_2(a_2)}^{g_2(x_{02})} \frac{\left\|\left(f_1 \circ g_1^{-1}\right)(z_1)\left(f_2 \circ g_2^{-1}\right)(z_2)\right\|_{\gamma} dz_1 dz_2}{\left(\frac{(g_1(x_{01})-z_1)^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(g_2(x_{02})-z_2)^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right)} \leq$$

$$\frac{(g_1(x_{01})-g_1(a_1))(g_2(x_{02})-g_2(a_2))}{(n_1-1)!(n_2-1)!} \tag{89}$$

$$\left\|\left\|\left(f_1 \circ g_1^{-1}\right)^{(n_1)}\right\|_{\gamma}\right\|_{L_q([g_1(a_1),g_1(x_{01})],\mathcal{B}_{\gamma}(H))}\left\|\left\|\left(f_2 \circ g_2^{-1}\right)^{(n_2)}\right\|_{\gamma}\right\|_{L_p([g_2(a_2),g_2(x_{02})],\mathcal{B}_{\gamma}(H))}.$$

Proof. As similar to Theorem 20 is omitted. ■

6 Applications

We start with $\mathcal{B}_2(H)$ Ostrowski type inequalities.

Corollary 24 *(to Theorem 15) All as in Theorem 15, with $g(t) = t$. Then*

1) for $\gamma, \delta > 1 : \frac{1}{\gamma}+\frac{1}{\delta} = 1$, we have

$$\|\Phi(A_1,A_2)(x_0)\|_1 \leq \frac{1}{(n-1)!\,(\gamma(n-1)+1)^{\frac{1}{\gamma}}}$$

$$\left\{ \left[\left\| \left\| A_1^{(n)} \right\|_p \right\|_{\delta, [a, x_0]} \int_a^{x_0} \|A_2(x)\|_q (x_0 - x)^{n - \frac{1}{\delta}} dx \right] + \right. \tag{90}$$

$$\left[\left\| \left\| A_1^{(n)} \right\|_p \right\|_{\delta, [x_0, b]} \int_{x_0}^{b} \|A_2(x)\|_q (x - x_0)^{n - \frac{1}{\delta}} dx \right] +$$

$$\left[\left\| \left\| A_2^{(n)} \right\|_q \right\|_{\delta, [a, x_0]} \int_a^{x_0} \|A_1(x)\|_p (x_0 - x)^{n - \frac{1}{\delta}} dx \right] +$$

$$\left. \left[\left\| \left\| A_2^{(n)} \right\|_q \right\|_{\delta, [x_0, b]} \int_{x_0}^{b} \|A_1(x)\|_p (x - x_0)^{n - \frac{1}{\delta}} dx \right] \right\},$$

2)

$$\|\Phi(A_1, A_2)(x_0)\|_1 \leq \frac{1}{(n-1)!}$$

$$\left\{ \left[\left\| \left\| A_1^{(n)} \right\|_p \right\|_{L_1([a, x_0])} \int_a^{x_0} \|A_2(x)\|_q (x_0 - x)^{n-1} dx \right] + \right.$$

$$\left[\left\| \left\| A_1^{(n)} \right\|_p \right\|_{L_1([x_0, b])} \int_{x_0}^{b} \|A_2(x)\|_q (x - x_0)^{n-1} dx \right] + \tag{91}$$

$$\left[\left\| \left\| A_2^{(n)} \right\|_q \right\|_{L_1([a, x_0])} \int_a^{x_0} \|A_1(x)\|_p (x_0 - x)^{n-1} dx \right] +$$

$$\left. \left[\left\| \left\| A_2^{(n)}) \right\|_q \right\|_{L_1([x_0, b])} \int_{x_0}^{b} \|A_1(x)\|_p (x - x_0)^{n-1} dx \right] \right\},$$

and

3)

$$\|\Phi(A_1, A_2)(x_0)\|_1 \leq \frac{1}{n!}$$

$$\left\{ \left[\left\| \left\| A_1^{(n)} \right\|_p \right\|_{\infty, [a, x_0]} \int_a^{x_0} \|A_2(x)\|_q (x_0 - x)^{n} dx \right] + \right.$$

$$\left[\left\| \left\| A_1^{(n)} \right\|_p \right\|_{\infty, [x_0, b]} \int_{x_0}^{b} \|A_2(x)\|_q (x - x_0)^{n} dx \right] + \tag{92}$$

$$\left[\left\| \left\| A_2^{(n)} \right\|_q \right\|_{\infty, [a, x_0]} \int_a^{x_0} \|A_1(x)\|_p (x_0 - x)^{n} dx \right] +$$

$$\left. \left[\left\| \left\| A_2^{(n)}) \right\|_q \right\|_{\infty, [x_0, b]} \int_{x_0}^{b} \|A_1(x)\|_p (x - x_0)^{n} dx \right] \right\}.$$

We continue with $\mathcal{B}_\gamma(H)$, $\gamma \geq 1$, Ostrowski type inequalities.

Corollary 25 *(to Theorems 12 - 14) All as in Theorem 12, with $g(t)=e^{t}$. Then*

1)

$$\left\|E\left(f_1,...,f_r\right)\left(x_0\right)\right\|_\gamma \leq \frac{1}{n!}\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ\log\right)^{(n)}\right\|_\gamma\right\|_{\infty,[e^a,e^{x_0}]}\right.\right.\right.$$

$$\left.\left(e^{x_0}-e^a\right)^n\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_\gamma\right)dx\right)\right]+$$

$$\left.\left.\left.\left[\left\|\left\|\left(f_i\circ\log\right)^{(n)}\right\|_\gamma\right\|_{\infty,[e^{x_0},e^b]}\left(e^b-e^{x_0}\right)^n\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_\gamma\right)dx\right)\right]\right]\right\}, \tag{93}$$

2)

$$\left\|E\left(f_1,...,f_r\right)\left(x_0\right)\right\|_\gamma \leq \frac{1}{(n-1)!}\left\{\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(f_i\circ\log\right)^{(n)}\right\|_\gamma\right\|_{L_1([e^a,e^{x_0}])}\right.\right.\right.$$

$$\left.\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_\gamma\right)\left(e^{x_0}-e^x\right)^{n-1}dx\right]+$$

$$\left.\left.\left.\left[\left\|\left\|\left(f_i\circ\log\right)^{(n)}\right\|_\gamma\right\|_{L_1([e^{x_0},e^b])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_\gamma\right)\left(e^x-e^{x_0}\right)^{n-1}dx\right]\right]\right\}, \tag{94}$$

and

3) if $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, we have

$$\left\|E\left(f_1,...,f_r\right)\left(x_0\right)\right\|_\gamma \leq \frac{1}{(n-1)!\left(p\left(n-1\right)+1\right)^{\frac{1}{p}}}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(f_i\circ\log\right)^{(n)}\right\|_\gamma\right\|_{L_q([e^a,e^{x_0}])}\left(\int_a^{x_0}\left(e^{x_0}-e^x\right)^{n-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_\gamma\right)dx\right)+\right.$$

$$\left.\left\|\left\|\left(f_i\circ\log\right)^{(n)}\right\|_\gamma\right\|_{L_q([e^{x_0},e^b])}\left(\int_{x_0}^{b}\left(e^x-e^{x_0}\right)^{n-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\left\|f_j\left(x\right)\right\|_\gamma\right)dx\right)\right]. \tag{95}$$

We continue with a $\mathcal{B}_2(H)$ Opial type inequality.

Corollary 26 *(to Theorem 16) All as in Theorem 16 with $g(t)=t$. Then*

$$\int_{x_0}^{x}\left\|f(z)\,f^{(n)}(z)\right\|_1 dz \leq$$

$$\frac{(x-x_0)^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}(n-1)!\left[(p(n-1)+1)(p(n-1)+2)\right]^{\frac{1}{p}}}\left(\int_{x_0}^{z}\left\|f^{(n)}(z)\right\|_2^q dz\right)^{\frac{2}{q}}, \tag{96}$$

for all $x_0 \leq x \leq b$.

Next comes a $\mathcal{B}_\gamma(H)$, $\gamma \geq 1$, Opial type inequality.

Corollary 27 *(to Theorem 18) All as in Theorem 18 with $g(t)=e^t$. Then*

$$\int_{e^{x_0}}^{e^x}\left\|((f\circ\log)(z))\,(f\circ\log)^{(n)}(z)\right\|_\gamma dz \leq$$

$$\frac{(e^x-e^{x_0})^{n+\frac{1}{p}-\frac{1}{q}}}{2^{\frac{1}{q}}(n-1)!\left[(p(n-1)+1)(p(n-1)+2)\right]^{\frac{1}{p}}}\left(\int_{e^{x_0}}^{e^x}\left\|(f\circ\log)^{(n)}(z)\right\|_\gamma^q dz\right)^{\frac{2}{q}}, \tag{97}$$

for all $x_0 \leq x \leq b$.

A $\mathcal{B}_2(H)$ Hilbert-Pachpatte type inequality follows.

Corollary 28 *(to Theorem 20) All as in Theorem 20 for $g_1(t)=g_2(t)=t$. Then*

$$\int_{x_{01}}^{b_1}\int_{x_{02}}^{b_2}\frac{\left\|f_1(z_1)\,f_2(z_2)\right\|_1 dz_1dz_2}{\left(\frac{(z_1-x_{01})^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(z_2-x_{02})^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right)} \leq$$

$$\frac{(b_1-x_{01})(b_2-x_{02})}{(n_1-1)!\,(n_2-1)!}\left\|\left\|f_1^{(n_1)}\right\|_2\right\|_{L_q([x_{01},b_1],\mathcal{B}_2(H))}\left\|\left\|f_2^{(n_2)}\right\|_2\right\|_{L_p([x_{02},b_2],\mathcal{B}_2(H))}. \tag{98}$$

We finish with a $\mathcal{B}_\gamma(H)$, $\gamma \geq 1$, Hilbert-Pachpatte type inequality.

Corollary 29 *(to Theorem 22) All as in Theorem 22 for $g_1(t)=g_2(t)=\log t$, and $[a_i,b_i]\subset\mathbb{R}_+-\{0\}$, $i=1,2$. Then*

$$\int_{\log x_{01}}^{\log b_1}\int_{\log x_{02}}^{\log b_2}\frac{\left\|(f_1\circ e^t)(z_1)\,(f_2\circ e^t)(z_2)\right\|_\gamma dz_1dz_2}{\left(\frac{(z_1-\log x_{01})^{p(n_1-1)+1}}{p(p(n_1-1)+1)}+\frac{(z_2-\log x_{02})^{q(n_2-1)+1}}{q(q(n_2-1)+1)}\right)} \leq$$

$$\frac{\left(\log\frac{b_1}{x_{01}}\right)\left(\log\frac{b_2}{x_{02}}\right)}{(n_1-1)!\,(n_2-1)!} \tag{99}$$

$$\left\|\left\|\left(f_1\circ e^t\right)^{(n_1)}\right\|_\gamma\right\|_{L_q([\log x_{01},\log b_1],\mathcal{B}_\gamma(H))}\left\|\left\|\left(f_2\circ e^t\right)^{(n_2)}\right\|_\gamma\right\|_{L_p([\log x_{02},\log b_2],\mathcal{B}_\gamma(H))}.$$

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