# $\gamma$-Schatten norm Multivariate Ostrowski type inequalities for several Neumann-Schatten class $\mathcal{B}_{\gamma}(H)$ valued functions

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**Abstract**

Here we are dealing with several smooth functions from a compact convex set of $\mathbb{R}^k$, $k \geq 2$ to a Neumann-Schatten class $\mathcal{B}_{\gamma}(H)$, $\gamma \geq 1$, which is a Banach algebra. For these we prove general multivariate Ostrowski type inequalities with estimates in norms $\|\cdot\|_p$, for all $1 \leq p \leq \infty$. We provide also interesting applications.

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## 1 Introduction

Our main motivation comes from [1].

We mention a general Ostrowski type inequality result regarding several Banach algebra valued functions.

**Theorem 1** *([1]) Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $(A, \|\cdot\|)$ a Banach algebra and $f_i \in C^{n+1}(Q, A)$, $i = 1, ..., r$; $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, and fixed $\overrightarrow{x_0} \in Q \subset \mathbb{R}^k$, $k \geq 2$, where $Q$ is a compact and convex subset. Here all vector partial derivatives $f_{i\alpha} := \frac{\partial^{\alpha} f_i}{\partial z^{\alpha}}$, where $\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_{\lambda} \in \mathbb{Z}^+$, $\lambda = 1, ..., k$, $|\alpha| = \sum_{\lambda=1}^{k} \alpha_{\lambda} = j$, $j = 1, ..., n$, fulfill $f_{i\alpha}(\overrightarrow{x_0}) = 0$, $i = 1, ..., r$.*

*Denote*

$$D_{n+1}(f_i) := \max_{\alpha:|\alpha|=n+1} \||| f_{i\alpha} \|||_{\infty,Q}, \tag{1}$$

$i = 1, ..., r,$ *and*

$$\|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1} := \sum_{\lambda=1}^{k} |z_\lambda - x_{0\lambda}|. \tag{2}$$

*Then*

$$\left\| \sum_{i=1}^{r} \int_Q \left( \prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} f_\rho(\overrightarrow{z}) \right) f_i(\overrightarrow{z})\, d\overrightarrow{z} - \sum_{i=1}^{r} \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} f_\rho(\overrightarrow{z}) \right) d\overrightarrow{z} \right) f_i(\overrightarrow{x_0}) \right\| \leq \tag{3}$$

$$\frac{\max_{i\in\{1,...,r\}} D_{n+1}(f_i)}{(n+1)!} \sum_{i=1}^{r} \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} \|f_\rho(\overrightarrow{z})\| \right) \|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1}^{n+1} d\overrightarrow{z} \right) \leq$$

$$\frac{\max_{i\in\{1,...,r\}} D_{n+1}(f_i)}{(n+1)!} \min \left\{ \left( \int_Q \|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1}^{n+1} d\overrightarrow{z} \right) \left[ \sum_{i=1}^{r} \left( \prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} \||| f_\rho \|||_{\infty,Q} \right) \right], \right.$$

$$\left\| \|\cdot - \overrightarrow{x_0}\|_{l_1}^{n+1} \right\|_{\infty,Q} \left[ \sum_{i=1}^{r} \left\| \left( \prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} \|f_\rho\| \right) \right\|_{L_1(Q,A)} \right],$$

$$\left. \left\| \|\cdot - \overrightarrow{x_0}\|_{l_1}^{n+1} \right\|_{L_p(Q,A)} \left[ \sum_{i=1}^{r} \left[ \left\| \left( \prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} \|f_\rho\| \right) \right\|_{L_q(Q,A)} \right] \right] \right\}. \tag{4}$$

In particular we obtained the following Ostrowski type inequality.

**Theorem 2** *([1]) Let* $(A, \|\cdot\|)$ *a Banach algebra and* $f_i \in C^{n+1}\left( \prod_{\lambda=1}^{k} [a_\lambda, b_\lambda], A \right)$, $i = 1, ..., r$; $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, *and fixed* $\overrightarrow{x_0} \in \prod_{\lambda=1}^{k} [a_\lambda, b_\lambda] \subset \mathbb{R}^k$, $k \geq 2$. *Here all vector partial derivatives* $f_{i\alpha} := \frac{\partial^\alpha f_i}{\partial z^\alpha}$, *where* $\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_\lambda \in \mathbb{Z}^+$, $\lambda = 1, ..., k$, $|\alpha| = \sum_{\lambda=1}^{k} \alpha_\lambda = j$, $j = 1, ..., n$, *fulfill* $f_{i\alpha}(\overrightarrow{x_0}) = 0$, $i = 1, ..., r$.

*Denote*

$$D_{n+1}(f_i) := \max_{\alpha:|\alpha|=n+1} \||| f_{i\alpha} \|||_{\infty, \prod_{\lambda=1}^{k} [a_\lambda, b_\lambda]}, \tag{5}$$

$i = 1, ..., r.$

*Then*

$$\left\| \sum_{i=1}^{r} \int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} \left( \prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right) \right) f_i\left(\overrightarrow{z}\right) d\overrightarrow{z} - \right.$$

$$\left. \sum_{i=1}^{r} \left( \int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} \left( \prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right) \right) d\overrightarrow{z} \right) f_i\left(\overrightarrow{x_0}\right) \right\| \leq \tag{6}$$

$$\left( \max_{i\in\{1,\ldots,r\}} D_{n+1}\left(f_i\right) \right) \left[ \sum_{i=1}^{r} \left( \prod_{\substack{\rho=1 \\ \rho\neq i}}^{r} \left\| \left\| f_\rho \right\| \right\|_{\infty, \prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} \right) \right]$$

$$\left[ \sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1} \frac{1}{\prod_{\lambda=1}^{k}\rho_\lambda! \prod_{\lambda=1}^{k}\left(\rho_\lambda+1\right)} \prod_{\lambda=1}^{k} \left( \left(b_\lambda - x_{0\lambda}\right)^{\rho_\lambda+1} + \left(x_{0\lambda} - a_\lambda\right)^{\rho_\lambda+1} \right) \right].$$

Ostrowski type inequalities have a lot of important applications in Numerical Analysis and Probability.

We are also inspired by [5].

In this article we establish multivariate Ostrowski type inequalities for several smooth functions from a compact convex subset of $\mathbb{R}^k$, $k \geq 2$, to von Neumann-Schatten class $\mathcal{B}_\gamma(H)$, $\gamma \geq 1$, which is a Banach algebra. These involve the norms $\|\cdot\|_p$, $1 \leq p \leq \infty$.

## 2 About Banach Algebras

All here come from [9].

We need

**Definition 3** *([9], p. 245) A complex algebra is a vector space $A$ over the complex field $\mathbb{C}$ in which a multiplication is defined that satisfies*

$$x\left(yz\right) = \left(xy\right)z, \tag{7}$$

$$\left(x+y\right)z = xz + yz, \quad x\left(y+z\right) = xy + xz, \tag{8}$$

*and*

$$\alpha\left(xy\right) = \left(\alpha x\right)y = x\left(\alpha y\right), \tag{9}$$

*for all $x, y$ and $z$ in $A$ and for all scalars $\alpha$.*

*Additionally if $A$ is a Banach space with respect to a norm that satisfies the multiplicative inequality*

$$\|xy\| \leq \|x\| \, \|y\| \quad (x \in A, \, y \in A) \tag{10}$$

*and if $A$ contains a unit element $e$ such that*

$$xe = ex = x \quad (x \in A) \tag{11}$$

*and*

$$\|e\| = 1, \tag{12}$$

*then $A$ is called a Banach algebra.*

*$A$ is commutative iff $xy = yx$ for all $x, y \in A$.*

We make

**Remark 4** *Commutativity of $A$ will be explicited stated when needed.*

*There exists at most one $e \in A$ that satisfies (11).*

*Inequality (10) makes multiplication to be continuous, more precisely left and right continuous, see [9], p. 246.*

*Multiplication in $A$ is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.*

*For nice examples about Banach algebras see [9], p. 247-248, § 10.3.*

We also make

**Remark 5** *Next we mention about integration of $A$-valued functions, see [9], p. 259, § 10.22:*

*If $A$ is a Banach algebra and $f$ is a continuous $A$-valued function on some compact Hausdorff space $Q$ on which a complex Borel measure $\mu$ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [9], simply because $A$ is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then*

$$x \int_Q f \ d\mu = \int_Q x f(p) \ d\mu(p) \tag{13}$$

*and*

$$\left( \int_Q f \ d\mu \right) x = \int_Q f(p) \, x \ d\mu(p) \, . \tag{14}$$

*The vector integrals we will involve in our article follow (13) and (14).*

## 3 $p$-Schatten norms background

In this advanced section all come from [5].

Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on $H$. If $\{e_i\}_{i\in I}$ an orthonormal basis of $H$, we say that $A \in \mathcal{B}(H)$ is of trace class if

$$\|A\|_1 := \sum_{i\in I} \langle |A| e_i, e_i\rangle < \infty. \tag{15}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthornormal basis $\{e_i\}_{i\in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

We define the trace of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$tr(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{16}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of $H$. Note that this coincides with the usual definition of the trace if $H$ is finite-dimensional. We observe that the series (16) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

**Theorem 6** *We have:*

*(i) If $A \in \mathcal{B}_1(H)$ then $A^* \in \mathcal{B}_1(H)$ and*

$$tr(A^*) = \overline{tr(A)}; \tag{17}$$

*(ii) If $A \in \mathcal{B}_1(H)$ and $T \in \mathcal{B}(H)$, then $AT$, $TA \in \mathcal{B}_1(H)$ and*

$$tr(AT) = tr(TA) \quad \text{and} \quad |tr(AT)| \le \|A\|_1 \|T\|; \tag{18}$$

*(iii) $tr(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(H)$ with $\|tr\| = 1$;*

*(iv) If $A, B \in \mathcal{B}_2(H)$ then $AB$, $BA \in \mathcal{B}_1(H)$ and $tr(AB) = tr(BA)$;*

*(v) $\mathcal{B}_{fin}(H)$ (finite rank operators) is a dense subspace of $\mathcal{B}_1(H)$.*

An operator $A \in \mathcal{B}(H)$ is said to belong to the von Neumann-Schatten class $\mathcal{B}_p(H)$, $1 \le p < \infty$ if the $p$-Schatten norm is finite [13, p. 60-64]

$$\|A\|_p := [tr(|A|^p)]^{\frac{1}{p}} < \infty,$$

$|A|^p$ is an operator notation and not a power.

For $1 < p < q < \infty$ we have that

$$\mathcal{B}_1(H) \subset \mathcal{B}_p(H) \subset \mathcal{B}_q(H) \subset \mathcal{B}(H) \tag{19}$$

and

$$\|A\|_1 \ge \|A\|_p \ge \|A\|_q \ge \|A\|. \tag{20}$$

For $p \geq 1$ the functional $\|\cdot\|_p$ is a norm on the $*$-ideal $\mathcal{B}_p(H)$, which is a Banach algebra, and $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$ is a Banach space.

Also, see for instance [13, p. 60-64], for $p \geq 1$,

$$\|A\|_p = \|A^*\|_p, \quad A \in \mathcal{B}_p(H) \tag{21}$$

$$\|AB\|_p \leq \|A\|_p \|B\|_p, \quad A, B \in \mathcal{B}_p(H) \tag{22}$$

and

$$\|AB\|_p \leq \|A\|_p \|B\|, \quad \|BA\|_p \leq \|B\| \|A\|_p, \quad A \in \mathcal{B}_p(H), B \in \mathcal{B}(H). \tag{23}$$

This implies that

$$\|CAB\|_p \leq \|C\| \|A\|_p \|B\|, \quad A \in \mathcal{B}_p(H), B, C \in \mathcal{B}(H). \tag{24}$$

In terms of $p$-Schatten norm we have the Hölder inequality for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ :

$$(|tr(AB)| \leq) \|AB\|_1 \leq \|A\|_p \|B\|_q, \quad A \in \mathcal{B}_p(H), B \in \mathcal{B}_q(H). \tag{25}$$

For the theory of trace functionals and their applications the interested reader is referred to [12] and [13].

For some classical trace inequalities see [3], [4] and [8], which are continuations of the work of Bellman [2].

# 4 Vector Analysis Background

(see [11], pp. 83-94)

Let $f(t)$ be a function defined on $[a, b] \subseteq \mathbb{R}$ taking values in a real or complex normed linear space $(X, \|\cdot\|)$, Then $f(t)$ is said to be differentiable at a point $t_0 \in [a, b]$ if the limit

$$f'(t_0) = \lim_{h \to 0} \frac{f(t_0 + h) - f(t_0)}{h} \tag{26}$$

exists in $X$, the convergence is in $\|\cdot\|$. This is called the derivative of $f(t)$ at $t = t_0$.

We call $f(t)$ differentiable on $[a, b]$, iff there exists $f'(t) \in X$ for all $t \in [a, b]$.

Similarly and inductively are defined higher order derivatives of $f$, denoted $f'', f^{(3)}, ..., f^{(k)}$, $k \in \mathbb{N}$, just as for numerical functions.

For all the properties of derivatives see [11], pp. 83-86.

Let now $(X, \|\cdot\|)$ be a Banach space, and $f : [a, b] \to X$.

We define the vector valued Riemann integral $\int_a^b f(t)\,dt \in X$ as the limit of the vector valued Riemann sums in $X$, convergence is in $\|\cdot\|$. The definition is as for the numerical valued functions.

If $\int_a^b f(t)\,dt \in X$ we call $f$ integrable on $[a,b]$. If $f \in C([a,b],X)$, then $f$ is integrable, [11], p. 87.

For all the properties of vector valued Riemann integrals see [11], pp. 86-91.

We define the space $C^n([a,b],X)$, $n \in \mathbb{N}$, of $n$-times continuously differentiable functions from $[a,b]$ into $X$; here continuity is with respect to $\|\cdot\|$ and defined in the usual way as for numerical functions·.

Let $(X,\|\cdot\|)$ be a Banach space and $f \in C^n([a,b],X)$, then we have the vector valued Taylor's formula, see [11], pp. 93-94, and also [10], (IV, 9; 47).

It holds

$$f(y)-f(x)-f'(x)(y-x)-\frac{1}{2}f''(x)(y-x)^2-...-\frac{1}{(n-1)!}f^{(n-1)}(x)(y-x)^{n-1} \tag{27}$$
$$=\frac{1}{(n-1)!}\int_x^y (y-t)^{n-1} f^{(n)}(t)\,dt, \quad \forall\, x,y \in [a,b].$$

In particular (27) is true when $X = \mathbb{R}^m, \mathbb{C}^m$, $m \in \mathbb{N}$, etc.

A function $f(t)$ with values in a normed linear space $X$ is said to be piecewise continuous (see [11], p. 85) on the interval $a \le t \le b$ if there exists a partition $a = t_0 < t_1 < t_2 < ... < t_n = b$ such that $f(t)$ is continuous on every open interval $t_k < t < t_{k+1}$ and has finite limits $f(t_0+0)$, $f(t_1-0)$, $f(t_1+0)$, $f(t_2-0)$, $f(t_2+0)$, ..., $f(t_n-0)$.

Here $f(t_k-0) = \lim_{t\uparrow t_k} f(t)$, $f(t_k+0) = \lim_{t\downarrow t_k} f(t)$.

The values of $f(t)$ at the points $t_k$ can be arbitrary or even undefined.

A function $f(t)$ with values in normed linear space $X$ is said to be piecewise smooth on $[a,b]$, if it is continuous on $[a,b]$ and has a derivative $f'(t)$ at all but a finite number of points of $[a,b]$, and if $f'(t)$ is piecewise continuous on $[a,b]$ (see [11], p. 85).

Let $u(t)$ and $v(t)$ be two piecewise smooth functions on $[a,b]$, one a numerical function and the other a vector function with values in Banach space $X$. Then we have the following integration by parts formula

$$\int_a^b u(t)\,dv(t) = u(t)\,v(t)\,|_a^b - \int_a^b v(t)\,du(t), \tag{28}$$

see [11], p. 93.

We mention also the mean value theorem for Banach space valued functions.

**Theorem 7** *(see [7], p. 3) Let $f \in C([a,b],X)$, where $X$ is a Banach space. Assume $f'$ exists on $[a,b]$ and $\|f'(t)\| \le K$, $a < t < b$, then*

$$\|f(b) - f(a)\| \le K(b-a). \tag{29}$$

Here the multiple Riemann integral of a function from a real box or a real compact and convex subset to a Banach space is defined similarly to numerical one however convergence is with respect to $\|\cdot\|$. Similarly are defined the vector valued partial derivatives as in the numerical case.

We mention the equality of vector valued mixed partiasl derivatives.

**Proposition 8** *(see Proposition 4.11 of [6], p. 90) Let* $Q = (a, b) \times (c, d) \subseteq \mathbb{R}^2$ *and* $f \in C(Q, X)$*, where* $(X, \|\cdot\|)$ *is a Banach space. Assume that* $\frac{\partial}{\partial t} f(s, t)$*,* $\frac{\partial}{\partial s} f(s, t)$ *and* $\frac{\partial^2}{\partial t \partial s} f(s, t)$ *exist and are continuous for* $(s, t) \in Q$*, then* $\frac{\partial^2}{\partial s \partial t} f(s, t)$ *exists for* $(s, t) \in Q$ *and*

$$\frac{\partial^2}{\partial s \partial t} f(s, t) = \frac{\partial^2}{\partial t \partial s} f(s, t), \quad \text{for } (s, t) \in Q. \tag{30}$$

# 5 Main Results

We present a general Ostrowski type inequality result regarding several $\mathcal{B}_2(H)$-Banach algebra valued functions.

**Theorem 9** *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*;* $\mathcal{B}_2(H)$ *is a* $*$*-ideal, which* $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Banach algebra, and* $f_i \in C^{n+1}(Q, \mathcal{B}_2(H))$*,* $i = 1, ..., r$*;* $r \in \mathbb{N}$*,* $n \in \mathbb{Z}_+$*, and fixed* $\overrightarrow{x_0} \in Q \subset \mathbb{R}^k$*,* $k \geq 2$*, where* $Q$ *is a compact and convex subset. Here all vector partial derivatives* $f_{i\alpha} := \frac{\partial^\alpha f_i}{\partial z^\alpha}$*, where* $\alpha = (\alpha_1, ..., \alpha_k)$*,* $\alpha_\lambda \in \mathbb{Z}^+$*,* $\lambda = 1, ..., k$*,* $|\alpha| = \sum_{\lambda=1}^{k} \alpha_\lambda = j$*,* $j = 1, ..., n$*, fulfill* $f_{i\alpha}(\overrightarrow{x_0}) = 0$*,* $i = 1, ..., r$*.*

*Denote*

$$D_{n+1}(f_i) := \max_{\alpha : |\alpha| = n+1} \left\| \|f_{i\alpha}\|_2 \right\|_{\infty, Q}, \tag{31}$$

$i = 1, ..., r$*, and*

$$\left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1} := \sum_{\lambda=1}^{k} |z_\lambda - x_{0\lambda}|. \tag{32}$$

*Then*

$$\left\| \sum_{i=1}^{r} \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_\rho(\overrightarrow{z}) \right) f_i(\overrightarrow{z}) \, d\overrightarrow{z} - \sum_{i=1}^{r} \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_\rho(\overrightarrow{z}) \right) d\overrightarrow{z} \right) f_i(\overrightarrow{x_0}) \right\|_1 \leq \tag{33}$$

$$\frac{\max_{i \in \{1, ..., r\}} D_{n+1}(f_i)}{(n+1)!} \sum_{i=1}^{r} \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \|f_\rho(\overrightarrow{z})\|_2 \right) \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \leq$$

$$\frac{\max_{i \in \{1, ..., r\}} D_{n+1}(f_i)}{(n+1)!} \min \left\{ \left( \int_Q \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \left[ \sum_{i=1}^{r} \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| \|f_\rho\|_2 \right\|_{\infty, Q} \right) \right], \right.$$

$$\left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{\infty,Q} \left[ \sum_{i=1}^{r} \left\| \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \| f_\rho \|_2 \right) \right\|_{L_1(Q)} \right],$$

$$\left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{L_p(Q)} \left[ \sum_{i=1}^{r} \left[ \left\| \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \| f_\rho \|_2 \right) \right\|_{L_q(Q)} \right] \right] \right\}. \quad (34)$$

**Proof.** Take $g_{i\overrightarrow{z}}(t) := f_i(\overrightarrow{x_0} + t(\overrightarrow{z} - \overrightarrow{x_0}))$, $0 \leq t \leq 1$; $i = 1, ..., r$. Notice that $g_{i\overrightarrow{z}}(0) = f_i(\overrightarrow{x_0})$ and $g_{i\overrightarrow{z}}(1) = f_i(\overrightarrow{z})$. The $j$th derivative of $g_{i\overrightarrow{z}}(t)$, based on Proposition 8, is given by

$$g_{i\overrightarrow{z}}^{(j)}(t) = \left[ \left( \sum_{\lambda=1}^{k} (z_\lambda - x_{0\lambda}) \frac{\partial}{\partial z_\lambda} \right)^j f_i \right] (x_{01} + t(z_1 - x_{01}), ..., x_{0k} + t(z_k - x_{0k})) \quad (35)$$

and

$$g_{i\overrightarrow{z}}^{(j)}(0) = \left[ \left( \sum_{\lambda=1}^{k} (z_\lambda - x_{0\lambda}) \frac{\partial}{\partial z_\lambda} \right)^j f_i \right] (\overrightarrow{x_0}), \quad (36)$$

for $j = 1, ..., n+1$; $i = 1, ..., r$.

Let $f_{i\alpha}$ be a partial derivative of $f_i \in C^{n+1}(Q, \mathcal{B}_2(H))$. Because by assumption of the theorem we have $f_{i\alpha}(\overrightarrow{x_0}) = 0$ for all $\alpha : |\alpha| = j$, $j = 1, ..., n$, we find that

$$g_{i\overrightarrow{z}}^{(j)}(0) = 0, \ \ j = 1, ..., n; \ \ i = 1, ..., r.$$

Hence by vector Taylor's theorem (27) we see that

$$f_i(\overrightarrow{z}) - f_i(\overrightarrow{x_0}) = \sum_{j=1}^{n} \frac{g_{i\overrightarrow{z}}^{(j)}(0)}{j!} + R_{in}(\overrightarrow{z}, 0) = R_{in}(\overrightarrow{z}, 0), \quad (37)$$

where

$$R_{in}(\overrightarrow{z}, 0) := \int_0^1 \left( \int_0^{t_1} ... \left( \int_0^{t_{n-1}} \left( g_{i\overrightarrow{z}}^{(n)}(t_n) - g_{i\overrightarrow{z}}^{(n)}(0) \right) dt_n \right) ... \right) dt_1, \quad (38)$$

$i = 1, ..., r$.

Therefore,

$$\| R_{in}(\overrightarrow{z}, 0) \|_2 \leq \int_0^1 \left( \int_0^{t_1} ... \left( \int_0^{t_{n-1}} \left\| \left\| g_{i\overrightarrow{z}}^{(n+1)}(\xi(t_n)) \right\|_2 \right\|_\infty t_n dt_n \right) ... \right) dt_1, \quad (39)$$

by the vector mean value Theorem 7 applied on $g_{i\overrightarrow{z}}^{(n)}$ over $(0, t_n)$. Moreover, we get

$$\| R_{in}(\overrightarrow{z}, 0) \|_2 \leq \left\| \left\| g_{i\overrightarrow{z}}^{(n+1)} \right\|_2 \right\|_{\infty,[0,1]} \int_0^1 \int_0^{t_1} ... \left( \int_0^{t_{n-1}} t_n dt_n \right) ... dt_1$$

$$= \frac{\left\| \left\| g_{i\overrightarrow{z}}^{(n+1)} \right\|_2 \right\|_{\infty,[0,1]}}{(n+1)!}. \tag{40}$$

However, there exists a $t_{i0} \in [0,1]$ such that $\left\| \left\| g_{i\overrightarrow{z}}^{(n+1)} \right\|_2 \right\|_{\infty,[0,1]} = \left\| g_{i\overrightarrow{z}}^{(n+1)} (t_{i0}) \right\|_2$.

That is

$$\left\| \left\| g_{i\overrightarrow{z}}^{(n+1)} \right\|_2 \right\|_{\infty,[0,1]} = \left\| \left[ \left( \sum_{\lambda=1}^{k} (z_\lambda - x_{0\lambda}) \frac{\partial}{\partial z_\lambda} \right)^{n+1} f_i \right] \left( \overrightarrow{x_0} + t_{i0} \left( \overrightarrow{z} - \overrightarrow{z_{0i}} \right) \right) \right\|_2$$

$$\leq \left[ \left( \sum_{\lambda=1}^{k} |z_\lambda - x_{0\lambda}| \left\| \frac{\partial}{\partial z_\lambda} \right\|_2 \right)^{n+1} f_i \right] \left( \overrightarrow{x_0} + t_{i0} \left( \overrightarrow{z} - \overrightarrow{z_{0i}} \right) \right).$$

I.e.,

$$\left\| \left\| g_{i\overrightarrow{z}}^{(n+1)} \right\|_2 \right\|_{\infty,[0,1]} \leq \left[ \left( \sum_{\lambda=1}^{k} |z_\lambda - x_{0\lambda}| \left\| \left\| \frac{\partial}{\partial z_\lambda} \right\|_2 \right\|_\infty \right)^{n+1} f_i \right], \tag{41}$$

$i = 1, ..., r$.

Hence by (41) we get

$$\| R_{in} (\overrightarrow{z}, 0) \|_2 \leq \frac{\left[ \left( \sum_{\lambda=1}^{k} |z_\lambda - x_{0\lambda}| \left\| \left\| \frac{\partial}{\partial z_\lambda} \right\|_2 \right\|_\infty \right)^{n+1} f_i \right]}{(n+1)!} \leq$$

$$\frac{D_{n+1}(f_i)}{(n+1)!} \left( \sum_{\lambda=1}^{k} |z_\lambda - x_{0\lambda}| \right)^{n+1} = \frac{D_{n+1}(f_i)}{(n+1)!} \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1}, \tag{42}$$

$i = 1, ..., r$.

Therefore it holds

$$\| R_{in} (\overrightarrow{z}, 0) \|_2 \leq \frac{\max_{i \in \{1,...,r\}} D_{n+1}(f_i)}{(n+1)!} \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1}, \tag{43}$$

for $i = 1, ..., r$.

By (37) we get that

$$\left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_\rho (\overrightarrow{z}) \right) f_i (\overrightarrow{z}) - \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_\rho (\overrightarrow{z}) \right) f_i (\overrightarrow{x_0}) = \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_\rho (\overrightarrow{z}) \right) R_{in} (\overrightarrow{z}, 0), \tag{44}$$

for all $i = 1, ..., r$.

Hence

$$\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)-\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{x_0}\right)$$

$$=\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right). \tag{45}$$

Therefore we find

$$E\left(f_1,...,f_r\right)\left(x_0\right):=$$

$$\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)d\overrightarrow{z}-\sum_{i=1}^{r}\left(\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right)d\overrightarrow{z}\right) f_i\left(\overrightarrow{x_0}\right)=$$

$$\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right)d\overrightarrow{z}. \tag{46}$$

Consequently, we have that

$$\left\|E\left(f_1,...,f_r\right)\left(x_0\right)\right\|_1=$$

$$\left\|\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) f_i\left(\overrightarrow{z}\right)d\overrightarrow{z}-\sum_{i=1}^{r}\left(\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right)d\overrightarrow{z}\right) f_i\left(\overrightarrow{x_0}\right)\right\|_1=$$

$$\left\|\sum_{i=1}^{r}\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right)d\overrightarrow{z}\right\|_1\leq \tag{47}$$

$$\sum_{i=1}^{r}\left\|\int_Q\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right)d\overrightarrow{z}\right\|_1\leq$$

$$\sum_{i=1}^{r}\left(\int_Q\left\|\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho\left(\overrightarrow{z}\right)\right) R_{in}\left(\overrightarrow{z},0\right)\right\|_1 d\overrightarrow{z}\right)\leq$$

(by (25), (22))

$$\sum_{i=1}^{r}\left(\int_{Q}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}\left\|f_{\rho}\left(\overrightarrow{z}\right)\right\|_{2}\right)\left\|R_{in}\left(\overrightarrow{z},0\right)\right\|_{2}d\overrightarrow{z}\right)\overset{(43)}{\leq} \tag{48}$$

$$\frac{\max\limits_{i\in\{1,...,r\}}D_{n+1}\left(f_{i}\right)}{(n+1)!}\sum_{i=1}^{r}\left(\int_{Q}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}\left\|f_{\rho}\left(\overrightarrow{z}\right)\right\|_{2}\right)\left\|\overrightarrow{z}-\overrightarrow{x_{0}}\right\|_{l_{1}}^{n+1}d\overrightarrow{z}\right).$$

So far we have proved

$$\left\|E\left(f_{1},...,f_{r}\right)\left(x_{0}\right)\right\|_{1}\leq$$

$$\frac{\max\limits_{i\in\{1,...,r\}}D_{n+1}\left(f_{i}\right)}{(n+1)!}\sum_{i=1}^{r}\left(\int_{Q}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}\left\|f_{\rho}\left(\overrightarrow{z}\right)\right\|_{2}\right)\left\|\overrightarrow{z}-\overrightarrow{x_{0}}\right\|_{l_{1}}^{n+1}d\overrightarrow{z}\right)=:(\xi). \tag{49}$$

Furthermore it holds

$$(\xi)\leq\frac{\max\limits_{i\in\{1,...,r\}}D_{n+1}\left(f_{i}\right)}{(n+1)!}\left(\int_{Q}\left\|\overrightarrow{z}-\overrightarrow{x_{0}}\right\|_{l_{1}}^{n+1}d\overrightarrow{z}\right)\left[\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}\left\|\left\|f_{\rho}\right\|_{2}\right\|_{\infty,Q}\right)\right], \tag{50}$$

and

$$(\xi)\leq\frac{\max\limits_{i\in\{1,...,r\}}D_{n+1}\left(f_{i}\right)}{(n+1)!}\left\|\left\|\cdot-\overrightarrow{x_{0}}\right\|_{l_{1}}^{n+1}\right\|_{\infty,Q}\left[\sum_{i=1}^{r}\left\|\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}\left\|f_{\rho}\right\|_{2}\right)\right\|_{L_{1}(Q)}\right], \tag{51}$$

and finally

$$(\xi)\leq\frac{\max\limits_{i\in\{1,...,r\}}D_{n+1}\left(f_{i}\right)}{(n+1)!}\left[\sum_{i=1}^{r}\left[\left\|\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}\left\|f_{\rho}\right\|_{2}\right)\right\|_{L_{q}(Q)}\right]\right]\left\|\left\|\cdot-\overrightarrow{x_{0}}\right\|_{l_{1}}^{n+1}\right\|_{L_{p}(Q)}, \tag{52}$$

proving (33), (34). ■

We give

**Corollary 10** *(to Theorem 9) All as in Theorem 9, with* $f_1 = ... = f_r = f$, $r \in \mathbb{N}$. *Then*

$$\left\|\int_{Q}f^{r}\left(\overrightarrow{z}\right)d\overrightarrow{z}-\left(\int_{Q}f^{r-1}\left(\overrightarrow{z}\right)d\overrightarrow{z}\right)f\left(\overrightarrow{x_{0}}\right)\right\|_{1}\leq$$

$$\frac{D_{n+1}(f)}{(n+1)!}\left(\int_Q \|f(\overrightarrow{z})\|_2^{r-1}\|\overrightarrow{z}-\overrightarrow{x_0}\|_{l_1}^{n+1}d\overrightarrow{z}\right)\leq \tag{53}$$

$$\frac{D_{n+1}(f)}{(n+1)!}\min\left\{\left(\int_Q \|\overrightarrow{z}-\overrightarrow{x_0}\|_{l_1}^{n+1}d\overrightarrow{z}\right)\left(\|\|f\|_2\|_{\infty,Q}\right)^{r-1},\right.$$

$$\left.\left\|\|\cdot-\overrightarrow{x_0}\|_{l_1}^{n+1}\right\|_{\infty,Q}\left\|\|f\|_2^{r-1}\right\|_{L_1(Q)},\left\|\|\cdot-\overrightarrow{x_0}\|_{l_1}^{n+1}\right\|_{L_p(Q)}\left\|\|f\|_2^{r-1}\right\|_{L_q(Q)}\right\}. \tag{54}$$

We make

**Remark 11** *Of great interest are applications of Theorem 9 when* $Q=\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]$, *where* $[a_\lambda,b_\lambda]\subset\mathbb{R}$, $\lambda=1,...,k$.

*We observe that by the multinomial theorem we get:*

$$\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}\left(\sum_{\lambda=1}^{k}|z_\lambda-x_{0\lambda}|\right)^{n+1}dz_1...dz_k=\sum_{\rho_1+\rho_2+...\rho_k=n+1}\frac{(n+1)!}{\rho_1!\rho_2!...\rho_k!}$$

$$\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}|z_1-x_{01}|^{\rho_1}|z_2-x_{02}|^{\rho_2}...|z_k-x_{0k}|^{\rho_k}dz_1...dz_k= \tag{55}$$

$$\sum_{\rho_1+\rho_2+...\rho_k=n+1}\frac{(n+1)!}{\rho_1!\rho_2!...\rho_k!}\prod_{\lambda=1}^{k}\left(\int_{a_\lambda}^{b_\lambda}|z_\lambda-x_{0\lambda}|^{\rho_\lambda}dz_\lambda\right)=$$

$$\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1}\frac{(n+1)!}{\prod_{\lambda=1}^{k}\rho_\lambda!}\prod_{\lambda=1}^{k}\left(\int_{a_\lambda}^{x_{0\lambda}}(x_{0\lambda}-z_\lambda)^{\rho_\lambda}dz_\lambda+\int_{x_{0\lambda}}^{b_\lambda}(z_\lambda-x_{0\lambda})^{\rho_\lambda}dz_\lambda\right)=$$

$$\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1}\frac{(n+1)!}{\prod_{\lambda=1}^{k}\rho_\lambda!}\prod_{\lambda=1}^{k}\left(\frac{(x_{0\lambda}-a_\lambda)^{\rho_\lambda+1}+(b_\lambda-x_{0\lambda})^{\rho_\lambda+1}}{\rho_\lambda+1}\right). \tag{56}$$

*We have found that*

$$\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]}\|\overrightarrow{z}-\overrightarrow{x_0}\|_{l_1}^{n+1}d\overrightarrow{z}= \tag{57}$$

$$\sum_{\sum_{\lambda=1}^{k}\rho_\lambda=n+1}\frac{(n+1)!}{\prod_{\lambda=1}^{k}\rho_\lambda!}\prod_{\lambda=1}^{k}\left(\frac{(b_\lambda-x_{0\lambda})^{\rho_\lambda+1}+(x_{0\lambda}-a_\lambda)^{\rho_\lambda+1}}{\rho_\lambda+1}\right).$$

Based on (33), (34) and (57) we conclude:

**Theorem 12** *Let* $(\mathcal{B}_2(H), \|\cdot\|_2)$ *the* $*$*-ideal and* $f_i \in C^{n+1}\left(\prod_{\lambda=1}^{k}[a_\lambda, b_\lambda], \mathcal{B}_2(H)\right)$, $i = 1, ..., r$; $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, *and fixed* $\overrightarrow{x_0} \in \prod_{\lambda=1}^{k}[a_\lambda, b_\lambda] \subset \mathbb{R}^k$, $k \geq 2$. *Here all vector partial derivatives* $f_{i\alpha} := \frac{\partial^\alpha f_i}{\partial z^\alpha}$, *where* $\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_\lambda \in \mathbb{Z}^+$, $\lambda = 1, ..., k$, $|\alpha| = \sum_{\lambda=1}^{k}\alpha_\lambda = j$, $j = 1, ..., n$, *fulfill* $f_{i\alpha}(\overrightarrow{x_0}) = 0$, $i = 1, ..., r$.

*Denote*

$$D_{n+1}(f_i) := \max_{\alpha:|\alpha|=n+1} \left\| \|f_{i\alpha}\|_2 \right\|_{\infty, \prod_{\lambda=1}^{k}[a_\lambda, b_\lambda]}, \tag{58}$$

$i = 1, ..., r$.

*Then*

$$\left\| \sum_{i=1}^{r} \int_{\prod_{\lambda=1}^{k}[a_\lambda, b_\lambda]} \left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho(\overrightarrow{z})\right) f_i(\overrightarrow{z})\, d\overrightarrow{z} - \right.$$

$$\left. \sum_{i=1}^{r}\left(\int_{\prod_{\lambda=1}^{k}[a_\lambda, b_\lambda]}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} f_\rho(\overrightarrow{z})\right) d\overrightarrow{z}\right) f_i(\overrightarrow{x_0}) \right\|_1 \leq \tag{59}$$

$$\left(\max_{i\in\{1,...,r\}} D_{n+1}(f_i)\right)\left[\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r} \left\| \|f_\rho\|_2 \right\|_{\infty, \prod_{\lambda=1}^{k}[a_\lambda, b_\lambda]}\right)\right]$$

$$\left[\sum_{\sum_{\lambda=1}^{k}\rho_\lambda = n+1} \frac{1}{\prod_{\lambda=1}^{k}\rho_\lambda!\prod_{\lambda=1}^{k}(\rho_\lambda+1)} \prod_{\lambda=1}^{k}\left((b_\lambda - x_{0\lambda})^{\rho_\lambda+1} + (x_{0\lambda} - a_\lambda)^{\rho_\lambda+1}\right)\right].$$

We continue with a general Ostrowski type inequality result regarding several $\mathcal{B}_\gamma(H)$, $\gamma \geq 1$, Banach algebra valued functions.

**Theorem 13** *Let* $\gamma \geq 1$, $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; $\mathcal{B}_\gamma(H)$ *is a* $*$*-ideal, which* $\left(\mathcal{B}_\gamma(H), \|\cdot\|_\gamma\right)$ *is a Banach algebra, and* $f_i \in C^{n+1}(Q, \mathcal{B}_\gamma(H))$, $i = 1, ..., r$; $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, *and fixed* $\overrightarrow{x_0} \in Q \subset \mathbb{R}^k$, $k \geq 2$, *where* $Q$ *is a compact and convex subset. Here all vector partial derivatives* $f_{i\alpha} := \frac{\partial^\alpha f_i}{\partial z^\alpha}$, *where* $\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_\lambda \in \mathbb{Z}^+$, $\lambda = 1, ..., k$, $|\alpha| = \sum_{\lambda=1}^{k}\alpha_\lambda = j$, $j = 1, ..., n$, *fulfill* $f_{i\alpha}(\overrightarrow{x_0}) = 0$, $i = 1, ..., r$.

*Denote*

$$D_{n+1}(f_i) := \max_{\alpha:|\alpha|=n+1} \left\| \|f_{i\alpha}\|_\gamma \right\|_{\infty, Q}, \tag{60}$$

$i = 1, ..., r,$ *and*

$$\left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1} := \sum_{\lambda=1}^{k} \left| z_\lambda - x_{0\lambda} \right|. \tag{61}$$

*Then*

$$\left\| \sum_{i=1}^{r} \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_\rho \left( \overrightarrow{z} \right) \right) f_i \left( \overrightarrow{z} \right) d\overrightarrow{z} - \sum_{i=1}^{r} \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} f_\rho \left( \overrightarrow{z} \right) \right) d\overrightarrow{z} \right) f_i \left( \overrightarrow{x_0} \right) \right\|_\gamma \leq \tag{62}$$

$$\frac{\max\limits_{i \in \{1,...,r\}} D_{n+1} (f_i)}{(n+1)!} \sum_{i=1}^{r} \left( \int_Q \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_\rho \left( \overrightarrow{z} \right) \right\|_\gamma \right) \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \leq$$

$$\frac{\max\limits_{i \in \{1,...,r\}} D_{n+1} (f_i)}{(n+1)!} \min \Bigg\{ \left( \int_Q \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \left[ \sum_{i=1}^{r} \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| \left\| f_\rho \right\|_\gamma \right\|_{\infty,Q} \right) \right],$$

$$\left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{\infty,Q} \left[ \sum_{i=1}^{r} \left\| \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_\rho \right\|_\gamma \right) \right\|_{L_1(Q)} \right],$$

$$\left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{L_p(Q)} \left[ \sum_{i=1}^{r} \left[ \left\| \left( \prod_{\substack{\rho=1 \\ \rho \neq i}}^{r} \left\| f_\rho \right\|_\gamma \right) \right\|_{L_q(Q)} \right] \right] \Bigg\}. \tag{63}$$

**Proof.** As similar to Theorem 9 is omitted. Use of (22). ■

We also give

**Corollary 14** *(to Theorem 13) All as in Theorem 13, with* $f_1 = ... = f_r = f$, $r \in \mathbb{N}$. *Then*

$$\left\| \int_Q f^r \left( \overrightarrow{z} \right) d\overrightarrow{z} - \left( \int_Q f^{r-1} \left( \overrightarrow{z} \right) d\overrightarrow{z} \right) f \left( \overrightarrow{x_0} \right) \right\|_\gamma \leq$$

$$\frac{D_{n+1}(f)}{(n+1)!} \left( \int_Q \left\| f \left( \overrightarrow{z} \right) \right\|_\gamma^{r-1} \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \leq \tag{64}$$

$$\frac{D_{n+1}(f)}{(n+1)!} \min \Bigg\{ \left( \int_Q \left\| \overrightarrow{z} - \overrightarrow{x_0} \right\|_{l_1}^{n+1} d\overrightarrow{z} \right) \left( \left\| \left\| f \right\|_\gamma \right\|_{\infty,Q} \right)^{r-1},$$

$$\left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{\infty,Q} \left\| \left\| f \right\|_\gamma^{r-1} \right\|_{L_1(Q)}, \left\| \left\| \cdot - \overrightarrow{x_0} \right\|_{l_1}^{n+1} \right\|_{L_p(Q)} \left\| \left\| f \right\|_\gamma^{r-1} \right\|_{L_q(Q)} \Bigg\}. \tag{65}$$

In particular we obtain:

**Theorem 15** *Let* $\left(\mathcal{B}_\gamma(H), \|\cdot\|_\gamma\right)$, $\gamma \geq 1$, *the* $*$*-ideal and* $f_i \in C^{n+1}\left(\prod_{\lambda=1}^{k}[a_\lambda, b_\lambda], \mathcal{B}_\gamma(H)\right)$, $i = 1, ..., r$; $r \in \mathbb{N}$, $n \in \mathbb{Z}_+$, *and fixed* $\overrightarrow{x_0} \in \prod_{\lambda=1}^{k}[a_\lambda, b_\lambda] \subset \mathbb{R}^k$, $k \geq 2$. *Here all vector partial derivatives* $f_{i\alpha} := \frac{\partial^\alpha f_i}{\partial z^\alpha}$, *where* $\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_\lambda \in \mathbb{Z}^+$, $\lambda = 1, ..., k$, $|\alpha| = \sum_{\lambda=1}^{k}\alpha_\lambda = j$, $j = 1, ..., n$, *fulfill* $f_{i\alpha}(\overrightarrow{x_0}) = 0$, $i = 1, ..., r$.

*Denote*

$$D_{n+1}(f_i) := \max_{\alpha:|\alpha|=n+1}\left\|\|f_{i\alpha}\|_\gamma\right\|_{\infty, \prod_{\lambda=1}^{k}[a_\lambda, b_\lambda]}, \tag{66}$$

$i = 1, ..., r$.

*Then*

$$\left\|\sum_{i=1}^{r}\int_{\prod_{\lambda=1}^{k}[a_\lambda, b_\lambda]}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}f_\rho(\overrightarrow{z})\right)f_i(\overrightarrow{z})\,d\overrightarrow{z} - \right.$$

$$\left.\sum_{i=1}^{r}\left(\int_{\prod_{\lambda=1}^{k}[a_\lambda, b_\lambda]}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}f_\rho(\overrightarrow{z})\right)d\overrightarrow{z}\right)f_i(\overrightarrow{x_0})\right\|_\gamma \leq \tag{67}$$

$$\left(\max_{i\in\{1,...,r\}}D_{n+1}(f_i)\right)\left[\sum_{i=1}^{r}\left(\prod_{\substack{\rho=1\\ \rho\neq i}}^{r}\left\|\|f_\rho\|_\gamma\right\|_{\infty, \prod_{\lambda=1}^{k}[a_\lambda, b_\lambda]}\right)\right]$$

$$\left[\sum_{\sum_{\lambda=1}^{k}\rho_\lambda = n+1}\frac{1}{\prod_{\lambda=1}^{k}\rho_\lambda!\prod_{\lambda=1}^{k}(\rho_\lambda+1)}\prod_{\lambda=1}^{k}\left((b_\lambda - x_{0\lambda})^{\rho_\lambda+1} + (x_{0\lambda} - a_\lambda)^{\rho_\lambda+1}\right)\right].$$

**Proof.** As similar to Theorem 12 is omitted. ■

When $r = 2$ we derive the following $p$-Schatten norm operator related Ostrowski type inequalities.

**Theorem 16** *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$; *and let the* $*$*-ideals* $\mathcal{B}_p(H)$, $\mathcal{B}_q(H)$, *for which* $\left(\mathcal{B}_p(H), \|\cdot\|_p\right)$, $\left(\mathcal{B}_q(H), \|\cdot\|_q\right)$ *are Banach algebras,* $n \in \mathbb{Z}_+$; $\overrightarrow{x_0} \in Q \subset \mathbb{R}^k$, $k \geq 2$, *where* $Q$ *is a compact and convex subset, and* $f_1 \in C^{n+1}(Q, \mathcal{B}_p(H))$, $f_2 \in C^{n+1}(Q, \mathcal{B}_q(H))$. *Here all vector partial derivatives* $f_{i\alpha} := \frac{\partial^\alpha f_i}{\partial z^\alpha}$, *where* $\alpha = (\alpha_1, ..., \alpha_k)$, $\alpha_\lambda \in \mathbb{Z}^+$, $\lambda = 1, ..., k$, $|\alpha| = \sum_{\lambda=1}^{k}\alpha_\lambda = j$, $j = 1, ..., n$, *fulfill* $f_{i\alpha}(\overrightarrow{x_0}) = 0$, $i = 1, 2$.

*Denote by*

$$D_{n+1}(f_1) := \max_{\alpha:|\alpha|=n+1} \left\| \|f_{1\alpha}\|_p \right\|_{\infty,Q}, \tag{68}$$

*and*

$$D_{n+1}(f_2) := \max_{\alpha:|\alpha|=n+1} \left\| \|f_{2\alpha}\|_q \right\|_{\infty,Q}, \tag{69}$$

*and*

$$\|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1} := \sum_{\lambda=1}^{k} |z_\lambda - x_{0\lambda}|\,; \;\; \gamma, \delta > 1 : \frac{1}{\gamma} + \frac{1}{\delta} = 1. \tag{70}$$

*Then*

$$\Bigg\| \int_Q f_2(\overrightarrow{z}) f_1(\overrightarrow{z}) d\overrightarrow{z} + \int_Q f_1(\overrightarrow{z}) f_2(\overrightarrow{z}) d\overrightarrow{z} -$$

$$\left(\int_Q f_2(\overrightarrow{z}) d\overrightarrow{z}\right) f_1(\overrightarrow{x_0}) - \left(\int_Q f_1(\overrightarrow{z}) d\overrightarrow{z}\right) f_2(\overrightarrow{x_0}) \Bigg\|_1 \leq$$

$$\frac{\max(D_{n+1}(f_1), D_{n+1}(f_2))}{(n+1)!}$$

$$\left\{ \int_Q \|f_2(\overrightarrow{z})\|_q \|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1}^{n+1} d\overrightarrow{z} + \int_Q \|f_1(\overrightarrow{z})\|_p \|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1}^{n+1} d\overrightarrow{z} \right\} \leq \tag{71}$$

$$\frac{\max(D_{n+1}(f_1), D_{n+1}(f_2))}{(n+1)!}$$

$$\min\Bigg\{ \left(\int_Q \|\overrightarrow{z} - \overrightarrow{x_0}\|_{l_1}^{n+1} d\overrightarrow{z}\right) \left[ \left\| \|f_2\|_q \right\|_{\infty,Q} + \left\| \|f_1\|_p \right\|_{\infty,Q} \right],$$

$$\left\| \|\cdot - \overrightarrow{x_0}\|_{l_1}^{n+1} \right\|_{\infty,Q} \left[ \left\| \|f_2\|_q \right\|_{L_1(Q)} + \left\| \|f_1\|_p \right\|_{L_1(Q)} \right],$$

$$\left\| \|\cdot - \overrightarrow{x_0}\|_{l_1}^{n+1} \right\|_{L_\gamma(Q)} \left[ \left\| \|f_2\|_q \right\|_{L_\delta(Q)} + \left\| \|f_1\|_p \right\|_{L_\delta(Q)} \right] \Bigg\}. \tag{72}$$

**Proof.** Similar to Theorem 9. Use of (25). ■

We finish with

**Corollary 17** *(to Theorem 16) All as in Theorem 16, with* $Q = \prod_{\lambda=1}^{k} [a_\lambda, b_\lambda]$.

*Then*

$$\Bigg\| \int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} f_2(\overrightarrow{z}) f_1(\overrightarrow{z}) d\overrightarrow{z} + \int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} f_1(\overrightarrow{z}) f_2(\overrightarrow{z}) d\overrightarrow{z} -$$

$$\left(\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} f_2(\overrightarrow{z}) d\overrightarrow{z}\right) f_1(\overrightarrow{x_0}) - \left(\int_{\prod_{\lambda=1}^{k}[a_\lambda,b_\lambda]} f_1(\overrightarrow{z}) d\overrightarrow{z}\right) f_2(\overrightarrow{x_0}) \Bigg\|_1 \leq$$

$$\max\left(D_{n+1}\left(f_{1}\right),D_{n+1}\left(f_{2}\right)\right)\left[\left\|\left\|f_{1}\right\|_{p}\right\|_{\infty,\prod_{\lambda=1}^{k}\left[a_{\lambda},b_{\lambda}\right]}+\left\|\left\|f_{2}\right\|_{q}\right\|_{\infty,\prod_{\lambda=1}^{k}\left[a_{\lambda},b_{\lambda}\right]}\right] \quad (73)$$

$$\left[\sum_{\sum_{\lambda=1}^{k}\rho_{k}=n+1}\frac{1}{\prod_{\lambda=1}^{k}\rho_{k}!\prod_{\lambda=1}^{k}\left(\rho_{\lambda}+1\right)}\prod_{\lambda=1}^{k}\left(\left(b_{\lambda}-x_{0\lambda}\right)^{\rho_{\lambda}+1}+\left(x_{0\lambda}-a_{\lambda}\right)^{\rho_{\lambda}+1}\right)\right].$$

**Proof.** Similar to Theorem 12. ■

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