SOME NEW INNER PRODUCT INEQUALITIES FOR THE ČEBYŠEV FUNCTIONAL IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. For two continuous functions $f,\ g:[a,b]\rightarrow H$ we define the *Čebyšev functional*

$$D(f,g):=(b-a)\int_a^b\langle f(t),g(t)\rangle dt-\left\langle \int_a^b f(t)\,dt,\int_a^b g(t)\,dt\right\rangle .$$

In this paper we show among others that if $f,\ g:[a,b]\rightarrow H$ are strongly differentiable functions on the interval (a,b), then

$$|D(f,g)|\le\frac{1}{2}(b-a)\|f'\|_{[a,b],\infty}\int_a^b(b-t)(t-a)\|g'(t)\|\,dt$$
$$\le\frac{1}{2}(b-a)^3\|f'\|_{[a,b],\infty}\times\begin{cases}\frac{1}{4}\|g'\|_{[a,b],1}\,,\\ \\ (b-a)^{1/q}\left[B(q+1,q+1)\right]^{1/q}\|g'\|_{[a,b],p}\,,\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \frac{1}{6}(b-a)^2\|g'\|_{[a,b],\infty}\,,\end{cases}$$

where

$$\|h'\|_{[a,b],p}:=\left(\int_a^b\|h'(u)\|^p du\right)^{1/p},\ p\ge1$$

and $\|h'\|_{[a,b],\infty}:=\sup_{t\in(a,b)}\|h'(u)\|$ for a strongly differentiable function h on (a,b), while $B(\cdot,\cdot)$ is Beta function. Some applications for operator monotone function with examples for power function are also given.

1. Introduction

For two Lebesgue integrable functions $f,\ g:[a,b]\rightarrow\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g):=(b-a)\int_a^b f(t)\,g(t)\,dt-\int_a^b f(t)\,dt\int_a^b g(t)\,dt.$$

In 1934, G. Grüss [15] showed that

$$|D(f,g)|\le\frac{1}{4}(b-a)^2(M-m)(N-n)\,, \tag{1.1}$$

provided $m,\ M,\ n,\ N$ are real numbers with the property that

$$-\infty<m\le f\le M<\infty,\quad -\infty<n\le g\le N<\infty\quad\text{a.e. on}\quad[a,b]\,. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

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Another lesser known inequality for $D(f,g)$ was derived in 1882 by Čebyšev [5] under the assumption that f', g' exist and are continuous on $[a,b]$, and is given by

$$|D(f,g)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^4, \tag{1.3}$$

where $\|f'\|_\infty := \sup_{t\in[a,b]}|f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if f, $g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and f', $g' \in L_\infty[a,b]$.

In 1970, A.M. Ostrowski [18] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(f,g)| \leq \frac{1}{8}(b-a)^3 (M-m)\|g'\|_\infty, \tag{1.4}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_\infty[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [16] (see also [17, p. 210]) obtained the following result as well:

$$|D(f,g)| \leq \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a)^3, \tag{1.5}$$

provided f, g are absolutely continuous and f', $g' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [2], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(f,g)| \leq (b-a)\times\begin{cases} \inf\limits_{\gamma\in\mathbb{R}}\|g-\gamma\|_\infty \int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt, \\ \\ \inf\limits_{\gamma\in\mathbb{R}}\|g-\gamma\|_q \left(\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}} \\ \qquad\qquad \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(f,g)| \leq (b-a)\|g\|_\infty \int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\|g - \frac{m+M}{2}\right\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [4]

$$|D(f,g)| \leq \frac{1}{2}(b-a)(M-m)\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [3].

The following result holds [14].

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(f,g)| \leq \frac{1}{2}(b-a)\bigvee_a^b(f)\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt \tag{1.9}$$

where $\bigvee_a^b (f)$ *denotes the total variation of* f *on the interval* $[a,b]$*. The constant* $\frac{1}{2}$ *is best possible in (1.9).*

For more recent upper bounds related to the Čebyšev functional see [2], [3] and [9]-[14].

An extension of this classical result to real or complex inner product spaces has been obtained by S. S. Dragomir in [6]:

Theorem 2. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $e\in H$, $\|e\|=1$*. If* φ, ϕ, γ, $\Gamma\in\mathbb{C}$ *and* x, $y\in H$ *are such that*

$$\operatorname{Re}\langle \phi e-x,x-\varphi e\rangle \geq 0 \quad \text{and} \quad \operatorname{Re}\langle \Gamma e-y,y-\gamma e\rangle \geq 0 \tag{1.10}$$

or, equivalently (see [8]*)*

$$\left\| x-\frac{\varphi+\phi}{2}e\right\| \leq \frac{1}{2}|\phi-\varphi| \quad \text{and} \quad \left\| y-\frac{\gamma+\Gamma}{2}e\right\| \leq \frac{1}{2}|\Gamma-\gamma|, \tag{1.11}$$

then

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle| \leq \frac{1}{4}|\phi-\varphi||\Gamma-\gamma|. \tag{1.12}$$

The constant $\frac{1}{4}$ *is best possible in (1.12).*

A further extension for Bochner integrals of vector-valued functions in real or complex Hilbert spaces was obtained by S. S. Dragomir in 2001, [7].

Theorem 3. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real or complex Hilbert space,* $\Omega\subset\mathbb{R}^n$ *be a Lebesgue measurable set and* $\rho:\Omega\to[0,\infty)$ *a Lebesgue measurable function with* $\int_\Omega \rho(s)\,ds=1$*. We denote by* $L_{2,\rho}(\Omega,H)$ *the set of all Bochner measurable functions* f *on* Ω *such that* $\|f\|_{2,\rho}^2:=\int_\Omega \rho(s)\|f(s)\|^2 ds<\infty$*. If* f, g *belong to* $L_{2,\rho}(\Omega,H)$ *and there exist the vectors* x, X, y, $Y\in H$ *such that*

$$\begin{aligned} &\int_\Omega \rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle\,dt \geq 0, \\ &\int_\Omega \rho(t)\operatorname{Re}\langle Y-g(t),g(t)-y\rangle\,dt \geq 0, \end{aligned} \tag{1.13}$$

then we have the inequality

$$\begin{aligned} &\left|\int_\Omega \rho(t)\langle f(t),g(t)\rangle\,dt-\left\langle \int_\Omega \rho(t)f(t)\,dt,\int_\Omega \rho(t)g(t)\,dt\right\rangle\right| \\ &\leq \frac{1}{4}\|X-x\|\|Y-y\|. \end{aligned} \tag{1.14}$$

The constant $\frac{1}{4}$ *is sharp in the sense mentioned above.*

Remark 1. *A practical sufficient condition for (1.13) to hold is*

$$\operatorname{Re}\langle X-f(t),f(t)-x\rangle \geq 0,\ \operatorname{Re}\langle Y-g(t),g(t)-y\rangle \geq 0$$

or, equivalently

$$\left\| f(t)-\frac{X+x}{2}\right\| \leq \frac{1}{2}\|X-x\| \quad \text{and} \quad \left\| g(t)-\frac{Y+y}{2}\right\| \leq \frac{1}{2}\|Y-y\|,$$

for a.e. $t\in\Omega$.

An interesting particular inequality that has not been mentioned in [7] can be obtained by considering $H=\mathbb{C}$, $\langle x,y\rangle := x\cdot\bar{y}$ and $g=\bar{f}$, to give

$$\left|\int_\Omega \rho(s) f^2(s)\,ds - \left(\int_\Omega \rho(s) f(s)\,ds\right)^2\right| \le \frac{1}{4}|A-a|^2, \tag{1.15}$$

provided

$$\int_\Omega \rho(s)\operatorname{Re}\left[(A-f(s))\left(\overline{f(s)}-\overline{a}\right)\right]ds \ge 0 \tag{1.16}$$

or, sufficiently,

$$\operatorname{Re}\left[(A-f(s))\left(\overline{f(s)}-\overline{a}\right)\right] \ge 0 \tag{1.17}$$

for a.e. $s\in\Omega$.

Note that the alternative result

$$0 \le \int_\Omega \rho(s)|f(s)|^2\,ds - \left|\int_\Omega \rho(s) f(s)\,ds\right|^2 \le \frac{1}{4}|A-a|^2, \tag{1.18}$$

provided (1.16) or (1.17) hold, has been stated in [7].

Let H be a complex Hilbert space. For two continuous functions $f,g:[a,b]\to H$ we define the *Čebyšev functional*

$$D(f,g) := (b-a)\int_a^b \langle f(t), g(t)\rangle\,dt - \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt\right\rangle.$$

In this paper we show among other that if $f,g:[a,b]\to H$ are strongly differentiable functions on the interval (a,b), then

$$|D(f,g)| \le \frac{1}{2}(b-a)\|f'\|_{[a,b],\infty}\int_a^b (b-t)(t-a)\|g'(t)\|\,dt$$
$$\le \frac{1}{2}(b-a)^3\|f'\|_{[a,b],\infty}\times\begin{cases}\frac{1}{4}\|g'\|_{[a,b],1}, \\ (b-a)^{1/q}[B(q+1,q+1)]^{1/q}\|g'\|_{[a,b],p}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{6}(b-a)^2\|g'\|_{[a,b],\infty},\end{cases}$$

where

$$\|h'\|_{[a,b],p} := \left(\int_a^b \|h'(u)\|^p\,du\right)^{1/p},\ p\ge 1$$

and $\|h'\|_{[a,b],\infty} := \sup_{t\in(a,b)}\|h'(u)\|$ for a strongly differentiable function h on (a,b). Some applications for operator monotone function with examples for power function are also given.

2. Main Results

We have the following result of interest:

Theorem 4. *Let $f,\ g:[a,b]\to H$ be a strongly differentiable functions on the interval (a,b). If $\|f'\|_{[a,b],\infty}:=\sup_{t\in(a,b)}\|f'(u)\|<\infty$, then*

$$|D(f,g)|\leq\|f'\|_{[a,b],\infty}D\left(\ell,\int_a^{\cdot}\|g'(u)\|\,du\right)\leq\frac{1}{8}(b-a)^3\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],1}, \tag{2.1}$$

where $\|h'\|_{[a,b],1}:=\int_a^b\|h'(u)\|\,du$ and $\ell(t)=t$.

Proof. Observe that

$$\begin{aligned}&\int_a^b\int_a^b\langle f(t)-f(s),g(t)-g(s)\rangle\,dtds\\&=\int_a^b\int_a^b(\langle f(t),g(t)\rangle-\langle f(s),g(t)\rangle-\langle f(t),g(s)\rangle+\langle f(s),g(s)\rangle)\,dtds\\&=(b-a)\int_a^b\langle f(t),g(t)\rangle\,dt-\left\langle\int_a^b f(s)\,ds,\int_a^b g(t)\,dt\right\rangle\\&-\left\langle\int_a^b f(t)\,dt,\int_a^b g(s)\,ds\right\rangle+(b-a)\int_a^b\langle f(s),g(s)\rangle\,ds\\&=2(b-a)\int_a^b\langle f(t),g(t)\rangle\,dt-2\left\langle\int_a^b f(t)\,dt,\int_a^b g(t)\,dt\right\rangle=2D(f,g),\end{aligned}$$

which give the Korkine's identity for functions with values in Hilbert spaces

$$D(f,g)=\frac{1}{2}\int_a^b\int_a^b\langle f(t)-f(s),g(t)-g(s)\rangle\,dtds.$$

For Korkine's classical identity for real-valued functions, see [17, p. 242].

If we take the modulus and use the integral's properties, we get by Schwarz inequality

$$\begin{aligned}|D(f,g)|&\leq\frac{1}{2}\int_a^b\int_a^b|\langle f(t)-f(s),g(t)-g(s)\rangle|\,dtds\\&\leq\frac{1}{2}\int_a^b\int_a^b\|f(t)-f(s)\|\,\|g(t)-g(s)\|\,dtds.\end{aligned} \tag{2.2}$$

Observe that for $s,\ t\in[a,b]$

$$f(t)-f(s)=\int_s^t f'(u)\,du,\ \ g(t)-g(s)=\int_s^t g'(u)\,du,$$

which implies that

$$\begin{aligned}\|f(t)-f(s)\|\|g(t)-g(s)\| &= \left\|\int_s^t f'(u)\,du\right\|\left\|\int_s^t g'(u)\,du\right\|\\ &\le \left|\int_s^t \|f'(u)\|\,du\right|\left|\int_s^t \|g'(u)\|\,du\right|\\ &\le \sup_{t\in(a,b)}\|f'(u)\|\,|t-s|\left|\int_s^t \|g'(u)\|\,du\right|\\ &= \sup_{t\in(a,b)}\|f'(u)\|\,(t-s)\int_s^t \|g'(u)\|\,du,\end{aligned}$$

for all $s,\ t\in[a,b]$.

By (**??**) we get

$$|D(f,g)|\le \sup_{t\in(a,b)}\|f'(u)\|\frac{1}{2}\int_a^b\int_a^b (t-s)\left(\int_s^t \|g'(u)\|\,du\right)dtds. \tag{2.3}$$

Since

$$(t-s)\left(\int_s^t \|g'(u)\|\,du\right)=(t-s)\left(\int_a^t \|g'(u)\|\,du-\int_a^s \|g'(u)\|\,du\right),$$

hence by Korkine's identity for real valued functions $f(t)=\ell(t)$ and $g(t)=\int_a^t\|g'(u)\|\,du$, we have

$$\begin{aligned}&\frac{1}{2}\int_a^b\int_a^b (t-s)\left(\int_s^t \|g'(u)\|\,du\right)\\ &=(b-a)\int_a^b \ell(t)\left(\int_a^t \|g'(u)\|\,du\right)dt-\int_a^b \ell(t)\,dt\int_a^b\left(\int_a^t \|g'(u)\|\,du\right)dt\\ &=D\left(\ell,\int_a^{\cdot}\|g'(u)\|\,du\right).\end{aligned} \tag{2.4}$$

By utilising (2.3) and (2.4), we deduce the first inequality in (2.1).

Observe that

$$0\le\int_a^t\|g'(u)\|\,du\le\int_a^b\|g'(u)\|\,du$$

for all $t\in[a,b]$, then by (1.8) for the functions $f(t)=\ell(t)$ and $g(t)=\int_a^t\|g'(u)\|\,du$, $t\in[a,b]$, we get

$$\begin{aligned}&\left|D\left(\ell,\int_a^{\cdot}\|g'(u)\|\,du\right)\right|\\ &\le\frac{1}{2}(b-a)\int_a^b\|g'(u)\|\,du\int_a^b\left|t-\frac{1}{b-a}\int_a^b sds\right|dt\\ &=\frac{1}{2}(b-a)\int_a^b\|g'(u)\|\,du\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{1}{8}(b-a)^3\int_a^b\|g'(u)\|\,du,\end{aligned}$$

which proves the first part of (2.1). □

Remark 2. *If we apply the same inequality (1.8) for the functions* $f(t)=\int_a^t \|g'(u)\|\,du$ *and* $g(t)=\ell(t)$, $t\in[a,b]$, *then we get*

$$\left|D\left(\ell,\int_a^{\cdot}\|g'(u)\|\,du\right)\right| \le \frac{1}{2}(b-a)^2\int_a^b\left|\int_a^t\|g'(u)\|\,du-\frac{1}{b-a}\int_a^b\left(\int_a^s\|g'(u)\|\,du\right)ds\right|dt. \tag{2.5}$$

Observe that

$$\begin{aligned}
&\int_a^b\left|\int_a^t\|g'(u)\|\,du-\frac{1}{b-a}\int_a^b\left(\int_a^s\|g'(u)\|\,du\right)ds\right|dt\\
&=\int_a^b\left|\int_a^t\|g'(u)\|\,du-\frac{1}{b-a}\left(\left(\int_a^b\|g'(u)\|\,du\right)b-\int_a^b\|g'(s)\|\,sds\right)\right|dt\\
&=\int_a^b\left|\int_a^t\|g'(u)\|\,du-\frac{1}{b-a}\left(\int_a^b(b-u)\|g'(u)\|\,du\right)\right|dt\\
&=\frac{1}{b-a}\int_a^b\left|(b-a)\int_a^t\|g'(u)\|\,du-\int_a^b(b-u)\|g'(u)\|\,du\right|dt\\
&=\frac{1}{b-a}\int_a^b\left|\int_a^t(u-a)\|g'(u)\|\,du-\int_t^b(b-u)\|g'(u)\|\,du\right|dt.
\end{aligned}$$

Then by (2.1) and (2.5) we obtain

$$\begin{aligned}
|D(f,g)| &\le \|f'\|_{[a,b],\infty}\,D\left(\ell,\int_a^{\cdot}\|g'(u)\|\,du\right)\\
&\le \frac{1}{2}(b-a)\int_a^b\left|\int_a^t(u-a)\|g'(u)\|\,du-\int_t^b(b-u)\|g'(u)\|\,du\right|dt.
\end{aligned} \tag{2.6}$$

Remark 3. *Using (1.3) we have*

$$0\le D\left(\ell,\int_a^{\cdot}\|g'(u)\|\,du\right)\le\frac{1}{12}\sup_{t\in(a,b)}\|g'(u)\|(b-a)^4,$$

and by (2.1) we derive

$$\begin{aligned}
|D(f,g)| &\le \|f'\|_{[a,b],\infty}\,D\left(\ell,\int_a^{\cdot}\|g'(u)\|\,du\right)\\
&\le \frac{1}{12}\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\infty}(b-a)^4
\end{aligned} \tag{2.7}$$

provided that $\|f'\|_{[a,b],\infty}$, $\|g'\|_{[a,b],\infty}<\infty$.

Using (1.4) we have

$$0\le D\left(\ell,\int_a^{\cdot}\|g'(u)\|\,du\right)\le\frac{1}{8}(b-a)^3\int_a^b\|g'(u)\|\,du,$$

and by (2.1) we obtain

$$|D(f,g)| \le \|f'\|_{[a,b],\infty} D\left(\ell, \int_a^{\cdot} \|g'(u)\| du\right) \le \frac{1}{8}(b-a)^3 \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],1}, \tag{2.8}$$

provided that $\|f'\|_{[a,b],\infty} < \infty$.

Corollary 1. *Let* $f, g : [a,b] \to H$ *be a strongly differentiable functions on the interval* (a,b). *If*

$$\|g'\|_{[a,b],r} := \left(\int_a^b \|g'(u)\|^r du\right)^{1/r}, \ r \ge 1,$$

then

$$|D(f,g)| \le \frac{1}{2}(b-a)\|f'\|_{[a,b],\infty} \int_a^b (b-t)(t-a)\|g'(t)\| dt \le \begin{cases} \frac{1}{8}(b-a)^3 \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],1}, \\ \frac{1}{2}(b-a)^{3+1/q}[B(q+1,q+1)]^{1/q} \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],p}, \\ \frac{1}{12}(b-a)^4 \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\infty}, \end{cases} \tag{2.9}$$

where $B(\cdot,\cdot)$ *is Beta function and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Observe that, integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\int_a^b (b-t)(t-a)\|g'(t)\| dt \\
&= \frac{1}{2}\int_a^b (b-t)(t-a)\, d\left(\int_a^t \|g'(u)\| du\right) \\
&= \frac{1}{2}\left[(b-t)(t-a)\int_a^t \|g'(u)\| du\Big|_a^b + \int_a^b (2t-a-b)\left(\int_a^t \|g'(u)\| du\right) dt\right] \\
&= \int_a^b \left(t - \frac{a+b}{2}\right)\left(\int_a^t \|g'(u)\| du\right) dt \\
&= \int_a^b t\left(\int_a^t \|g'(u)\| du\right) dt - \frac{a+b}{2}\int_a^b \left(\int_a^t \|g'(u)\| du\right) dt \\
&= \frac{1}{b-a} D\left(\ell, \int_a^{\cdot} \|g'(u)\| du\right),
\end{aligned}$$

namely

$$D\left(\ell, \int_a^{\cdot} \|g'(u)\| du\right) = \frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\|g'(t)\| dt. \tag{2.10}$$

By utilising the first inequality (2.1) we deduce the first inequality in (2.9).

By Hölder's integral inequality we have for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1,$

$$\begin{aligned}
&\int_a^b (b-t)(t-a)\|g'(t)\|\,dt\\
&\leq \begin{cases} \sup_{t\in[a,b]}[(b-t)(t-a)]\int_a^b \|g'(t)\|\,dt, \\[2ex] \left(\int_a^b[(b-t)(t-a)]^q\,dt\right)^{1/q}\left(\int_a^b\|g'(t)\|^p\,dt\right)^{1/p}, \\[2ex] \int_a^b (b-t)(t-a)\,dt\sup_{t\in[a,b]}\|g'(t)\|, \end{cases}\\
&= \begin{cases} \frac{1}{4}(b-a)^2\int_a^b\|g'(t)\|\,dt, \\[2ex] (b-a)^{2+1/q}[B(q+1,q+1)]^{1/q}\left(\int_a^b\|g'(t)\|^p\,dt\right)^{1/p}, \\[2ex] \frac{1}{6}(b-a)^3\sup_{t\in[a,b]}\|g'(t)\|, \end{cases}
\end{aligned}$$

which proves the last part of (2.9). □

Theorem 5. *Let $f,\ g:[a,b]\to H$ be a strongly differentiable functions on the interval (a,b). If $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\begin{aligned}
|D(f,g)| &\leq \left[D\left(\ell,\int_a^{\cdot}\|f'(u)\|^p\,du\right)\right]^{1/p}\left[D\left(\ell,\int_a^{\cdot}\|f'(u)\|^q\,du\right)\right]^{1/q} \\
&= \frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\|f'(t)\|^p\,dt\right]^{1/p} \\
&\quad\times\left[\int_a^b (b-t)(t-a)\|g'(t)\|^q\,dt\right]^{1/q} \\
&\leq \frac{1}{8}(b-a)^3\|f'\|_{[a,b],p}\|g'\|_{[a,b],q}\,.
\end{aligned}\tag{2.11}$$

In particular, we have for $p=q=2$

$$\begin{aligned}
|D(f,g)| &\leq \left[D\left(\ell,\int_a^{\cdot}\|f'(u)\|^2\,du\right)\right]^{1/2}\left[D\left(\ell,\int_a^{\cdot}\|f'(u)\|^2\,du\right)\right]^{1/2} \\
&= \frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\|f'(t)\|^2\,dt\right]^{1/2} \\
&\quad\times\left[\int_a^b (b-t)(t-a)\|g'(t)\|^2\,dt\right]^{1/2} \\
&\leq \frac{1}{8}(b-a)^3\|f'\|_{[a,b],2}\|g'\|_{[a,b],2}\,.
\end{aligned}\tag{2.12}$$

Proof. Using Hölder's inequality for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\begin{aligned}
&\|f(t)-f(s)\|\,\|g(t)-g(s)\| \\
&=\left\|\int_s^t f'(u)\,du\right\|\left\|\int_s^t g'(u)\,du\right\| \\
&\le\left|\int_s^t \|f'(u)\|\,du\right|\left|\int_s^t \|g'(u)\|\,du\right| \\
&\le |t-s|^{1/q}\left|\int_s^t \|f'(u)\|^p\,du\right|^{1/p}|t-s|^{1/p}\left|\int_s^t \|g'(u)\|^q\,du\right|^{1/q} \\
&=|t-s|\left|\int_s^t \|f'(u)\|^p\,du\right|^{1/p}\left|\int_s^t \|g'(u)\|^q\,du\right|^{1/q}.
\end{aligned}$$

By the weighted Hölder's inequality for double integral, we also have

$$\begin{aligned}
&\int_a^b\int_a^b \|f(t)-f(s)\|\,\|g(t)-g(s)\|\,dtds \\
&\le\int_a^b\int_a^b |t-s|\left|\int_s^t \|f'(u)\|^p\,du\right|^{1/p}\left|\int_s^t \|g'(u)\|^q\,du\right|^{1/q}dtds \\
&\le\left(\int_a^b\int_a^b |t-s|\left(\left|\int_s^t \|f'(u)\|^p\,du\right|^{1/p}\right)^p dtds\right)^{1/p} \\
&\times\left(\int_a^b\int_a^b |t-s|\left(\left|\int_s^t \|g'(u)\|^q\,du\right|^{1/q}\right)^q dtds\right)^{1/q} \\
&=\left(\int_a^b\int_a^b |t-s|\left|\int_s^t \|f'(u)\|^p\,du\right|dtds\right)^{1/p} \\
&\times\left(\int_a^b\int_a^b |t-s|\left|\int_s^t \|g'(u)\|^q\,du\right|dtds\right)^{1/q}.
\end{aligned} \tag{2.13}$$

Observe that

$$\begin{aligned}
&\int_a^b\int_a^b |t-s|\left|\int_s^t \|f'(u)\|^p\,du\right|dtds \\
&=\int_a^b\int_a^b (t-s)\left(\int_s^t \|f'(u)\|^p\,du\right)dtds \\
&=\int_a^b\int_a^b (t-s)\left(\int_a^t \|f'(u)\|^p\,du-\int_a^s \|f'(u)\|^p\,du\right)dtds \\
&=2D\left(\ell,\int_a^{\cdot} \|f'(u)\|^p\,du\right)
\end{aligned}$$

and

$$\int_a^b\int_a^b |t-s|\left|\int_s^t \|g'(u)\|^q\,du\right|dtds=2D\left(\ell,\int_a^{\cdot} \|g'(u)\|^q\,du\right).$$

Therefore, by (**??**)

$$\begin{aligned}|D(f,g)| &\le \frac{1}{2}\int_a^b\int_a^b \|f(t)-f(s)\|\,\|g(t)-g(s)\|\,dtds\\ &\le \frac{1}{2}\left[2D\left(\ell,\int_a^{\cdot}\|f'(u)\|^p du\right)\right]^{1/p}\left[2D\left(\ell,\int_a^{\cdot}\|g'(u)\|^q du\right)\right]^{1/q}\\ &= \left[D\left(\ell,\int_a^{\cdot}\|f'(u)\|^p du\right)\right]^{1/p}\left[D\left(\ell,\int_a^{\cdot}\|g'(u)\|^q du\right)\right]^{1/q}.\end{aligned}$$

From (2.10) we have

$$D\left(\ell,\int_a^{\cdot}\|f'(u)\|^p du\right)=\frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\|f'(t)\|^p dt$$

and

$$D\left(\ell,\int_a^{\cdot}\|g'(u)\|^q du\right)=\frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\|g'(t)\|^q dt.$$

Therefore

$$\begin{aligned}&\left[D\left(\ell,\int_a^{\cdot}\|f'(u)\|^p du\right)\right]^{1/p}\left[D\left(\ell,\int_a^{\cdot}\|g'(u)\|^q du\right)\right]^{1/q}\\ &=\left[\frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\|f'(t)\|^p dt\right]^{1/p}\\ &\times\left[\frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\|g'(t)\|^q dt\right]^{1/q}\\ &=\frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\|f'(t)\|^p dt\right]^{1/p}\left[\int_a^b (b-t)(t-a)\|g'(t)\|^q dt\right]^{1/q}\end{aligned}$$

and the first part of the theorem is proved.

Now, observe that

$$\int_a^b (b-t)(t-a)\|f'(t)\|^p dt \le \frac{1}{4}(b-a)^2\int_a^b \|f'(t)\|^p dt$$

and

$$\int_a^b (b-t)(t-a)\|g'(t)\|^q dt \le \frac{1}{4}(b-a)^2\int_a^b \|g'(t)\|^q dt,$$

which gives the last part of (2.11). □

Remark 4. *Assume that* $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *and* $\gamma,\ \delta>1$ *with* $\frac{1}{\gamma}+\frac{1}{\delta}=1$. *Then by Hölder's inequality we get*

$$\begin{aligned}&\int_a^b (b-t)(t-a)\|f'(t)\|^p dt\\ &\le (b-a)^{2+1/\beta}\left[B(\beta+1,\beta+1)\right]^{1/\beta}\left(\int_a^b \|f'(t)\|^{\alpha p} dt\right)^{1/\alpha}\end{aligned}$$

and

$$\int_a^b (b-t)(t-a)\|g'(t)\|^q dt$$
$$\leq (b-a)^{2+1/\delta}[B(\delta+1,\delta+1)]^{1/\delta}\left(\int_a^b \|g'(t)\|^{\gamma q} dt\right)^{1/\gamma}.$$

Then

$$\left[\int_a^b (b-t)(t-a)\|f'(t)\|^p dt\right]^{1/p}$$
$$\leq (b-a)^{(2\beta+1)/(\beta p)}[B(\beta+1,\beta+1)]^{1/(\beta p)}\left(\int_a^b \|f'(t)\|^{\alpha p} dt\right)^{1/(\alpha p)}$$

and

$$\left[\int_a^b (b-t)(t-a)\|g'(t)\|^q dt\right]^{1/q}$$
$$\leq (b-a)^{(2\delta+1)/(\delta q)}[B(\delta+1,\delta+1)]^{1/(\delta q)}\left(\int_a^b \|g'(t)\|^{\gamma q} dt\right)^{1/(\gamma q)}.$$

Therefore

$$\frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\|f'(t)\|^p dt\right]^{1/p}$$
$$\times\left[\int_a^b (b-t)(t-a)\|g'(t)\|^q dt\right]^{1/q}$$
$$\leq \frac{1}{2}(b-a)(b-a)^{(2\beta+1)/(\beta p)}[B(\beta+1,\beta+1)]^{1/(\beta p)}\left(\int_a^b \|f'(t)\|^{\alpha p} dt\right)^{1/(\alpha p)}$$
$$\times(b-a)^{(2\delta+1)/(\delta q)}[B(\delta+1,\delta+1)]^{1/(\delta q)}\left(\int_a^b \|g'(t)\|^{\gamma q} dt\right)^{1/(\gamma q)}$$
$$= \frac{1}{2}[B(\beta+1,\beta+1)]^{1/(\beta p)}[B(\delta+1,\delta+1)]^{1/(\delta q)}$$
$$\times(b-a)^{1+(2\beta+1)/(\beta p)+(2\delta+1)/(\delta q)}$$
$$\times\left(\int_a^b \|f'(t)\|^{\alpha p} dt\right)^{1/(\alpha p)}\left(\int_a^b \|g'(t)\|^{\gamma q} dt\right)^{1/(\gamma q)}$$

and by (2.11) we get

$$|D(f,g)| \leq \frac{1}{2}[B(\beta+1,\beta+1)]^{1/(\beta p)}[B(\delta+1,\delta+1)]^{1/(\delta q)} \times(b-a)^{1+(2\beta+1)/(\beta p)+(2\delta+1)/(\delta q)}\|f'\|_{[a,b],\alpha p}\|g'\|_{[a,b],\gamma q}, \tag{2.14}$$

where $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *and* $\gamma,\ \delta>1$ *with* $\frac{1}{\gamma}+\frac{1}{\delta}=1$.

3. Applications for Operator Monotone Functions

Assume that $f : I \to \mathbb{R}$ is continuous on the interval I and the selfadjoint operators A, B with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(A) \subset I$. Using the continuos functional calculus for selfadjoint operators on Hilbert spaces, we can define the functions

$$f_{A,B}(t) := f((1-t)A + tB), \; g_{A,B}(t) := g((1-t)A + tB)$$

for $t \in [0,1]$. For $x, y \in H$ we define the Čebyšev functional

$$\begin{aligned} D(f,g,A,B;x,y) &:= \int_0^1 \langle f((1-t)A+tB)x, g((1-t)A+tB)y\rangle dt \\ &\quad - \left\langle \int_0^1 f((1-t)A+tB)xdt, \int_0^1 g((1-t)A+tB)ydt \right\rangle . \end{aligned}$$

A real valued continuous function f on $[0,\infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B \geq 0$.

We have the following representation of operator monotone functions, see for instance [1, p. 144-145]:

Theorem 6. *A function $f : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$f(t) = f(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{3.1}$$

where $b \geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{3.2}$$

Lemma 1. *Let $f : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq 0$, then for all selfadjoint operators V we have*

$$Df(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1}\right] d\mu(\lambda). \tag{3.3}$$

Proof. From (3.1) we get

$$f(t) = f(0) + bt + \int_0^\infty \left(\lambda - \frac{\lambda^2}{t+\lambda}\right) d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, then by the representation of f we have for t in a small open interval around 0 that

$$\begin{aligned} &f(U+tV) - f(U) \\ &= btV + \int_0^\infty \left(\lambda - \lambda^2 (U+tV+\lambda)^{-1}\right) d\mu(\lambda) - \int_0^\infty \left(\lambda - \lambda^2 (U+\lambda)^{-1}\right) d\mu(\lambda) \\ &= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} - (\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\ &= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} (\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\ &= btV + t\int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda). \end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{f(U+tV)-f(U)}{t} = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Df(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1}\right] d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Theorem 7. *Let $f : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators V we have*

$$\|Df(U)(V)\| \leq f'(u)\|V\|. \tag{3.4}$$

Proof. From (3.3) we get

$$\|Df(U)(V) - bV\| \leq \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1} V (\lambda+U)^{-1}\right\| d\mu(\lambda) \tag{3.5}$$
$$\leq \|V\| \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1}\right\|^2 d\mu(\lambda).$$

Observe that $\lambda + U \geq \lambda + u > 0$ for $\lambda \in [0,\infty)$. Then $0 < (\lambda+U)^{-1} \leq (\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\| \leq (\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2 \leq (\lambda+u)^{-2}$ for $\lambda \in [0,\infty)$.

Therefore by (3.5) we get

$$\|Df(U)(V) - bV\| \leq \|V\| \int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$f'(t) = b + \int_0^\infty \frac{\lambda(t+\lambda) - \lambda t}{(t+\lambda)^2} d\mu(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) \tag{3.7}$$

for $t > 0$.

From (3.7) we get

$$\int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda) = f'(u) - b,$$

and by (3.6) we derive

$$\|Df(U)(V) - bV\| \leq \|V\| f'(u) - b\|V\|.$$

Finally, by the triangle inequality and by the fact that $b \geq 0$, we obtain that

$$\|Df(U)(V)\| - b\|V\| \leq \|Df(U)(V) - bV\|,$$

which proves the desired result (3.4). □

For a continuous function f on $(0,\infty)$ and A, $B > 0$ we consider the auxiliary function $f_{A,B} : [0,1] \to \mathbb{R}$ defined by

$$f_{A,B}(t) := f((1-t)A + tB), \ t \in [0,1].$$

We have the following representations of the derivatives:

Lemma 2. *Assume that the operator function generated by f is Fréchet differentiable in each $A \geq 0$, then for $B \geq 0$ we have that $f_{A,B}$ is differentiable on $[0,1]$ and*

$$f'_{A,B}(t) = D(f)((1-t)A + tB)(B - A) \tag{3.8}$$

for $t \in [0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} & \frac{f_{A,B}(t+h) - f(t)}{h} \\ & = \frac{f((1-(t+h))A + (t+h)B) - f((1-t)A + tB)}{h} \\ & = \frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} f'_{A,B}(t) &= \lim_{h\to 0} \frac{f_{A,B}(t+h) - f(t)}{h} \\ &= \lim_{h\to 0} \left[\frac{f((1-t)A + tB + h(B-A)) - f((1-t)A + tB)}{h} \right] \\ &= D(f)((1-t)A + tB)(B-A), \end{aligned}$$

which proves (3.8). □

Corollary 2. *Let $f : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A \geq a > 0$, $B \geq b > 0$ we have*

$$\|D(f)((1-t)A + tB)(B-A)\| \leq f'((1-t)a + tb)\|B - A\| \tag{3.9}$$

for all $t \in [0,1]$.

The proof follows by Theorem 7 and Lemma 2.

One can observe that the inequality (3.9) remains valid for operator monotone functions on $(0,\infty)$. This follows by considering the function $f_{\varepsilon}(t) := f(t+\varepsilon)$ for $\varepsilon > 0$, which is operator monotone on $[0,\infty)$ and then by letting $\varepsilon \to 0+$ and using the continuity of f and f'.

We have the following result:

Theorem 8. *Let f, $g:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A\geq a>0$, $B\geq b>0$ with $a\neq b$, we have*

$$\begin{aligned}(3.10)\qquad & \left|D\left(f,g,A,B;x,y\right)\right| \\ & \leq \left\Vert B-A\right\Vert^{2}\left\Vert x\right\Vert\left\Vert y\right\Vert\sup_{t\in[0,1]}f'\left(\left(1-t\right)a+tb\right) \\ & \times\frac{1}{b-a}\int_{0}^{1}\left(t-\frac{1}{2}\right)g\left(\left(1-t\right)a+tb\right)dt \\ & \leq\frac{1}{2}\left\Vert B-A\right\Vert^{2}\left\Vert x\right\Vert\left\Vert y\right\Vert\sup_{t\in[0,1]}f'\left(\left(1-t\right)a+tb\right) \\ & \times\begin{cases}\frac{1}{4}\frac{g(b)-g(a)}{b-a}, \\ \\ \left[B\left(q+1,q+1\right)\right]^{1/q}\left(\int_{0}^{1}\left[g'\left(\left(1-t\right)a+tb\right)\right]^{p}dt\right)^{1/p}, \\ \\ \frac{1}{6}\sup_{t\in[0,1]}g'\left(\left(1-t\right)a+tb\right),\end{cases}\end{aligned}$$

for all x, $y\in H$, where $B\left(\cdot,\cdot\right)$ is Beta function and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Observe that

$$f'_{A,B}\left(t\right)x=D\left(f\right)\left(\left(1-t\right)A+tB\right)\left(B-A\right)x,$$
$$g'_{A,B}\left(t\right)y=D\left(g\right)\left(\left(1-t\right)A+tB\right)\left(B-A\right)y$$

and by (3.9) we get

$$\left\Vert f'_{A,B}\left(t\right)x\right\Vert\leq f'\left(\left(1-t\right)a+tb\right)\left\Vert B-A\right\Vert\left\Vert x\right\Vert$$

and

$$\left\Vert g'_{A,B}\left(t\right)y\right\Vert\leq g'\left(\left(1-t\right)a+tb\right)\left\Vert B-A\right\Vert\left\Vert y\right\Vert,$$

for all x, $y\in H$.

If we use inequality (2.9), then we get

$$\begin{aligned}(3.11)\qquad & \left|D\left(f,g,A,B;x,y\right)\right| \\ & \leq\frac{1}{2}\left\Vert B-A\right\Vert^{2}\left\Vert x\right\Vert\left\Vert y\right\Vert\sup_{t\in[0,1]}f'\left(\left(1-t\right)a+tb\right) \\ & \times\int_{0}^{1}\left(1-t\right)tg'\left(\left(1-t\right)a+tb\right)dt \\ & \leq\frac{1}{2}\left\Vert B-A\right\Vert^{2}\left\Vert x\right\Vert\left\Vert y\right\Vert\sup_{t\in[0,1]}f'\left(\left(1-t\right)a+tb\right) \\ & \times\begin{cases}\frac{1}{4}\int_{0}^{1}g'\left(\left(1-t\right)a+tb\right)dt, \\ \\ \left[B\left(q+1,q+1\right)\right]^{1/q}\left(\int_{0}^{1}\left[g'\left(\left(1-t\right)a+tb\right)\right]^{p}dt\right)^{1/p}, \\ \\ \frac{1}{6}\sup_{t\in[0,1]}g'\left(\left(1-t\right)a+tb\right),\end{cases}\end{aligned}$$

and since

$$\int_{0}^{1}g'\left(\left(1-t\right)a+tb\right)dt=\frac{g\left(b\right)-g\left(a\right)}{b-a}$$

and for $b \neq a$,

$$\begin{aligned}
&\int_0^1 (1-t)\, t g'((1-t)\, a + tb)\, dt \\
&= \frac{1}{b-a}\int_0^1 (1-t)\, t d\left[g((1-t)\, a + tb)\right] \\
&= \frac{1}{b-a}(1-t)\, t g((1-t)\, a + tb)\big|_0^1 - \frac{1}{b-a}\int_0^1 (1-2t)\, g((1-t)\, a + tb)\, dt \\
&= \frac{2}{b-a}\int_0^1 \left(t - \frac{1}{2}\right) g((1-t)\, a + tb)\, dt,
\end{aligned}$$

hence by (3.11) we derive (3.10). □

Remark 5. *If $b = a$ in Theorem 8, then we get*

$$|D(f, g, A, B; x, y)| \leq \frac{1}{12}\|B - A\|^2 \|x\| \|y\| f'(a)\, g'(a)\,. \tag{3.12}$$

Corollary 3. *Let $f : [0, \infty) \to \mathbb{R}$ be operator monotone in $[0, \infty)$. Then for all $A \geq a > 0$, $B \geq b > 0$, $a \neq b$, we have*

$$\begin{aligned}
0 &\leq \int_0^1 \|f((1-t)\, A + tB)\, x\|^2\, dt - \left\|\int_0^1 f((1-t)\, A + tB)\, x dt\right\|^2 \\
&\leq \|B - A\|^2 \|x\|^2 \sup_{t\in[0,1]} f'((1-t)\, a + tb) \\
&\times \frac{1}{b-a}\int_0^1 \left(t - \frac{1}{2}\right) f((1-t)\, a + tb)\, dt \\
&\leq \frac{1}{2}\|B - A\|^2 \|x\|^2 \sup_{t\in[0,1]} f'((1-t)\, a + tb) \\
&\times \begin{cases} \frac{1}{4}\frac{f(b)-f(a)}{b-a}, \\ [B(q+1, q+1)]^{1/q}\left(\int_0^1 [f'((1-t)\, a + tb)]^p\, dt\right)^{1/p}, \\ \frac{1}{6}\sup_{t\in[0,1]} f'((1-t)\, a + tb)\,, \end{cases}
\end{aligned} \tag{3.13}$$

for all $x \in H$.

If $b = a$, then

$$\begin{aligned}
0 &\leq \int_0^1 \|f((1-t)\, A + tB)\, x\|^2\, dt - \left\|\int_0^1 f((1-t)\, A + tB)\, x dt\right\|^2 \\
&\leq \frac{1}{12}\|B - A\|^2 \|x\|^2 [f'(a)]^2
\end{aligned} \tag{3.14}$$

for all $x \in H$.

The proof follows by Theorem 8 for $g = f$ and $y = x$.

In the following we assume that $A \geq a > 0$, $B \geq b > 0$ with $a \neq b$. Consider the power function $\ell^r(t) = t^r$, $r \in (0, 1)$ which is operator monotone on $[0, \infty)$, then for r, $s\in(0,1)$ we have by Theorem 8 that

$$\begin{aligned}(3.15)\quad & |D(\ell^r,\ell^p,A,B;x,y)| \\ & \le \frac{r}{\min\{a^{1-r},b^{1-r}\}}\|B-A\|^2\|x\|\|y\| \\ & \times\frac{1}{b-a}\int_0^1\left(t-\frac{1}{2}\right)((1-t)a+tb)^s\,dt \\ & \le \frac{r}{2\min\{a^{1-r},b^{1-r}\}}\|B-A\|^2\|x\|\|y\|\times\begin{cases}\frac{1}{4}\frac{b^s-a^s}{b-a}, \\ \\ \frac{1}{6}\frac{s}{\min\{a^{1-s},b^{1-s}\}},\end{cases}\end{aligned}$$

for all x, $y\in H$.

For $b=a$ we have the simpler inequality

$$|D(\ell^r,\ell^p,A,B;x,y)|\le\frac{1}{12}rsa^{r+s-2}\|B-A\|^2\|x\|\|y\|, \tag{3.16}$$

for all x, $y\in H$.

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