SOME INNER PRODUCT OSTROWSKI'S TYPE INEQUALITIES IN HILBERT SPACES

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Abstract. Let H be a complex Hilbert space. Assume that $f,\ g: I \to \mathbb{R}$ are continuous on the interval I and the selfadjoint operators A, B are with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(A) \subset I$. Using the continuos functional calculus for selfadjoint operators on Hilbert spaces, we can define the *Ostrowski type functional*

$$O(f,g,A,B,x,y;u) := \int_0^1 \langle f((1-t)A+tB)x, g((1-t)A+tB)y\rangle dt - \left\langle \int_0^1 f((1-s)A+sB)xds, g((1-u)A+uB)y \right\rangle,$$

where $x,\ y \in H$ and $u \in [0,1]$.

In this paper we show among others that, if f is continuous and g is operator monotone on $[0,\infty)$, then for all $A \geq a > 0$, $B \geq b > 0$,

$$|O(f,g,A,B,x,y;u)| \leq \|B-A\| \|x\| \|y\| \times \begin{cases} \sup_{t\in[0,1]} g'((1-t)a+tb),\ b \neq a \\ g'(a),\ b = a \end{cases}$$

$$\times \begin{cases} \left[\frac{1}{2} + \left|u-\frac{1}{2}\right|\right] \int_0^1 \|f((1-t)A+tB)\| dt, \\ \left[\frac{u^{q+1}+(b-1)^{q+1}}{q+1}\right]^{1/q} \left(\int_0^1 \|f((1-t)A+tB)\|^p dt\right)^{1/p}, \\ p,\ q > 1,\ \frac{1}{p}+\frac{1}{q} = 1, \\ \left[\frac{1}{4} + \left(u-\frac{1}{2}\right)^2\right] \sup_{t\in[0,1]} \|f((1-t)A+tB)\|, \end{cases}$$

for all $u \in [0,1]$.

1. Introduction

In 1938, A. Ostrowski [13], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1. *Let $f:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$. Then*

$$\left| f(u) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{4} + \left(\frac{u-\frac{a+b}{2}}{b-a}\right)^2\right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $u \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

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For a recent survey on Ostrowski's inequality for scalar functions and Lebesgue integral see [8]. For related results see also [1], [2], [7] and [9]-[12].

Let X be a Banach space and $-\infty < a < b < \infty$. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms of vectors or operators acting on X will be denoted by $\|\cdot\|$.

A function $f : [a, b] \to X$ is called *measurable* if there exists a sequence of simple functions $f_n : [a, b] \to X$ which converges punctually almost everywhere on $[a, b]$ at f. We recall also that a measurable function $f : [a, b] \to X$ is *Bochner integrable* if and only if its norm function (i.e. the function $t \mapsto \|f(t)\| : [a, b] \to \mathbb{R}_+$) is Lebesgue integrable on $[a, b]$.

The following generalization of Ostrowski scalar inequality holds [3].

Theorem 2. *Assume that $B : [a, b] \to \mathcal{L}(X)$ is Hölder continuous on $[a, b]$, i.e.,*

$$\|B(t) - B(s)\| \le H|t-s|^{\alpha} \quad \text{for all} \quad t, \ s \in [a, b], \tag{1.2}$$

where $H > 0$ and $\alpha \in (0, 1]$.

If $f : [a, b] \to X$ is Bochner integrable on $[a, b]$, then we have the inequality:

$$\left\| \int_a^b B(s) f(s)\, ds - B(t) \int_a^b f(s)\, ds \right\| \le H \int_a^b |t-s|^{\alpha} \|f(s)\|\, ds \tag{1.3}$$

$$\le H \times \begin{cases} \dfrac{(b-t)^{\alpha+1} + (t-a)^{\alpha+1}}{\alpha+1} \operatorname{esssup}_{t\in[a,b]} \|f(t)\|, \\[2ex] \left[\dfrac{(b-t)^{q\alpha+1} + (t-a)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}} \left(\int_a^b \|f(t)\|^p\, dt\right)^{1/p}, \ p, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \\[2ex] \left[\dfrac{1}{2}(b-a) + \left|t - \dfrac{a+b}{2}\right|\right]^{\alpha} \int_a^b \|f(t)\|\, dt \end{cases}$$

for any $t \in [a, b]$, provided the integrals and $\operatorname{esssup}_{t\in[a,b]}$ *from the right hand side are finite.*

Assume that $f, \ g : I \to \mathbb{R}$ are continuous on the interval I and the selfadjoint operators A, B are with spectra $\operatorname{Sp}(A), \operatorname{Sp}(A) \subset I$. Using the continuos functional calculus for selfadjoint operators on Hilbert spaces, we can define the functions

$$f_{A,B}(t) := f((1-t)A + tB), \ g_{A,B}(t) := g((1-t)A + tB)$$

for $t \in [0, 1]$.

We can define the *Ostrowski type functional*

$$O(f, g, A, B, x, y; u) := \int_0^1 \langle f((1-t)A + tB)x, g((1-t)A + tB)y\rangle\, dt - \left\langle \int_0^1 f((1-s)A + sB)x ds, g((1-u)A + uB)y \right\rangle, \tag{1.4}$$

where $x, \ y \in H$ and $u \in [0, 1]$.

For $u=1/2$, we consider the *mid-point functional*

$$M(f,g,A,B,x,y):=\int_0^1\langle f((1-t)A+tB)x,g((1-t)A+tB)y\rangle dt - \left\langle\int_0^1 f((1-s)A+sB)xds,g\left(\frac{A+B}{2}\right)y\right\rangle, \tag{1.5}$$

where x, $y\in H$.

In this paper we show among others that, if f is continuous and g is operator monotone on $[0,\infty)$, then for all $A\geq a>0$, $B\geq b>0$,

$$\begin{aligned}&|O(f,g,A,B,x,y;u)|\\ &\leq\|B-A\|\,\|x\|\,\|y\|\times\begin{cases}\sup_{t\in[0,1]}g'((1-t)a+tb), & b\neq a,\\ g'(a), & b=a,\end{cases}\\ &\times\begin{cases}\left[\frac{1}{2}+\left|u-\frac{1}{2}\right|\right]\int_0^1\|f((1-t)A+tB)\|\,dt,\\ \left[\frac{u^{q+1}+(b-1)^{q+1}}{q+1}\right]^{1/q}\left(\int_0^1\|f((1-t)A+tB)\|^p\,dt\right)^{1/p},\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \left[\frac{1}{4}+\left(u-\frac{1}{2}\right)^2\right]\sup_{t\in[0,1]}\|f((1-t)A+tB)\|,\end{cases}\end{aligned}$$

for all $u\in[0,1]$. Some applications for power and logarithmic functions are also provided.

2. General Ostrowski Inequalities

We have the following weighted version of Ostrowski's inequality for two functions with values in Hilbert spaces, see also [6]:

Theorem 3. *Assume that f, $g:[a,b]\to\mathcal{B}$, are continuous and g is strongly differentiable on (a,b), then for all $u\in[a,b]$,*

$$\left|\int_a^b\langle f(t),g(t)\rangle dt-\left\langle\int_a^b f(s)\,ds,g(u)\right\rangle\right|\leq B(f,g,u), \tag{2.1}$$

where

$$B(f,g,u):=\int_u^b\left(\int_t^b\|f(s)\|\,ds\right)\|g'(t)\|\,dt+\int_a^u\left(\int_a^t\|f(s)\|\,ds\right)\|g'(t)\|\,dt.$$

We also have the bounds

$$B(f,g,u) \tag{2.2}$$
$$\leq \begin{cases} \int_u^b \|f(s)\| \, ds \int_u^b \|g'(t)\| \, dt, \\ \left[\int_u^b \left(\int_t^b \|f(s)\| \, ds\right)^p dt\right]^{1/p} \left(\int_u^b \|g'(t)\|^q \, dt\right)^{1/q}, \\ \int_u^b \left(\int_t^b \|f(s)\| \, ds\right) dt \sup_{t\in[u,b]} \|g'(t)\|, \end{cases}$$
$$+ \begin{cases} \int_a^u \|f(s)\| \, ds \int_a^u \|g'(t)\| \, dt, \\ \left[\int_a^u \left(\int_a^t \|f(s)\| \, ds\right)^p dt\right]^{1/p} \left(\int_a^u \|g'(t)\|^q \, dt\right)^{1/q}, \\ \int_a^u \left(\int_a^t \|f(s)\| \, ds\right) dt \sup_{t\in[a,u]} \|g'(t)\|, \end{cases}$$

where $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $u \in [a,b]$. Using the integration by parts formula for inner products

$$\int_a^b \langle h'(t), l(t) \rangle \, dt = \langle h(b), l(b) \rangle - \langle h(a), l(a) \rangle - \int_a^b \langle h(t), l'(t) \rangle \, dt,$$

where h, l are strongly differentiable on (a,b), we have

$$\begin{aligned} \int_u^b \left\langle \int_t^b f(s) \, ds, g'(t) \right\rangle dt &= \left\langle \int_t^b f(s) \, ds, g(t) \right\rangle \Bigg]_u^b + \int_u^b \langle f(t), g(t) \rangle \, dt \\ &= -\left\langle \int_u^b f(s) \, ds, g(u) \right\rangle + \int_u^b \langle f(t), g(t) \rangle \, dt \end{aligned}$$

and

$$\begin{aligned} \int_a^u \left\langle \int_a^t f(s) \, ds, g'(t) \right\rangle dt &= \left\langle \int_a^t f(s) \, ds, g(t) \right\rangle \Bigg]_a^u - \int_a^u \langle f(t), g(t) \rangle \, dt \\ &= \left\langle \int_a^u f(s) \, ds, g(u) \right\rangle - \int_a^u \langle f(t), g(t) \rangle \, dt. \end{aligned}$$

By subtracting the second identity from the first, we get

$$\begin{aligned} &\int_u^b \left\langle \int_t^b f(s) \, ds, g'(t) \right\rangle dt - \int_a^u \left\langle \int_a^t f(s) \, ds, g'(t) \right\rangle dt \\ &= \int_u^b \langle f(t), g(t) \rangle \, dt + \int_a^u \langle f(t), g(t) \rangle \, dt \\ &- \left\langle \int_u^b f(s) \, ds, g(u) \right\rangle - \left\langle \int_a^u f(s) \, ds, g(u) \right\rangle \\ &= \int_a^b \langle f(t), g(t) \rangle \, dt - \left\langle \int_a^b f(s) \, ds, g(u) \right\rangle. \end{aligned}$$

Therefore, we get the following identity of interest

$$\begin{aligned}(2.3)\qquad &\int_a^b \langle f(t), g(t)\rangle \, dt - \left\langle \int_a^b f(s)\, ds, g(u)\right\rangle \\ &= \int_u^b \left\langle \int_t^b f(s)\, ds, g'(t)\right\rangle dt - \int_a^u \left\langle \int_a^t f(s)\, ds, g'(t)\right\rangle dt\end{aligned}$$

for all $u \in [a,b]$.

If we take the modulus in (2.3) and using Schwarz's inequality, then we get

$$\begin{aligned}(2.4)\qquad &\left|\int_a^b \langle f(t), g(t)\rangle \, dt - \left\langle \int_a^b f(s)\, ds, g(u)\right\rangle\right| \\ &\le \left|\int_u^b \left\langle \int_t^b f(s)\, ds, g'(t)\right\rangle dt\right| + \left|\int_a^u \left\langle \int_a^t f(s)\, ds, g'(t)\right\rangle dt\right| \\ &\le \int_u^b \left|\left\langle \int_t^b f(s)\, ds, g'(t)\right\rangle\right| dt + \int_a^u \left|\left\langle \int_a^t f(s)\, ds, g'(t)\right\rangle\right| dt \\ &\le \int_u^b \left\|\int_t^b f(s)\, ds\right\| \|g'(t)\|\, dt + \int_a^u \left\|\int_a^t f(s)\, ds\right\| \|g'(t)\|\, dt \\ &\le \int_u^b \left(\int_t^b \|f(s)\|\, ds\right) \|g'(t)\|\, dt + \int_a^u \left(\int_a^t \|f(s)\|\, ds\right) \|g'(t)\|\, dt \\ &= B(f,g,u),\end{aligned}$$

which proves (2.1).

Using Hölder's inequality, we get for $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, that

$$\begin{aligned}&\int_u^b \left(\int_t^b \|f(s)\|\, ds\right) \|g'(t)\|\, dt \\ &\le \begin{cases} \sup_{t\in[u,b]} \left(\int_t^b \|f(s)\|\, ds\right) \int_u^b \|g'(t)\|\, dt, \\ \left[\int_u^b \left(\int_t^b \|f(s)\|\, ds\right)^p dt\right]^{1/p} \left(\int_u^b \|g'(t)\|^q\, dt\right)^{1/q}, \\ \int_u^b \left(\int_t^b \|f(s)\|\, ds\right) dt \sup_{t\in[u,b]} \|g'(t)\|, \end{cases} \\ &= \begin{cases} \left(\int_u^b \|f(s)\|\, ds\right) \int_u^b \|g'(t)\|\, dt, \\ \left[\int_u^b \left(\int_t^b \|f(s)\|\, ds\right)^p dt\right]^{1/p} \left(\int_u^b \|g'(t)\|^q\, dt\right)^{1/q}, \\ \int_u^b \left(\int_t^b \|f(s)\|\, ds\right) dt \sup_{t\in[u,b]} \|g'(t)\| \end{cases}\end{aligned}$$

and

$$\int_a^u \left(\int_a^t \|f(s)\|\,ds\right)\|g'(t)\|\,dt$$
$$\leq \begin{cases} \sup_{t\in[a,u]}\left(\int_a^t \|f(s)\|\,ds\right)\int_a^u \|g'(t)\|\,dt, \\ \left[\int_a^u\left(\int_a^t \|f(s)\|\,ds\right)^p dt\right]^{1/p}\left(\int_a^u \|g'(t)\|^q\,dt\right)^{1/q}, \\ \int_a^u\left(\int_a^t \|f(s)\|\,ds\right)dt\sup_{t\in[a,u]}\|g'(t)\|, \end{cases}$$

$$= \begin{cases} \left(\int_a^u \|f(s)\|\,ds\right)\int_a^u \|g'(t)\|\,dt, \\ \left[\int_a^u\left(\int_a^t \|f(s)\|\,ds\right)^p dt\right]^{1/p}\left(\int_a^u \|g'(t)\|^q\,dt\right)^{1/q}, \\ \int_a^u\left(\int_a^t \|f(s)\|\,ds\right)dt\sup_{t\in[a,u]}\|g'(t)\|. \end{cases}$$

By making use of (2.4), we derive (2.2). □

Corollary 1. *With the assumptions of Theorem 3, we have*

$$\left|\int_a^b \langle f(t), g(t)\rangle\,dt - \left\langle \int_a^b f(s)\,ds, g(u)\right\rangle\right| \tag{2.5}$$
$$\leq \int_u^b \|f(s)\|\,ds\int_u^b \|g'(t)\|\,dt + \int_a^u \|f(s)\|\,ds\int_a^u \|g'(t)\|\,dt$$
$$\leq \begin{cases} \max\left\{\int_u^b \|f(s)\|\,ds, \int_a^u \|f(s)\|\,ds\right\}\int_a^b \|g'(t)\|\,dt \\ \int_a^b \|f(s)\|\,ds\max\left\{\int_u^b \|g'(t)\|\,dt, \int_a^u \|g'(t)\|\,dt\right\} \end{cases}$$
$$\leq \int_a^b \|f(s)\|\,ds\int_a^b \|g'(t)\|\,dt,$$

for all $u \in [a,b]$.

The proof follows by the first branches in the bounds (2.2).

Remark 1. *If* $m \in (a,b)$ *is such that*

$$\int_a^u \|f(s)\|\,ds = \int_u^b \|f(s)\|\,ds = \frac{1}{2}\int_a^b \|f(s)\|\,ds, \tag{2.6}$$

then by (2.5) we get

$$\left|\int_a^b \langle f(t), g(t)\rangle\,dt - \left\langle \int_a^b f(s)\,ds, g(m)\right\rangle\right| \leq \frac{1}{2}\int_a^b \|f(s)\|\,ds\int_a^b \|g'(t)\|\,dt. \tag{2.7}$$

Corollary 2. *With the assumptions of Theorem 3, we have*

$$\begin{aligned}
&\left|\int_a^b \langle f(t), g(t)\rangle dt - \left\langle \int_a^b f(s)\,ds, g(u)\right\rangle\right| \\
&\leq \int_u^b \left(\int_t^b \|f(s)\|\,ds\right) dt \sup_{t\in[u,b]} \|g'(t)\| \\
&+ \int_a^u \left(\int_a^t \|f(s)\|\,ds\right) dt \sup_{t\in[a,u]} \|g'(t)\| \\
&\leq \sup_{t\in[a,b]} \|g'(t)\| \int_a^b |t-u|\,\|f(t)\|\,dt,
\end{aligned} \tag{2.8}$$

for all $u \in [a, b]$.

Proof. From the third branches in the bounds (2.2) we have

$$\begin{aligned}
&\left|\int_a^b \langle f(t), g(t)\rangle dt - \left\langle \int_a^b f(s)\,ds, g(u)\right\rangle\right| \\
&\leq \int_u^b \left(\int_t^b \|f(s)\|\,ds\right) dt \sup_{t\in[u,b]} \|g'(t)\| \\
&+ \int_a^u \left(\int_a^t \|f(s)\|\,ds\right) dt \sup_{t\in[a,u]} \|g'(t)\| \\
&\leq \sup_{t\in[a,b]} \|g'(t)\| \left[\int_u^b \left(\int_t^b \|f(s)\|\,ds\right) dt + \int_a^u \left(\int_a^t \|f(s)\|\,ds\right) dt\right].
\end{aligned} \tag{2.9}$$

Using integration by parts, we have for $u \in [a, b]$ that

$$\begin{aligned}
\int_u^b \left(\int_t^b \|f(s)\|\,ds\right) dt &= \left(\int_t^b \|f(s)\|\,ds\right) t\,\bigg|_u^b + \int_u^b t\,\|f(t)\|\,dt \\
&= \int_u^b t\,\|f(t)\|\,dt - \left(\int_u^b \|f(s)\|\,ds\right) u \\
&= \int_u^b (t-u)\,\|f(t)\|\,dt = \int_u^b |t-u|\,\|f(t)\|\,dt
\end{aligned}$$

and

$$\begin{aligned}
\int_a^u \left(\int_a^t \|f(s)\|\,ds\right) dt &= \left(\int_a^t \|f(s)\|\,ds\right) t\,\bigg|_a^u - \int_a^u t\,\|f(t)\|\,dt \\
&= \left(\int_a^u \|f(s)\|\,ds\right) u - \int_a^u t\,\|f(t)\|\,dt \\
&= \int_a^u (u-t)\,\|f(t)\|\,dt = \int_a^u |t-u|\,\|f(t)\|\,dt,
\end{aligned}$$

which gives that

$$\int_u^b \left(\int_t^b \|f(s)\| \, ds \right) dt + \int_a^u \left(\int_a^t \|f(s)\| \, ds \right) dt$$
$$= \int_u^b |t-u| \|f(t)\| \, dt + \int_a^u |t-u| \|f(t)\| \, dt = \int_a^b |t-u| \|f(t)\| \, dt.$$

By making use of (2.9) we derive (2.8). □

Remark 2. *By making use of Hölder's integral inequality, we have for* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *that*

$$\int_a^b |t-u| \|f(t)\| \, dt \le \begin{cases} \sup_{t\in[a,b]} |t-u| \int_a^b \|f(t)\| \, dt, \\ \left(\int_a^b |t-u|^q \, dt \right)^{1/q} \left(\int_a^b \|f(t)\|^p \, dt \right)^{1/p}, \\ \int_a^b |t-u| \, dt \sup_{t\in[a,b]} \|f(t)\|. \end{cases}$$

Since

$$\sup_{t\in[a,b]} |t-u| = \max\{u-a, b-u\} = \frac{1}{2}(b-a) + \left| u - \frac{a+b}{2} \right|,$$
$$\left(\int_a^b |t-u|^q \, dt \right)^{1/q} = \left[\frac{(u-a)^{q+1} + (b-u)^{q+1}}{q+1} \right]^{1/q}$$

and

$$\int_a^b |t-u| \, dt = \frac{(u-a)^2 + (b-u)^2}{2} = \frac{1}{4}(b-a)^2 + \left(u - \frac{a+b}{2} \right)^2.$$

Then by (2.8) we derive the Ostrowsky type inequalities for functions in Hilbert spaces

$$\left| \int_a^b \langle f(t), g(t) \rangle \, dt - \left\langle \int_a^b f(s) \, ds, g(u) \right\rangle \right| \tag{2.10}$$
$$\le \sup_{t\in[a,b]} \|g'(t)\| \begin{cases} \left[\frac{1}{2}(b-a) + \left| u - \frac{a+b}{2} \right| \right] \int_a^b \|f(t)\| \, dt, \\ \left[\frac{(u-a)^{q+1} + (b-u)^{q+1}}{q+1} \right]^{1/q} \left(\int_a^b \|f(t)\|^p \, dt \right)^{1/p}, \\ \left[\frac{1}{4}(b-a)^2 + \left(u - \frac{a+b}{2} \right)^2 \right] \sup_{t\in[a,b]} \|f(t)\|, \end{cases}$$

for all $u \in [a,b]$.

Corollary 3. *Assume that* f, $h : [a,b] \to \mathcal{B}$, *are continuous and* h *is strongly differentiable on* (a,b). *If there exist vectors* x, $X \in H$ *such that*

$$\operatorname{Re} \langle X - h'(t), h'(t) - x \rangle \ge 0 \text{ for all } t \in (a,b),$$

or, equivalently [5],

$$\left\| h'(t) - \frac{x+X}{2} \right\| \le \frac{1}{2} \|X - x\| \text{ for all } t \in (a,b),$$

then

$$\left|\int_a^b \langle f(t), h(t)\rangle dt - \left\langle \int_a^b (t-u) f(t), \frac{x+X}{2}\right\rangle - \left\langle \int_a^b f(s)\, ds, h(u)\right\rangle\right| \tag{2.11}$$
$$\leq \frac{1}{2}\|X-x\| \times \begin{cases} \left[\frac{1}{2}(b-a) + \left|u - \frac{a+b}{2}\right|\right] \int_a^b \|f(t)\|\, dt, \\ \left[\frac{(u-a)^{q+1}+(b-u)^{q+1}}{q+1}\right]^{1/q} \left(\int_a^b \|f(t)\|^p dt\right)^{1/p}, \\ \left[\frac{1}{4}(b-a)^2 + \left(u-\frac{a+b}{2}\right)^2\right] \sup_{t\in[a,b]} \|f(t)\|, \end{cases}$$

for all $u \in [a,b]$.

In particular,

$$\left|\int_a^b \langle f(t), h(t)\rangle dt - \left\langle \int_a^b \left(t - \frac{a+b}{2}\right) f(t), \frac{x+X}{2}\right\rangle - \left\langle \int_a^b f(s)\, ds, h\left(\frac{a+b}{2}\right)\right\rangle\right| \tag{2.12}$$
$$\leq \frac{1}{4}(b-a)\|X-x\| \times \begin{cases} \int_a^b \|f(t)\|\, dt, \\ \frac{1}{(q+1)^{1/q}}(b-a)^1 \left(\int_a^b \|f(t)\|^p dt\right)^{1/p}, \\ \frac{1}{2}(b-a) \sup_{t\in[a,b]} \|f(t)\|. \end{cases}$$

Proof. Follows by using the inequality (2.10) for $g(t) = h(t) - t\frac{x+X}{2}$, $t \in [a,b]$ and observing that $g'(t) = h'(t) - \frac{x+X}{2}$ while

$$\begin{aligned} &\int_a^b \left\langle f(t), h(t) - t\frac{x+X}{2}\right\rangle dt - \left\langle \int_a^b f(s)\, ds, h(u) - u\frac{x+X}{2}\right\rangle \\ &= \int_a^b \langle f(t), h(t)\rangle dt - \left\langle \int_a^b t f(t), \frac{x+X}{2}\right\rangle \\ &- \left\langle \int_a^b f(s)\, ds, h(u)\right\rangle + u\left\langle \int_a^b f(s)\, ds, \frac{x+X}{2}\right\rangle \\ &= \int_a^b \langle f(t), h(t)\rangle dt - \left\langle \int_a^b (t-u) f(t), \frac{x+X}{2}\right\rangle - \left\langle \int_a^b f(s)\, ds, h(u)\right\rangle. \end{aligned}$$

□

Remark 3. *If, in addition to the assumptions in Corollary 3, we have that* f *is symmetric in* $[a,b]$, *namely* $f(a+b-t) = f(t)$ *for all* $t \in [a,b]$, *then*

$$\int_a^b \left(t - \frac{a+b}{2}\right) f(t) = 0$$

and by (2.12) we derive the mid-point inequality

$$\left|\int_a^b \langle f(t), h(t)\rangle dt - \left\langle \int_a^b f(s)\,ds, h\left(\frac{a+b}{2}\right)\right\rangle\right| \tag{2.13}$$
$$\leq \frac{1}{4}(b-a)\|X-x\| \times \begin{cases} \int_a^b \|f(t)\|\,dt, \\ \frac{1}{(q+1)^{1/q}}(b-a)^{1/q}\left(\int_a^b \|f(t)\|^p dt\right)^{1/p}, \\ \frac{1}{2}(b-a)\sup_{t\in[a,b]}\|f(t)\|. \end{cases}$$

We also have:

Corollary 4. *With the assumptions of Theorem 3, we have for all* $u \in [a,b]$,

$$\left|\int_a^b \langle f(t), g(t)\rangle dt - \left\langle \int_a^b f(s)\,ds, g(u)\right\rangle\right| \tag{2.14}$$
$$\leq \left[\left(\int_u^b \|f(s)\|\,ds\right)^p (b-u) + \left(\int_a^u \|f(s)\|\,ds\right)^p (u-a)\right]^{1/p} \times \left(\int_a^b \|g'(t)\|^q dt\right)^{1/q}$$
$$\leq (b-a)^{1/p}\left[\left(\int_u^b \|f(s)\|\,ds\right)^p + \left(\int_a^u \|f(s)\|\,ds\right)^p\right]^{1/p} \times \left(\int_a^b \|g'(t)\|^q dt\right)^{1/q}$$

for $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Observe that, by the elementary inequality for $a,\ b,\ c,\ d \geq 0$ and $p,\ q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$(ab+cd) \leq (a^p+c^p)^{1/p}(b^q+d^q)^{1/q},$$

we have

$$\left[\int_u^b \left(\int_t^b \|f(s)\|\,ds\right)^p dt\right]^{1/p}\left(\int_u^b \|g'(t)\|^q dt\right)^{1/q} \tag{2.15}$$
$$+\left[\int_a^u \left(\int_a^t \|f(s)\|\,ds\right)^p dt\right]^{1/p}\left(\int_a^u \|g'(t)\|^q dt\right)^{1/q}$$
$$\leq \left[\int_u^b \left(\int_t^b \|f(s)\|\,ds\right)^p dt + \int_a^u \left(\int_a^t \|f(s)\|\,ds\right)^p dt\right]^{1/p} \times \left[\int_u^b \|g'(t)\|^q dt + \int_a^u \|g'(t)\|^q dt\right]^{1/q}$$

$$= \left[\int_u^b \left(\int_t^b \|f(s)\| \, ds \right)^p dt + \int_a^u \left(\int_a^t \|f(s)\| \, ds \right)^p dt \right]^{1/p}$$
$$\times \left(\int_a^b \|g'(t)\|^q \, dt \right)^{1/q}$$
$$\leq \left[\left(\int_u^b \|f(s)\| \, ds \right)^p \int_u^b dt + \left(\int_a^u \|f(s)\| \, ds \right)^p \int_a^u dt \right]^{1/p}$$
$$\times \left(\int_a^b \|g'(t)\|^q \, dt \right)^{1/q}$$

$$= \left[\left(\int_u^b \|f(s)\| \, ds \right)^p (b-u) + \left(\int_a^u \|f(s)\| \, ds \right)^p (u-a) \right]^{1/p}$$
$$\times \left(\int_a^b \|g'(t)\|^q \, dt \right)^{1/q}$$
$$\leq (b-a)^{1/p} \left[\left(\int_u^b \|f(s)\| \, ds \right)^p + \left(\int_a^u \|f(s)\| \, ds \right)^p \right]^{1/p}$$
$$\times \left(\int_a^b \|g'(t)\|^q \, dt \right)^{1/q},$$

which proves (2.14). □

Remark 4. *If $m \in (a,b)$ is such that (2.6) is valid, then by (2.14) we get*

$$\left| \int_a^b \langle f(t), g(t) \rangle \, dt - \left\langle \int_a^b f(s) \, ds, g(m) \right\rangle \right| \leq \frac{1}{2} (b-a)^{1/p} \int_a^b \|f(s)\| \, ds \left(\int_a^b \|g'(t)\|^q \, dt \right)^{1/q}. \tag{2.16}$$

Remark 5. *With the assumptions of Theorem 3 we have the mid-point inequality*

$$\left| \int_a^b \langle f(t), g(t) \rangle \, dt - \left\langle \int_a^b f(s) \, ds, g\left(\frac{a+b}{2} \right) \right\rangle \right| \leq M(f,g), \tag{2.17}$$

where

$$M(f,g) := \int_{\frac{a+b}{2}}^b \left(\int_t^b \|f(s)\| \, ds \right) \|g'(t)\| \, dt + \int_a^{\frac{a+b}{2}} \left(\int_a^t \|f(s)\| \, ds \right) \|g'(t)\| \, dt.$$

We also have the bounds

$$M(f,g) \leq \begin{cases} \left(\int_{\frac{a+b}{2}}^{b} \|f(s)\| \, ds\right) \int_{\frac{a+b}{2}}^{b} \|g'(t)\| \, dt, \\ \left[\int_{\frac{a+b}{2}}^{b} \left(\int_{t}^{b} \|f(s)\| \, ds\right)^{p} dt\right]^{1/p} \left(\int_{\frac{a+b}{2}}^{b} \|g'(t)\|^{q} \, dt\right)^{1/q}, \\ \int_{\frac{a+b}{2}}^{b} \left(\int_{t}^{b} \|f(s)\| \, ds\right) dt \sup_{t\in\left[\frac{a+b}{2},b\right]} \|g'(t)\|, \end{cases} + \begin{cases} \left(\int_{a}^{\frac{a+b}{2}} \|f(s)\| \, ds\right) \int_{a}^{\frac{a+b}{2}} \|g'(t)\| \, dt, \\ \left[\int_{a}^{\frac{a+b}{2}} \left(\int_{a}^{t} \|f(s)\| \, ds\right)^{p} dt\right]^{1/p} \left(\int_{a}^{\frac{a+b}{2}} \|g'(t)\|^{q} \, dt\right)^{1/q}, \\ \int_{a}^{\frac{a+b}{2}} \left(\int_{a}^{t} \|f(s)\| \, ds\right) dt \sup_{t\in\left[a,\frac{a+b}{2}\right]} \|g'(t)\|. \end{cases} \tag{2.18}$$

Making use of (2.5), we get

$$\begin{aligned} &\left\|\int_{a}^{b} f(t)\, g(t)\, dt - \left(\int_{a}^{b} f(s)\, ds\right) g\left(\frac{a+b}{2}\right)\right\| \\ &\leq \left(\int_{\frac{a+b}{2}}^{b} \|f(s)\| \, ds\right) \int_{\frac{a+b}{2}}^{b} \|g'(t)\| \, dt \\ &+ \left(\int_{a}^{\frac{a+b}{2}} \|f(s)\| \, ds\right) \int_{a}^{\frac{a+b}{2}} \|g'(t)\| \, dt \\ &\leq \begin{cases} \max\left\{\int_{\frac{a+b}{2}}^{b} \|f(s)\| \, ds, \int_{a}^{\frac{a+b}{2}} \|f(s)\| \, ds\right\} \int_{a}^{b} \|g'(t)\| \, dt \\ \int_{a}^{b} \|f(s)\| \, ds \max\left\{\int_{\frac{a+b}{2}}^{b} \|g'(t)\| \, dt, \int_{a}^{\frac{a+b}{2}} \|g'(t)\| \, dt\right\} \end{cases} \\ &\leq \int_{a}^{b} \|f(s)\| \, ds \int_{a}^{b} \|g'(t)\| \, dt \end{aligned} \tag{2.19}$$

and by (2.8),

$$\begin{aligned} &\left\|\int_{a}^{b} f(t)\, g(t)\, dt - \left(\int_{a}^{b} f(s)\, ds\right) g\left(\frac{a+b}{2}\right)\right\| \\ &\leq \int_{\frac{a+b}{2}}^{b} \left(\int_{t}^{b} \|f(s)\| \, ds\right) dt \sup_{t\in\left[\frac{a+b}{2},b\right]} \|g'(t)\| \\ &+ \int_{a}^{\frac{a+b}{2}} \left(\int_{a}^{t} \|f(s)\| \, ds\right) dt \sup_{t\in\left[a,\frac{a+b}{2}\right]} \|g'(t)\| \\ &\leq \sup_{t\in[a,b]} \|g'(t)\| \int_{a}^{b} \left|t - \frac{a+b}{2}\right| \|f(t)\| \, dt. \end{aligned}$$

From (2.10) we derive the mid-point type inequalities for functions with values in Hilbert spaces

$$\left|\int_a^b \langle f(t), g(t)\rangle\, dt - \left\langle \int_a^b f(s)\, ds, g\left(\frac{a+b}{2}\right)\right\rangle\right| \tag{2.20}$$
$$\leq \sup_{t\in[a,b]} \|g'(t)\| \begin{cases} \frac{1}{2}(b-a)\int_a^b \|f(t)\|\, dt, \\ \\ \frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}}\left(\int_a^b \|f(t)\|^p\, dt\right)^{1/p}, \\ \\ \frac{1}{4}(b-a)^2 \sup_{t\in[a,b]}\|f(t)\|. \end{cases}$$

By (2.14) we obtain that

$$\left|\int_a^b \langle f(t), g(t)\rangle\, dt - \left\langle \int_a^b f(s)\, ds, g\left(\frac{a+b}{2}\right)\right\rangle\right| \tag{2.21}$$
$$\leq \frac{(b-a)^{1/p}}{2^{1/p}}\left[\left(\int_{\frac{a+b}{2}}^b \|f(s)\|\, ds\right)^p + \left(\int_a^{\frac{a+b}{2}} \|f(s)\|\, ds\right)^p\right]^{1/p}$$
$$\times\left(\int_a^b \|g'(t)\|^q\, dt\right)^{1/q}.$$

3. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A)\geq h(B)$ holds for any $A\geq B\geq 0$.

We have the following representation of operator monotone functions, see for instance [4, p. 144-145]:

Theorem 4. *A function $h:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$h(t) = h(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda}\, d\mu(\lambda), \tag{3.1}$$

where $b\geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}\, d\mu(\lambda) < \infty. \tag{3.2}$$

Lemma 1. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U\geq 0$, then for all selfadjoint operators V we have*

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2\left[(\lambda+U)^{-1} V (\lambda+U)^{-1}\right] d\mu(\lambda). \tag{3.3}$$

Proof. From (3.1) we get

$$h(t) = h(0) + bt + \int_0^\infty \left(\lambda - \frac{\lambda^2}{t+\lambda}\right) d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned} & h(U+tV) - h(U) \\ & = btV + \int_0^\infty \left(\lambda - \lambda^2 (U+tV+\lambda)^{-1}\right) d\mu(\lambda) - \int_0^\infty \left(\lambda - \lambda^2 (U+\lambda)^{-1}\right) d\mu(\lambda) \\ & = btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} - (\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\ & = btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} (\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\ & = btV + t\int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda). \end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{h(U+tV) - h(U)}{t} = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1}\right] d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Theorem 5. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators V we have*

$$\|Dh(U)(V)\| \leq h'(u) \|V\|. \tag{3.4}$$

Proof. From (3.3) we get

$$\begin{aligned} \|Dh(U)(V) - bV\| & \leq \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1} V (\lambda+U)^{-1}\right\| d\mu(\lambda) \\ & \leq \|V\| \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1}\right\|^2 d\mu(\lambda). \end{aligned} \tag{3.5}$$

Observe that $\lambda + U \geq \lambda + u > 0$ for $\lambda \in [0,\infty)$. Then $0 < (\lambda+U)^{-1} \leq (\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\| \leq (\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2 \leq (\lambda+u)^{-2}$ for $\lambda \in [0,\infty)$.

Therefore by (3.5) we get

$$\|Dh(U)(V) - bV\| \leq \|V\| \int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1), then we have

$$h'(t) = b + \int_0^\infty \frac{\lambda(t+\lambda) - \lambda t}{(t+\lambda)^2} d\mu(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) \tag{3.7}$$

for $t > 0$.

From (3.7) we get

$$\int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda) = h'(u) - b,$$

and by (3.6) we derive

$$\|Dh(U)(V) - bV\| \leq \|V\| h'(u) - b\|V\|.$$

Finally, by the triangle inequality and by the fact that $b \geq 0$, we obtain that

$$\|Dh(U)(V)\| - b\|V\| \leq \|Dh(U)(V) - bV\|,$$

which proves the desired result (3.4). □

For a continuous function h on $(0,\infty)$ and $A,\ B > 0$ we consider the auxiliary function $h_{A,B} : [0,1] \to \mathbb{R}$ defined by

$$h_{A,B}(t) := h((1-t)A + tB),\ t \in [0,1].$$

We have the following representations of the derivatives:

Lemma 2. *Assume that the operator function generated by h is Fréchet differentiable in each $A \geq 0$, then for $B \geq 0$ we have that $h_{A,B}$ is differentiable on $[0,1]$ and*

$$h'_{A,B}(t) = D(h)((1-t)A + tB)(B - A) \tag{3.8}$$

for $t \in [0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned}
&\frac{h_{A,B}(t+h) - h(t)}{h} \\
&= \frac{h((1-(t+h))A + (t+h)B) - h((1-t)A + tB)}{h} \\
&= \frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}
h'_{A,B}(t) &= \lim_{h\to 0} \frac{h_{A,B}(t+h) - h(t)}{h} \\
&= \lim_{h\to 0} \left[\frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h}\right] \\
&= D(h)((1-t)A + tB)(B - A),
\end{aligned}$$

which proves (3.8). □

Corollary 5. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A \geq a > 0$, $B \geq b > 0$ we have*

$$\begin{aligned}
\left\|h'_{A,B}(t)\right\| &= \|D(h)((1-t)A + tB)(B - A)\| \\
&\leq h'((1-t)a + tb)\|B - A\|
\end{aligned} \tag{3.9}$$

for all $t \in [0,1]$.

The proof follows by Theorem 5 and Lemma 2.

One can observe that the inequality (3.9) remains valid for operator monotone functions on $(0,\infty)$. This follows by considering the function $h_\varepsilon(t) := h(t+\varepsilon)$ for $\varepsilon > 0$, which is operator monotone on $[0,\infty)$ and then by letting $\varepsilon \to 0+$ and using the continuity of h and h'.

We have the following result:

Theorem 6. *Let f be continuous on $[0,\infty)$ and g be operator monotone in $[0,\infty)$. Then for all $A\geq a>0,\ B\geq b>0$, we have for all $x,\ y\in H$ that*

$$\begin{aligned}
&|O(f,g,A,B,x,y;u)| \\
&\leq \|B-A\|\,\|x\|\,\|y\| \\
&\times \max\left\{\int_u^1 \|f((1-t)A+tB)\|\,ds, \int_0^u \|f((1-t)A+tB)\|\,ds\right\} \\
&\times \begin{cases} \frac{g(b)-g(a)}{b-a} & \text{if } b\neq a, \\ g'(a) & \text{if } b=a \end{cases}
\end{aligned} \tag{3.10}$$

for all $u\in[0,1]$.

In particular, we have for all $x,\ y\in H$ that

$$\begin{aligned}
&|M(f,g,A,B,x,y)| \\
&\leq \|B-A\|\,\|x\|\,\|y\| \\
&\times \max\left\{\int_{1/2}^1 \|f((1-t)A+tB)\|\,ds, \int_0^{1/2} \|f((1-t)A+tB)\|\,ds\right\} \\
&\times \begin{cases} \frac{g(b)-g(a)}{b-a} & \text{if } b\neq a, \\ g'(a) & \text{if } b=a. \end{cases}
\end{aligned} \tag{3.11}$$

Proof. From the inequality (2.5) we get

$$\begin{aligned}
&\left|\int_0^1 \langle f((1-t)A+tB)x, g((1-t)A+tB)y\rangle\,dt\right. \\
&\left.-\left\langle \int_0^1 f((1-s)A+sB)x\,ds, g((1-u)A+uB)y\right\rangle\right| \\
&\leq \max\left\{\int_u^1 \|f((1-t)A+tB)x\|\,ds, \int_a^u \|f((1-t)A+tB)x\|\,ds\right\} \\
&\times \int_a^b \left\|g'_{A,B}(t)y\right\|\,dt \\
&\leq \|x\| \max\left\{\int_u^1 \|f((1-t)A+tB)x\|\,ds, \int_a^u \|f((1-t)A+tB)x\|\,ds\right\} \\
&\times \|y\| \int_0^1 \left\|g'_{A,B}(t)\right\|\,dt \\
&\leq \|B-A\|\,\|x\|\,\|y\| \\
&\times \max\left\{\int_u^1 \|f((1-t)A+tB)x\|\,ds, \int_a^u \|f((1-t)A+tB)x\|\,ds\right\} \\
&\times \int_0^1 g'((1-t)a+tb)\,dt.
\end{aligned} \tag{3.12}$$

Since

$$\int_0^1 g'((1-t)a+tb)\,dt = \begin{cases} \frac{g(b)-g(a)}{b-a} & \text{if } b\neq a, \\ g'(a) & \text{if } b=a, \end{cases}$$

hence by (3.12) we get (3.10). □

By using (2.10) we can state the following result as well:

Theorem 7. *Let f be continuous on $[0,\infty)$ and g be operator monotone in $[0,\infty)$. Then for all $A\geq a>0$, $B\geq b>0$, we have for all x, $y\in H$ that*

$$
\begin{aligned}
&|O(f,g,A,B,x,y;u)| \\
&\leq \|B-A\|\,\|x\|\,\|y\|\times\begin{cases}\sup_{t\in[0,1]} g'((1-t)a+tb), & b\neq a,\\ g'(a), & b=a,\end{cases}\\
&\times\begin{cases}\left[\frac{1}{2}+\left|u-\frac{1}{2}\right|\right]\int_0^1\|f((1-t)A+tB)\|\,dt,\\ \left[\frac{u^{q+1}+(b-1)^{q+1}}{q+1}\right]^{1/q}\left(\int_0^1\|f((1-t)A+tB)\|^p\,dt\right)^{1/p},\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \left[\frac{1}{4}+\left(u-\frac{1}{2}\right)^2\right]\sup_{t\in[0,1]}\|f((1-t)A+tB)\|,\end{cases}
\end{aligned}
\tag{3.13}
$$

for all $u\in[0,1]$.

In particular, we have for all x, $y\in H$ that

$$
\begin{aligned}
&|M(f,g,A,B,x,y)| \\
&\leq \frac{1}{2}\|B-A\|\,\|x\|\,\|y\|\times\begin{cases}\sup_{t\in[0,1]} g'((1-t)a+tb), & b\neq a,\\ g'(a), & b=a,\end{cases}\\
&\times\begin{cases}\int_0^1\|f((1-t)A+tB)\|\,dt,\\ \frac{1}{(q+1)^{1/q}}\left(\int_0^1\|f((1-t)A+tB)\|^p\,dt\right)^{1/p},\\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \frac{1}{2}\sup_{t\in[0,1]}\|f((1-t)A+tB)\|.\end{cases}
\end{aligned}
\tag{3.14}
$$

4. Some Examples

We consider the function $f(t)=\ell^r(t)=t^r$ for $r\in(0,1)$. Then for A, $B\geq 0$ and $t\in[0,1]$ we have

$$\|((1-t)A+tB)^r\|\leq\|(1-t)A+tB\|^r\leq[(1-t)\|A\|+t\|B\|]^r.$$

Therefore

$$
\begin{aligned}
\int_0^1\|f((1-t)A+tB)\|\,dt &\leq \int_0^1[(1-t)\|A\|+t\|B\|]^r\\
&= \begin{cases}\frac{\|B\|^{r+1}-\|A\|^{r+1}}{(r+1)(\|B\|-\|A\|)} \text{ if } \|B\|\neq\|A\|\\ \|A\|^r \text{ if } \|B\|=\|A\|.\end{cases}
\end{aligned}
$$

Also

$$\int_0^1 \|((1-t)A+tB)^r\|^p dt \le \int_0^1 [(1-t)\|A\|+t\|B\|]^{pr} = \begin{cases} \frac{\|B\|^{pr+1}-\|A\|^{pr+1}}{(pr+1)(\|B\|-\|A\|)} \text{ if } \|B\|\neq\|A\| \\ \\ \|A\|^{pr} \text{ if } \|B\|=\|A\| \end{cases}$$

and

$$\sup_{t\in[0,1]}\|((1-t)A+tB)^r\| \le \sup_{t\in[0,1]}[(1-t)\|A\|+t\|B\|]^r = \max\{\|A\|^r,\|B\|^r\}.$$

Let $g:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A\ge a>0$, $B\ge b>0$, we have by Theorem 7 for all x, $y\in H$ that

$$\begin{aligned}(4.1)\qquad &\left|\int_0^1 \langle((1-t)A+tB)^r x, g((1-t)A+tB)y\rangle dt\right.\\ &\left.-\left\langle\int_0^1((1-s)A+sB)^r xds, g((1-u)A+uB)y\right\rangle\right|\end{aligned}$$

$$\le \|B-A\|\|x\|\|y\|\times\begin{cases}\sup_{t\in[0,1]} g'((1-t)a+tb),\ b\neq a,\\ \\ g'(a),\ b=a,\end{cases}$$

$$\times\begin{cases}\left[\frac{1}{2}+\left|u-\frac{1}{2}\right|\right]\times\begin{cases}\frac{\|B\|^{r+1}-\|A\|^{r+1}}{(r+1)(\|B\|-\|A\|)} \text{ if } \|B\|\neq\|A\|,\\ \\ \|A\|^r \text{ if } \|B\|=\|A\|,\end{cases}, \\ \\ \left[\frac{u^{q+1}+(b-1)^{q+1}}{q+1}\right]^{1/q}\times\begin{cases}\left(\frac{\|B\|^{pr+1}-\|A\|^{pr+1}}{(pr+1)(\|B\|-\|A\|)}\right)^{1/p} \text{ if } \|B\|\neq\|A\| \\ \\ \|A\|^r \text{ if } \|B\|=\|A\|\end{cases}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\frac{1}{4}+\left(u-\frac{1}{2}\right)^2\right]\max\{\|A\|^r,\|B\|^r\},\end{cases}$$

and, in particular, the mid-point inequality

$$\begin{aligned}(4.2)\qquad &\left|\int_0^1 \langle((1-t)A+tB)^r x, g((1-t)A+tB)y\rangle dt\right.\\ &\left.-\left\langle\int_0^1((1-s)A+sB)^r xds, g\left(\frac{A+B}{2}\right)y\right\rangle\right|\end{aligned}$$

$$\leq \frac{1}{2}\|B-A\|\|x\|\|y\| \times \begin{cases} \sup_{t\in[0,1]} g'((1-t)a+tb), & b\neq a, \\ g'(a), & b=a, \end{cases}$$

$$\times \begin{cases} \begin{cases} \frac{\|B\|^{r+1}-\|A\|^{r+1}}{(r+1)(\|B\|-\|A\|)} \text{ if } \|B\|\neq\|A\|, \\ \|A\|^r \text{ if } \|B\|=\|A\|, \end{cases}, \\ \frac{1}{(q+1)^{1/q}} \begin{cases} \left(\frac{\|B\|^{pr+1}-\|A\|^{pr+1}}{(pr+1)(\|B\|-\|A\|)}\right)^{1/p} \text{ if } \|B\|\neq\|A\|, \\ \|A\|^r \text{ if } \|B\|=\|A\|, \end{cases} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}\max\{\|A\|^r,\|B\|^r\}. \end{cases}$$

Further, if we take $g(t)=\ln t$, which is operator monotone on $(0,\infty)$, then we derive

$$\left|\int_0^1 \langle((1-t)A+tB)^r x, \ln((1-t)A+tB)y\rangle dt - \left\langle \int_0^1 ((1-s)A+sB)^r x ds, \ln((1-u)A+uB)y\right\rangle\right| \tag{4.3}$$

$$\leq \|B-A\|\|x\|\|y\| \times \begin{cases} \frac{1}{\min\{a,b\}} \text{ if } b\neq a, \\ \frac{1}{a}, \text{ if } b=a, \end{cases}$$

$$\times \begin{cases} \left[\frac{1}{2}+\left|u-\frac{1}{2}\right|\right] \times \begin{cases} \frac{\|B\|^{r+1}-\|A\|^{r+1}}{(r+1)(\|B\|-\|A\|)} \text{ if } \|B\|\neq\|A\|, \\ \|A\|^r \text{ if } \|B\|=\|A\|, \end{cases} \\ \left[\frac{u^{q+1}+(b-1)^{q+1}}{q+1}\right]^{1/q} \times \begin{cases} \left(\frac{\|B\|^{pr+1}-\|A\|^{pr+1}}{(pr+1)(\|B\|-\|A\|)}\right)^{1/p} \text{ if } \|B\|\neq\|A\|, \\ \|A\|^r \text{ if } \|B\|=\|A\|, \end{cases} \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \left[\frac{1}{4}+\left(u-\frac{1}{2}\right)^2\right]\max\{\|A\|^r,\|B\|^r\}, \end{cases}$$

and, in particular, the mid-point inequality

$$\left|\int_0^1 \langle((1-t)A+tB)^r x, \ln((1-t)A+tB)y\rangle dt - \left\langle \int_0^1 ((1-s)A+sB)^r x ds, \ln\left(\frac{A+B}{2}\right)y\right\rangle\right| \tag{4.4}$$

$$\leq \frac{1}{2}\left\Vert B-A\right\Vert \left\Vert x\right\Vert \left\Vert y\right\Vert \times \left\{ \begin{array}{l} \frac{1}{\min\{a,b\}},\ \text{if } b\neq a, \\ \\ \frac{1}{a},\ \text{if } b=a, \end{array}\right.$$

$$\times \left\{ \begin{array}{l} \left\{ \begin{array}{l} \frac{\left\Vert B\right\Vert^{r+1}-\left\Vert A\right\Vert^{r+1}}{(r+1)(\left\Vert B\right\Vert -\left\Vert A\right\Vert)} \text{ if } \left\Vert B\right\Vert \neq \left\Vert A\right\Vert, \\ \\ \left\Vert A\right\Vert^{r} \text{ if } \left\Vert B\right\Vert = \left\Vert A\right\Vert, \end{array}\right. \\ \\ \frac{1}{(q+1)^{1/q}}\left\{ \begin{array}{l} \left(\frac{\left\Vert B\right\Vert^{pr+1}-\left\Vert A\right\Vert^{pr+1}}{(pr+1)(\left\Vert B\right\Vert -\left\Vert A\right\Vert)}\right)^{1/p} \text{ if } \left\Vert B\right\Vert \neq \left\Vert A\right\Vert, \\ \\ \left\Vert A\right\Vert^{r} \text{ if } \left\Vert B\right\Vert = \left\Vert A\right\Vert, \end{array}\right. \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{1}{2}\max\left\{\left\Vert A\right\Vert^{r},\left\Vert B\right\Vert^{r}\right\}. \end{array}\right.$$

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