INNER PRODUCT TRAPEZOID TYPE INEQUALITIES IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a complex Hilbert space. In this paper we show among others that, if $f:[a,b]\to H$ is continuous and $g:[a,b]\to H$ is strongly differentiable on (a,b), then

$$\begin{aligned}&\left|\left\langle \int_a^b f(s)\,ds, \frac{g(a)+g(b)}{2}\right\rangle - \int_a^b \langle f(t), g(t)\rangle\,dt\right|\\ &\le \frac{1}{2}\int_a^b \left\|\int_a^t f(s)\,ds - \int_t^b f(s)\,ds\right\| \|g'(t)\|\,dt\\ &\le \frac{1}{2}\int_a^b \|f(s)\|\,dt \times \begin{cases} \int_a^b \|g'(t)\|\,dt, \\ (b-a)^{1/p}\left(\int_a^b \|g'(t)\|^q\,dt\right)^{1/q}, \\ (b-a)\sup_{t\in[a,b]}\|g'(t)\|. \end{cases}\end{aligned}$$

Applications for operator monotone functions with examples for power and logarithmic functions are also given.

1. Introduction

In Classical Analysis a *trapezoidal type inequality* is an inequality that provides upper and/or lower bounds for the quantity

$$\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)\,dt,$$

that is the error in approximating the integral by a trapezoidal rule, for various classes of integrable functions f defined on the compact interval $[a,b]$.

If we would assume absolute continuity for the function f, then the following estimates in terms of the Lebesgue norms of the derivative f' hold [3, p. 93].

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function on $[a,b]$. Then we have*

$$\left|\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2}(b-a)\right| \tag{1.1}$$

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$$\leq \begin{cases} \frac{1}{4}(b-a)^2 \|f'\|_\infty & \text{if} \quad f' \in L_\infty[a,b]; \\ \\ \frac{1}{2(q+1)^{\frac{1}{q}}}(b-a)^{1+1/q} \|f'\|_p & \text{if} \quad f' \in L_p[a,b], \\ & \quad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{2}(b-a)\|f'\|_1, \end{cases}$$

where $\|\cdot\|_p$ $(p \in [1,\infty])$ *are the Lebesgue norms, i.e.,*

$$\|f'\|_\infty = ess \sup_{s\in[a,b]} |f'(s)|$$

and

$$\|f'\|_p := \left(\int_a^b |f'(s)|\, ds \right)^{\frac{1}{p}}, \quad p \geq 1.$$

For some recent results related to the trapezoid type inequalities, see [1] and [10]-[14].

Let U be a selfadjoint operator on the complex Hilbert space $(H, \langle .,. \rangle)$ with the spectrum $\mathrm{Sp}(U)$ included in the interval $[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f:[m,M]\to\mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$\langle f(U)x,y\rangle = \int_{m-0}^{M} f(\lambda)\, d(\langle E_\lambda x,y\rangle), \tag{1.2}$$

for any x, $y \in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x,y\rangle$ is of *bounded variation* on the interval $[m,M]$ and

$$g_{x,y}(m-0)=0 \text{ and } g_{x,y}(M)=\langle x,y\rangle$$

for any x, $y\in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x,x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m,M]$.

With the notations introduced above, we have considered in the recent paper [5] the problem of bounding the error

$$\frac{f(M)+f(m)}{2}\langle x,y\rangle - \langle f(A)x,y\rangle$$

in approximating $\langle f(A)x,y\rangle$ by the trapezoidal type formula $\frac{f(M)+f(m)}{2}\langle x,y\rangle$, where x,y are vectors in the Hilbert space H and f is a continuous functions of the selfadjoint operator A with the spectrum in the compact interval of real numbers $[m,M]$.

We recall here only two such results. The first deals with the case of continuous functions of bounded variation and is incorporated in the following theorem [5]:

Theorem 2. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $\mathrm{Sp}(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{C}$ is a continuous function of bounded variation on $[m,M]$,*

then we have the inequality

$$\left|\frac{f(M)+f(m)}{2}\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{1.3}$$
$$\leq\frac{1}{2}\max_{\lambda\in[m,M]}\Big[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}$$
$$+\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\Big]\bigvee_m^M(f)$$
$$\leq\frac{1}{2}\|x\|\|y\|\bigvee_m^M(f)$$

for any $x,\ y\in H$.

The case of Lipschitzian functions is as follows [5]:

Theorem 3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $\operatorname{Sp}(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$, then we have the inequality*

$$\left|\frac{f(M)+f(m)}{2}\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{1.4}$$
$$\leq\frac{1}{2}L\int_{m-0}^{M}\Big[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}$$
$$+\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\Big]d\lambda$$
$$\leq\frac{1}{2}(M-m)L\|x\|\|y\|$$

for any $x,\ y\in H$.

For some trapezoid operator inequalities, see [8], [9], [6] and [7].

In this paper we show among others that, if $f:[a,b]\to H$ is continuous and $g:[a,b]\to H$ is strongly differentiable on (a,b), then

$$\left|\left\langle\int_a^b f(s)\,ds,\frac{g(a)+g(b)}{2}\right\rangle-\int_a^b\langle f(t),g(t)\rangle\,dt\right|$$
$$\leq\frac{1}{2}\int_a^b\left\|\int_a^t f(s)\,ds-\int_t^b f(s)\,ds\right\|\|g'(t)\|\,dt$$
$$\leq\frac{1}{2}\int_a^b\|f(s)\|\,dt\times\begin{cases}\int_a^b\|g'(t)\|\,dt,\\ \\ (b-a)^{1/p}\left(\int_a^b\|g'(t)\|^q\,dt\right)^{1/q},\\ \\ (b-a)\sup_{t\in[a,b]}\|g'(t)\|.\end{cases}$$

Applications for operator monotone functions with examples for power and logarithmic functions are also given.

2. Main Results

We have the following weighted version of generalized trapezoid inequality for two functions with values in Hilbert spaces:

Theorem 4. *Assume that $f,\ g:[a,b]\to\mathcal{B}$ are continuous and g is strongly differentiable on (a,b), then*

$$\left|\left\langle \int_a^b f(s)\,ds, \frac{g(a)+g(b)}{2}\right\rangle - \int_a^b \langle f(t),g(t)\rangle\,dt\right| \tag{2.1}$$
$$\leq \frac{1}{2}\int_a^b \left\|\int_a^t f(s)\,ds - \int_t^b f(s)\,ds\right\| \|g'(t)\|\,dt =: G(f,g)$$
$$\leq \frac{1}{2}\int_a^b \|f(s)\|\,ds \int_a^b \|g'(t)\|\,dt.$$

We also have the bounds,

$$G(f,g) \tag{2.2}$$
$$\leq \frac{1}{2}\times\begin{cases} \sup_{t\in[a,b]}\left\|\int_a^t f(s)\,ds - \int_t^b f(s)\,ds\right\| \int_a^b \|g'(t)\|\,dt, \\ \left(\int_a^b \left\|\int_a^t f(s)\,ds - \int_t^b f(s)\,ds\right\|^p dt\right)^{1/p}\left(\int_a^b \|g'(t)\|^q\,dt\right)^{1/q}, \\ \int_a^b \left\|\int_a^t f(s)\,ds - \int_t^b f(s)\,ds\right\| dt \sup_{t\in[a,b]}\|g'(t)\|, \end{cases}$$
$$\leq \frac{1}{2}\times\begin{cases} \int_a^b \|f(s)\|\,dt \int_a^b \|g'(t)\|\,dt, \\ (b-a)^{1/p}\int_a^b \|f(s)\|\,dt \left(\int_a^b \|g'(t)\|^q\,dt\right)^{1/q}, \\ (b-a)\int_a^b \|f(s)\|\,dt \sup_{t\in[a,b]}\|g'(t)\| \end{cases}$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using the integration by parts formula for inner products

$$\int_a^b \langle h(t),l'(t)\rangle\,dt = \langle h(b),l(b)\rangle - \langle h(a),l(a)\rangle - \int_a^b \langle h'(t),l(t)\rangle\,dt,$$

where $h,\ l$ are strongly differentiable on (a,b), we have

$$\int_a^b \left\langle \left[\int_a^t f(s)\,ds - \frac{1}{2}\int_a^b f(s)\,ds\right], g'(t)\right\rangle dt$$
$$= \left\langle \left[\int_a^t f(s)\,ds - \frac{1}{2}\int_a^b f(s)\,ds\right], g(t)\right\rangle\Bigg|_a^b$$
$$- \int_a^b \left\langle \left[\int_a^t f(s)\,ds - \frac{1}{2}\int_a^b f(s)\,ds\right]', g(t)\right\rangle dt$$

$$= \left\langle \left[\int_a^b f(s)\,ds - \frac{1}{2}\int_a^b f(s)\,ds \right], g(b) \right\rangle$$
$$- \left\langle \left[\int_a^a f(s)\,ds - \frac{1}{2}\int_a^b f(s)\,ds \right], g(a) \right\rangle - \int_a^b \langle f(t), g(t)\rangle\,dt$$
$$= \left\langle \int_a^b f(s)\,ds, \frac{g(a)+g(b)}{2} \right\rangle - \int_a^b \langle f(t), g(t)\rangle\,dt$$

and

$$\int_a^b \left\langle \left[\int_a^t f(s)\,ds - \frac{1}{2}\int_a^b f(s)\,ds \right], g'(t) \right\rangle dt$$
$$= \frac{1}{2}\int_a^b \left\langle \left(\int_a^t f(s)\,ds - \int_t^b f(s)\,ds \right), g'(t) \right\rangle dt.$$

Therefore we have the following identity of interest

$$\left\langle \int_a^b f(s)\,ds, \frac{g(a)+g(b)}{2} \right\rangle - \int_a^b \langle f(t), g(t)\rangle\,dt \tag{2.3}$$
$$= \frac{1}{2}\int_a^b \left\langle \left(\int_a^t f(s)\,ds - \int_t^b f(s)\,ds \right), g'(t) \right\rangle dt.$$

Taking the norm in (2.3), using the Schwarz inequality and the integral's properties we get

$$\left| \left\langle \int_a^b f(s)\,ds, \frac{g(a)+g(b)}{2} \right\rangle - \int_a^b \langle f(t), g(t)\rangle\,dt \right|$$
$$\leq \frac{1}{2}\int_a^b \left| \left\langle \left(\int_a^t f(s)\,ds - \int_t^b f(s)\,ds \right), g'(t) \right\rangle \right| dt$$
$$\leq \frac{1}{2}\int_a^b \left\| \int_a^t f(s)\,ds - \int_t^b f(s)\,ds \right\| \|g'(t)\|\,dt$$
$$\leq \frac{1}{2}\int_a^b \left(\left\| \int_a^t f(s)\,ds \right\| + \left\| \int_t^b f(s)\,ds \right\| \right) \|g'(t)\|\,dt$$
$$\leq \frac{1}{2}\int_a^b \left(\int_a^t \|f(s)\|\,ds + \int_t^b \|f(s)\|\,ds \right) \|g'(t)\|\,dt$$
$$= \frac{1}{2}\int_a^b \|f(s)\|\,ds \int_a^b \|g'(t)\|\,dt.$$

The first inequality in (2.2) follows by Hölder's inequality applied to the integral of the product

$$\int_a^b \left\| \int_a^t f(s)\,ds - \int_t^b f(s)\,ds \right\| \|g'(t)\|\,dt.$$

The last part follows by the fact that

$$\left\| \int_a^t f(s)\, ds - \int_t^b f(s)\, ds \right\| \leq \int_a^b \|f(s)\|\, ds$$

for all $t \in [a,b]$.

This implies that

$$\sup_{t\in[a,b]} \left\| \int_a^t f(s)\, ds - \int_t^b f(s)\, ds \right\| \leq \int_a^b \|f(s)\|\, ds,$$

$$\begin{aligned} \int_a^b \left\| \int_a^t f(s)\, ds - \int_t^b f(s)\, ds \right\|^p dt &\leq \int_a^b \left(\int_a^b \|f(s)\|\, ds \right)^p dt \\ &= (b-a) \left(\int_a^b \|f(s)\|\, ds \right)^p \end{aligned}$$

and

$$\int_a^b \left\| \int_a^t f(s)\, ds - \int_t^b f(s)\, ds \right\| dt \leq (b-a) \int_a^b \|f(s)\|\, ds,$$

and the second part of (2.2) is thus proved. □

Corollary 1. *Assume that $f,\ h : [a,b] \to \mathcal{B}$ are continuous and h is strongly differentiable on (a,b) and such that*

$$\|h'(t) - v\| \leq M \text{ for all } t \in (a,b)$$

for some element $v \in \mathcal{B}$ and $M > 0$, then

$$\begin{aligned} &\left| \left\langle \int_a^b f(s)\, ds, \frac{h(a)+h(b)}{2} \right\rangle + \left\langle \int_a^b \left(t - \frac{a+b}{2} \right) f(t)\, dt, v \right\rangle \right. \\ &\left. - \int_a^b \langle f(t), h(t) \rangle\, dt \right| \\ &\leq \frac{1}{2} M \int_a^b \left\| \int_a^t f(s)\, ds - \int_t^b f(s)\, ds \right\| dt \leq \frac{1}{2} M (b-a) \int_a^b \|f(s)\|\, ds. \end{aligned} \tag{2.4}$$

Proof. Put $g(t) = h(t) - tv$, $t \in [0,1]$, then

$$\left\langle \int_a^b f(s)\,ds, \frac{g(a)+g(b)}{2} \right\rangle - \int_a^b \langle f(t), g(t)\rangle\,dt$$
$$= \left\langle \int_a^b f(s)\,ds, \frac{h(a) - av + h(b) - bv}{2} \right\rangle - \int_a^b \langle f(t), [h(t) - tv]\rangle\,dt$$
$$= \left\langle \int_a^b f(s)\,ds, \frac{h(a)+h(b)}{2} \right\rangle - \frac{a+b}{2}\left\langle \int_a^b f(s)\,ds, v \right\rangle$$
$$- \int_a^b \langle f(t), h(t)\rangle\,dt + \int_a^b t\langle f(t), v\rangle\,dt$$
$$= \left\langle \int_a^b f(s)\,ds, \frac{h(a)+h(b)}{2} \right\rangle + \int_a^b t\langle f(t), v\rangle\,dt$$
$$- \frac{a+b}{2}\left\langle \int_a^b f(s)\,ds, v \right\rangle - \int_a^b \langle f(t), h(t)\rangle\,dt$$
$$= \left\langle \int_a^b f(s)\,ds, \frac{h(a)+h(b)}{2} \right\rangle + \int_a^b \left(t - \frac{a+b}{2}\right)\langle f(t), v\rangle\,dt$$
$$- \int_a^b \langle f(t), h(t)\rangle\,dt.$$

Also

$$\int_a^b \left\| \int_a^t f(s)\,ds - \int_t^b f(s)\,ds \right\| \|g'(t)\|\,dt$$
$$= \int_a^b \left\| \int_a^t f(s)\,ds - \int_t^b f(s)\,ds \right\| \|h'(t) - v\|\,dt$$
$$\leq M \int_a^b \left\| \int_a^t f(s)\,ds - \int_t^b f(s)\,ds \right\| dt \leq M(b-a)\int_a^b \|f(s)\|\,ds.$$

By utilising (2.1) we derive the desired result (2.4). □

Remark 1. *Assume that f, $h : [a,b] \to \mathcal{B}$ are continuous and h is strongly differentiable on (a,b). If there exist vectors x, $X \in H$ such that*

$$\operatorname{Re}\langle X - h'(t), h'(t) - x\rangle \geq 0 \text{ for all } t \in (a,b),$$

or, equivalently [9],

$$\left\| h'(t) - \frac{x+X}{2} \right\| \leq \frac{1}{2}\|X - x\| \text{ for all } t \in (a,b),$$

then

$$\left|\left\langle \int_a^b f(s)\,ds, \frac{h(a)+h(b)}{2}\right\rangle + \left\langle \int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt, \frac{x+X}{2}\right\rangle - \int_a^b \langle f(t), h(t)\rangle\,dt\right| \le \frac{1}{4}\|X-x\| \int_a^b \left\| \int_a^t f(s)\,ds - \int_t^b f(s)\,ds \right\| dt \le \frac{1}{4}\|X-x\|(b-a)\int_a^b \|f(s)\|\,ds. \tag{2.5}$$

If, in addition, we have that f is symmetric in $[a,b]$, namely $f(a+b-t) = f(t)$ for all $t \in [a,b]$, then

$$\int_a^b \left(t-\frac{a+b}{2}\right) f(t) = 0$$

and by (2.5) we derive

$$\left|\left\langle \int_a^b f(s)\,ds, \frac{h(a)+h(b)}{2}\right\rangle - \int_a^b \langle f(t), h(t)\rangle\,dt\right| \le \frac{1}{4}\|X-x\| \int_a^b \left\| \int_a^t f(s)\,ds - \int_t^b f(s)\,ds \right\| dt \le \frac{1}{4}\|X-x\|(b-a)\int_a^b \|f(s)\|\,ds. \tag{2.6}$$

3. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A) \ge h(B)$ holds for any $A \ge B \ge 0$.

We have the following representation of operator monotone functions, see for instance [2, p. 144-145]:

Theorem 5. *A function $h : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$h(t) = h(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda}\,d\mu(\lambda), \tag{3.1}$$

where $b \ge 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}\,d\mu(\lambda) < \infty. \tag{3.2}$$

Lemma 1. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \ge 0$, then for all selfadjoint operators V we have*

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1}\right] d\mu(\lambda). \tag{3.3}$$

Proof. From (3.1) we get

$$h(t) = h(0) + bt + \int_0^\infty \left(\lambda - \frac{\lambda^2}{t+\lambda}\right) d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned} & h(U+tV) - h(U) \\ &= btV + \int_0^\infty \left(\lambda - \lambda^2 (U+tV+\lambda)^{-1}\right) d\mu(\lambda) - \int_0^\infty \left(\lambda - \lambda^2 (U+\lambda)^{-1}\right) d\mu(\lambda) \\ &= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} - (\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\ &= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1}(\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\ &= btV + t\int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda). \end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{h(U+tV) - h(U)}{t} = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1}\right] d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Theorem 6. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators V we have*

$$\|Dh(U)(V)\| \leq h'(u) \|V\|. \tag{3.4}$$

Proof. From (3.3) we get

$$\begin{aligned} \|Dh(U)(V) - bV\| &\leq \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1} V (\lambda+U)^{-1}\right\| d\mu(\lambda) \\ &\leq \|V\| \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1}\right\|^2 d\mu(\lambda). \end{aligned} \tag{3.5}$$

Observe that $\lambda + U \geq \lambda + u > 0$ for $\lambda \in [0,\infty)$. Then $0 < (\lambda+U)^{-1} \leq (\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\| \leq (\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2 \leq (\lambda+u)^{-2}$ for $\lambda \in [0,\infty)$.

Therefore by (3.5) we get

$$\|Dh(U)(V) - bV\| \leq \|V\| \int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$h'(t) = b + \int_0^\infty \frac{\lambda(t+\lambda) - \lambda t}{(t+\lambda)^2} d\mu(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) \tag{3.7}$$

for $t > 0$.

From (3.7) we get

$$\int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda) = h'(u) - b,$$

and by (3.6) we derive

$$\|Dh(U)(V) - bV\| \le \|V\| h'(u) - b\|V\|.$$

Finally, by the triangle inequality and by the fact that $b \ge 0$, we obtain that

$$\|Dh(U)(V)\| - b\|V\| \le \|Dh(U)(V) - bV\|,$$

which proves the desired result (3.4). □

For a continuous function h on $(0, \infty)$ and $A,\ B > 0$ we consider the auxiliary function $h_{A,B} : [0,1] \to \mathbb{R}$ defined by

$$h_{A,B}(t) := h((1-t)A + tB),\ t \in [0,1].$$

We have the following representations of the derivatives:

Lemma 2. *Assume that the operator function generated by h is Fréchet differentiable in each $A \ge 0$, then for $B \ge 0$ we have that $h_{A,B}$ is differentiable on $[0,1]$ and*

$$h'_{A,B}(t) = D(h)((1-t)A + tB)(B - A) \tag{3.8}$$

for $t \in [0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \ne 0$,

$$\begin{aligned}
&\frac{h_{A,B}(t+h) - h(t)}{h} \\
&= \frac{h((1-(t+h))A + (t+h)B) - h((1-t)A + tB)}{h} \\
&= \frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}
h'_{A,B}(t) &= \lim_{h\to 0} \frac{h_{A,B}(t+h) - h(t)}{h} \\
&= \lim_{h\to 0} \left[\frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h}\right] \\
&= D(h)((1-t)A + tB)(B - A),
\end{aligned}$$

which proves (3.8). □

Corollary 2. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A \ge a > 0$, $B \ge b > 0$ we have*

$$\begin{aligned}
\left\|h'_{A,B}(t)\right\| &= \|D(h)((1-t)A + tB)(B - A)\| \\
&\le h'((1-t)a + tb)\|B - A\|
\end{aligned} \tag{3.9}$$

for all $t \in [0,1]$.

The proof follows by Theorem 6 and Lemma 2.

One can observe that the inequality (3.9) remains valid for operator monotone functions on $(0,\infty)$. This follows by considering the function $h_\varepsilon(t) := h(t+\varepsilon)$ for $\varepsilon > 0$, which is operator monotone on $[0,\infty)$ and then by letting $\varepsilon \to 0+$ and using the continuity of h and h'.

We have the following result:

Theorem 7. *Let f be continuous on $[0,\infty)$ and g be operator monotone in $[0,\infty)$. Then for all $A \geq a > 0$, $B \geq b > 0$, we have for all x, $y \in H$ that*

$$\begin{aligned} &|T(f,g,A,B,x,y)| \\ &\leq \frac{1}{2}\|B-A\|\|x\|\|y\|\int_0^1 \|f((1-s)A+sB)\|\,dt \\ &\times \begin{cases} \begin{cases} \frac{g(b)-g(a)}{b-a} & \text{if } b \neq a, \\ g'(a) & \text{if } b = a, \end{cases} \\ \left(\int_0^1 [g'((1-t)a+tb)]^q\,dt\right)^{1/q}, \text{ for all } q>1, \\ \sup_{t\in[0,1]} g'((1-t)a+tb). \end{cases} \end{aligned} \tag{3.10}$$

Proof. From the inequality (2.2) for $f_{A,B}x$ and $g_{A,B}y$ we get

$$\begin{aligned} &|T(f,g,A,B,x,y)| \\ &\leq \frac{1}{2}\|B-A\|\|x\|\|y\| \\ &\times \begin{cases} \sup_{t\in[0,1]}\left\|\int_0^t f((1-s)A+sB)\,ds - \int_t^1 f((1-s)A+sB)\,ds\right\| \\ \times \int_0^1 g'((1-t)a+tb)\,dt, \\ \left(\int_0^1 \left\|\int_0^t f((1-s)A+sB)\,ds - \int_t^1 f((1-s)A+sB)\,ds\right\|^p dt\right)^{1/p} \\ \times \left(\int_0^1 [g'((1-t)a+tb)]^q\,dt\right)^{1/q}, \\ \int_0^1 \left\|\int_0^t f((1-s)A+sB)\,ds - \int_t^1 f((1-s)A+sB)\,ds\right\| dt \\ \times \sup_{t\in[0,1]} g'((1-t)a+tb), \end{cases} \\ &\leq \frac{1}{2}\|B-A\|\|x\|\|y\| \\ &\times \begin{cases} \int_0^1 \|f((1-s)A+sB)\|\,dt \int_0^1 g'((1-t)a+tb)\,dt, \\ \int_0^1 \|f((1-s)A+sB)\|\,dt \left(\int_0^1 [g'((1-t)a+tb)]^q\,dt\right)^{1/q}, \\ \int_0^1 \|f((1-s)A+sB)\|\,dt \sup_{t\in[0,1]} g'((1-t)a+tb) \end{cases} \end{aligned}$$

which proves (3.10). □

4. Some Examples

We consider the function $f(t) = \ell^r(t) = t^r$ for $r \in (0,1)$. Then for A, $B \geq 0$ and $t \in [0,1]$ we have

$$\|((1-t)A+tB)^r\| \leq \|(1-t)A+tB\|^r \leq [(1-t)\|A\|+t\|B\|]^r.$$

Therefore

$$\int_0^1 \|f((1-t)A+tB)^r\|\,dt \le \int_0^1 [(1-t)\|A\|+t\|B\|]^r = \begin{cases} \frac{\|B\|^{r+1}-\|A\|^{r+1}}{(r+1)(\|B\|-\|A\|)} \text{ if } \|B\|\ne\|A\| \\ \\ \|A\|^r \text{ if } \|B\|=\|A\|. \end{cases}$$

Let $g:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A\ge a>0$, $B\ge b>0$, we have by Theorem 7 for all x, $y\in H$ that

$$\begin{aligned}(4.1)\quad &\left|\left\langle \int_0^1 ((1-t)A+tB)^r x dt, \frac{g(A)y+g(B)y}{2}\right\rangle - \int_0^1 \langle ((1-t)A+tB)^r x, g((1-t)A+tB)y\rangle\, dt\right| \\ &\le \frac{1}{2}\|B-A\|\|x\|\|y\|\int_0^1 \|((1-s)A+sB)^r\|\,dt \\ &\times \begin{cases} \begin{cases} \frac{g(b)-g(a)}{b-a} \text{ if } b\ne a, \\ \\ g'(a) \text{ if } b=a, \end{cases} \\ \\ \left(\int_0^1 [g'((1-t)a+tb)]^q dt\right)^{1/q}, \text{ for all } q>1, \\ \\ \sup_{t\in[0,1]} g'((1-t)a+tb). \end{cases}\end{aligned}$$

$$\le \frac{1}{2}\|B-A\|\|x\|\|y\| \times \begin{cases} \frac{\|B\|^{r+1}-\|A\|^{r+1}}{(r+1)(\|B\|-\|A\|)} \text{ if } \|B\|\ne\|A\| \\ \\ \|A\|^r \text{ if } \|B\|=\|A\|, \end{cases}$$

$$\times \begin{cases} \begin{cases} \frac{g(b)-g(a)}{b-a} \text{ if } b\ne a, \\ \\ g'(a) \text{ if } b=a, \end{cases} \\ \\ \left(\int_0^1 [g'((1-t)a+tb)]^q dt\right)^{1/q}, \text{ for all } q>1, \\ \\ \sup_{t\in[0,1]} g'((1-t)a+tb). \end{cases}$$

If we take in (4.1) $g(t)=\ln t$, which is operator monotone on $(0,\infty)$, then for all $A\ge a>0$, $B\ge b>0$, we have for all x, $y\in H$ that

$$\begin{aligned}(4.2)\quad &\left|\left\langle \int_0^1 ((1-t)A+tB)^r x dt, \frac{\ln(A)y+\ln(B)y}{2}\right\rangle - \int_0^1 \langle ((1-t)A+tB)^r x, \ln((1-t)A+tB)y\rangle\, dt\right| \\ &\le \frac{1}{2}\|B-A\|\|x\|\|y\| \times \begin{cases} \frac{\|B\|^{r+1}-\|A\|^{r+1}}{(r+1)(\|B\|-\|A\|)} \text{ if } \|B\|\ne\|A\| \\ \\ \|A\|^r \text{ if } \|B\|=\|A\|, \end{cases}\end{aligned}$$

$$\times \begin{cases} \begin{cases} \frac{\ln b - \ln a}{b-a} \text{ if } b \neq a, \\ \\ \frac{1}{a} \text{ if } b = a, \end{cases} \\ \\ \begin{cases} \left(\frac{b^{1-q} - a^{1-q}}{(1-q)(b-a)}\right)^{1/q} \text{ if } b \neq a, \\ \\ \frac{1}{a} \text{ if } b = a, \end{cases} & \text{for all } q > 1, \\ \\ \frac{1}{\min\{a,b\}}. \end{cases} \times$$

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa