INNER PRODUCT GENERALIZED TRAPEZOID TYPE INEQUALITIES IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let H be a complex Hilbert space. In this paper we show among others that, if $f:[a,b]\to H$ is continuous and $g:[a,b]\to H$ is strongly differentiable on (a,b), then

$$\left|\left\langle \int_u^b f(s)\,ds, g(b)\right\rangle + \left\langle \int_a^u f(s)\,ds, g(a)\right\rangle - \int_a^b \langle f(t), g(t)\rangle dt\right|$$
$$\le \sup_{t\in[a,b]}\|g'(t)\|$$
$$\times\begin{cases}\left[(b-u)\int_u^b\|f(t)\|dt+(u-a)\int_a^u\|f(t)\|dt\right],\\ \frac{1}{(q+1)^{1/q}}\left[(b-u)^{1+1/q}\left(\int_u^b\|f(t)\|^p dt\right)^{1/p}\right.\\ \left.+(u-a)^{1+1/q}\left(\int_a^u\|f(t)\|^p dt\right)^{1/p}\right],\ p,\ q>1 \text{ with } \frac{1}{p}+\frac{1}{q}=1,\\ \frac{1}{2}\left[(b-u)^2\sup_{t\in[u,b]}\|f(t)\|+(u-a)^2\sup_{t\in[a,u]}\|f(t)\|\right]\end{cases}$$

for all $u\in[a,b]$. Applications for operator monotone functions with examples for power and logarithmic functions are also given.

1. Introduction

In 2001, Dragomir et al. [6] obtained the following generalized trapezoid inequality:

Theorem 1. *If $g:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ and $f:[a,b]\to\mathbb{R}$ is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b f(t)g(t)\,dt - f(b)\int_u^b g(t)\,dt - f(a)\int_a^u g(t)\,dt\right| \le \left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f) \tag{1.1}$$

for all $u\in[a,b]$, where $\bigvee_a^b(f)$ is the total variation of f on $[a,b]$.

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In particular, we have the mid-point trapezoid inequality

$$\left|\int_a^b f(t)\,g(t)\,dt - f(b)\int_{\frac{a+b}{2}}^b g(t)\,dt - f(a)\int_a^{\frac{a+b}{2}} g(t)\,dt\right| \leq \frac{1}{2}(b-a)\sup_{t\in[a,b]}|g(t)|\bigvee_a^b(f). \tag{1.2}$$

The constant $1/2$ *is sharp in the sense that it cannot be replaced by a smaller quantity.*

For some recent results related to the trapezoid type inequalities, see [1] and [14]-[18].

Let U be a selfadjoint operator on the complex Hilbert space $(H,\langle .,.\rangle)$ with the spectrum $\operatorname{Sp}(U)$ included in the interval $[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its *spectral family.* Then for any continuous function $f:[m,M]\to\mathbb{C}$, it is well known that we have the following *spectral representation in terms of the Riemann-Stieltjes integral*:

$$\langle f(U)\,x,y\rangle = \int_{m-0}^{M} f(\lambda)\,d(\langle E_\lambda x,y\rangle), \tag{1.3}$$

for any $x,\ y\in H$. The function $g_{x,y}(\lambda) := \langle E_\lambda x,y\rangle$ is of *bounded variation* on the interval $[m,M]$ and

$$g_{x,y}(m-0)=0 \text{ and } g_{x,y}(M)=\langle x,y\rangle$$

for any $x,\ y\in H$. It is also well known that $g_x(\lambda) := \langle E_\lambda x,x\rangle$ is *monotonic nondecreasing* and *right continuous* on $[m,M]$.

With the notations introduced above, we have considered in the recent paper [8] the problem of bounding the error

$$\frac{f(M)+f(m)}{2}\langle x,y\rangle - \langle f(A)\,x,y\rangle$$

in approximating $\langle f(A)\,x,y\rangle$ by the trapezoidal type formula $\frac{f(M)+f(m)}{2}\langle x,y\rangle$, where x,y are vectors in the Hilbert space H and f is a continuous functions of the selfadjoint operator A with the spectrum in the compact interval of real numbers $[m,M]$.

We recall here only two such results. The first deals with the case of continuous functions of bounded variation and is incorporated in the following theorem [8]:

Theorem 2. *Let* A *be a selfadjoint operator in the Hilbert space* H *with the spectrum* $\operatorname{Sp}(A)\subseteq[m,M]$ *for some real numbers* $m<M$ *and let* $\{E_\lambda\}_\lambda$ *be its spectral family. If* $f:[m,M]\to\mathbb{C}$ *is a continuous function of bounded variation on* $[m,M]$,

then we have the inequality

$$\left|\frac{f(M)+f(m)}{2}\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{1.4}$$
$$\leq \frac{1}{2}\max_{\lambda\in[m,M]}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}\right.$$
$$\left.+\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\right]\bigvee_m^M(f)$$
$$\leq \frac{1}{2}\|x\|\|y\|\bigvee_m^M(f)$$

for any x, $y\in H$.

The case of Lipschitzian functions is as follows [8]:

Theorem 3. *Let A be a selfadjoint operator in the Hilbert space H with the spectrum $\operatorname{Sp}(A)\subseteq[m,M]$ for some real numbers $m<M$ and let $\{E_\lambda\}_\lambda$ be its spectral family. If $f:[m,M]\to\mathbb{C}$ is Lipschitzian with the constant $L>0$ on $[m,M]$, then we have the inequality*

$$\left|\frac{f(M)+f(m)}{2}\langle x,y\rangle-\langle f(A)x,y\rangle\right| \tag{1.5}$$
$$\leq \frac{1}{2}L\int_{m-0}^{M}\left[\langle E_\lambda x,x\rangle^{1/2}\langle E_\lambda y,y\rangle^{1/2}\right.$$
$$\left.+\langle(1_H-E_\lambda)x,x\rangle^{1/2}\langle(1_H-E_\lambda)y,y\rangle^{1/2}\right]d\lambda$$
$$\leq \frac{1}{2}(M-m)L\|x\|\|y\|$$

for any x, $y\in H$.

For some trapezoid operator inequalities, see [11], [12], [9] and [10].

In this paper we show among others that, if $f:[a,b]\to H$ is continuous and $g:[a,b]\to H$ is strongly differentiable on (a,b), then

$$\left|\left\langle\int_u^b f(s)\,ds,g(b)\right\rangle+\left\langle\int_a^u f(s)\,ds,g(a)\right\rangle-\int_a^b\langle f(t),g(t)\rangle\,dt\right|$$
$$\leq \sup_{t\in[a,b]}\|g'(t)\|$$
$$\times\begin{cases}\left[(b-u)\int_u^b\|f(t)\|\,dt+(u-a)\int_a^u\|f(t)\|\,dt\right],\\[2ex] \frac{1}{(q+1)^{1/q}}\left[(b-u)^{1+1/q}\left(\int_u^b\|f(t)\|^p\,dt\right)^{1/p}\right.\\ \left.+\,(u-a)^{1+1/q}\left(\int_a^u\|f(t)\|^p\,dt\right)^{1/p}\right],\\[2ex] \frac{1}{2}\left[(b-u)^2\sup_{t\in[u,b]}\|f(t)\|+(u-a)^2\sup_{t\in[a,u]}\|f(t)\|\right]\end{cases}$$

for all $u\in[a,b]$, where p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Applications for operator monotone functions with examples for power and logarithmic functions are also given.

2. Main Results

We have the following weighted version of generalized trapezoid inequality for two functions with values in Hilbert spaces:

Theorem 4. *Assume that $f,\ g : [a,b] \to H$ are continuous and g is strongly differentiable on (a,b), then for all $u \in [a,b]$*

$$\left| \left\langle \int_u^b f(s)\,ds, g(b) \right\rangle + \left\langle \int_a^u f(s)\,ds, g(a) \right\rangle - \int_a^b \langle f(t), g(t)\rangle\,dt \right| \leq C(f,g,u), \tag{2.1}$$

where

$$C(f,g,u) := \int_u^b \left(\int_u^t \|f(s)\|\,ds \right) \|g'(t)\|\,dt + \int_a^u \left(\int_t^u \|f(s)\|\,ds \right) \|g'(t)\|\,dt.$$

We also have the bounds

$$C(f,g,u) \leq \begin{cases} \int_u^b \|f(s)\|\,ds \int_u^b \|g'(t)\|\,dt, \\ \left[\int_u^b \left(\int_u^t \|f(s)\|\,ds \right)^p dt \right]^{1/p} \left(\int_u^b \|g'(t)\|^q\,dt \right)^{1/q}, \\ \int_u^b \left(\int_u^t \|f(s)\|\,ds \right) dt \sup_{t\in[u,b]} \|g'(t)\|, \end{cases} + \begin{cases} \left(\int_a^u \|f(s)\|\,ds \right) \int_a^u \|g'(t)\|\,dt, \\ \left[\int_a^u \left(\int_t^u \|f(s)\|\,ds \right)^p dt \right]^{1/p} \left(\int_a^u \|g'(t)\|^q\,dt \right)^{1/q}, \\ \int_a^u \left(\int_t^u \|f(s)\|\,ds \right) dt \sup_{t\in[a,u]} \|g'(t)\|, \end{cases} \tag{2.2}$$

where $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Let $u \in [a,b]$. Using the integration by parts formula for inner products

$$\int_a^b \langle h(t), l'(t)\rangle\,dt = \langle h(b), l(b)\rangle - \langle h(a), l(a)\rangle - \int_a^b \langle h'(t), l(t)\rangle\,dt,$$

where $h,\ l$ are strongly differentiable on (a,b), we have

$$\begin{aligned} &\int_a^b \left\langle \left(\int_a^t f(s)\,ds - \int_a^u f(s)\,ds \right), g'(t) \right\rangle dt \\ &= \left\langle \left(\int_a^t f(s)\,ds - \int_a^u f(s)\,ds \right), g(t) \right\rangle \Big|_a^b \\ &- \int_a^b \left\langle \left(\int_a^t f(s)\,ds - \int_a^u f(s)\,ds \right)', g(t) \right\rangle dt \end{aligned} \tag{2.3}$$

$$= \left\langle \left(\int_a^b f(s)\,ds - \int_a^u f(s)\,ds \right), g(b) \right\rangle$$
$$- \left\langle \left(\int_a^a f(s)\,ds - \int_a^u f(s)\,ds \right), g(a) \right\rangle$$
$$- \int_a^b \left\langle \left(\int_a^t f(s)\,ds - \int_a^u f(s)\,ds \right)', g(t) \right\rangle dt$$
$$= \left\langle \left(\int_u^b f(s)\,ds \right), g(b) \right\rangle + \left\langle \left(\int_a^u f(s)\,ds \right), g(a) \right\rangle - \int_a^b \langle f(t), g(t) \rangle\,dt.$$

Also,

$$\begin{aligned} &\int_a^b \left\langle \left(\int_a^t f(s)\,ds - \int_a^u f(s)\,ds \right), g'(t) \right\rangle dt \\ &= \int_a^u \left\langle \left(\int_a^t f(s)\,ds - \int_a^u f(s)\,ds \right), g'(t) \right\rangle dt \\ &+ \int_u^b \left\langle \left(\int_a^t f(s)\,ds - \int_a^u f(s)\,ds \right), g'(t) \right\rangle dt \\ &= -\int_a^u \left\langle \left(\int_t^u f(s)\,ds \right), g'(t) \right\rangle dt + \int_u^b \left\langle \left(\int_u^t f(s)\,ds \right), g'(t) \right\rangle dt. \end{aligned} \tag{2.4}$$

By utilising (2.3) and (2.4) we derive the following identity of interest

$$\begin{aligned} &\left\langle \int_u^b f(s)\,ds, g(b) \right\rangle + \left\langle \int_a^u f(s)\,ds, g(a) \right\rangle - \int_a^b \langle f(t), g(t) \rangle\,dt \\ &= \int_u^b \left\langle \int_u^t f(s)\,ds, g'(t) \right\rangle dt - \int_a^u \left\langle \int_t^u f(s)\,ds, g'(t) \right\rangle dt \end{aligned} \tag{2.5}$$

for all $u \in [a,b]$.

Taking the norm in (2.5) and using the properties of the integral and Schwarz's inequality, we get

$$\begin{aligned} &\left| \left\langle \int_u^b f(s)\,ds, g(b) \right\rangle + \left\langle \int_a^u f(s)\,ds, g(a) \right\rangle - \int_a^b \langle f(t), g(t) \rangle\,dt \right| \\ &\le \left| \int_u^b \left\langle \int_u^t f(s)\,ds, g'(t) \right\rangle dt \right| + \left| \int_a^u \left\langle \int_t^u f(s)\,ds, g'(t) \right\rangle dt \right| \\ &\le \int_u^b \left| \left\langle \int_u^t f(s)\,ds, g'(t) \right\rangle \right| dt + \int_a^u \left| \left\langle \int_t^u f(s)\,ds, g'(t) \right\rangle \right| dt \\ &\le \int_u^b \left\| \int_u^t f(s)\,ds \right\| \|g'(t)\|\,dt + \int_a^u \left\| \int_t^u f(s)\,ds \right\| \|g'(t)\|\,dt \\ &\le \int_u^b \left(\int_u^t \|f(s)\|\,ds \right) \|g'(t)\|\,dt + \int_a^u \left(\int_t^u \|f(s)\|\,ds \right) \|g'(t)\|\,dt \\ &= C(f,g,u), \end{aligned} \tag{2.6}$$

which proves (2.1).

Using Hölder's inequality, we get for p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, that

$$\begin{aligned}
&\int_u^b \left(\int_u^t \|f(s)\|\,ds\right)\|g'(t)\|\,dt \\
&\leq \begin{cases} \sup_{t\in[u,b]}\left(\int_u^t \|f(s)\|\,ds\right)\int_u^b \|g'(t)\|\,dt, \\ \\ \left[\int_u^b\left(\int_u^t \|f(s)\|\,ds\right)^p dt\right]^{1/p}\left(\int_u^b \|g'(t)\|^q\,dt\right)^{1/q}, \\ \\ \int_u^b\left(\int_u^t \|f(s)\|\,ds\right)dt\,\sup_{t\in[u,b]}\|g'(t)\|, \end{cases} \\
&= \begin{cases} \left(\int_u^b \|f(s)\|\,ds\right)\int_u^b \|g'(t)\|\,dt, \\ \\ \left[\int_u^b\left(\int_u^t \|f(s)\|\,ds\right)^p dt\right]^{1/p}\left(\int_u^b \|g'(t)\|^q\,dt\right)^{1/q}, \\ \\ \int_u^b\left(\int_u^t \|f(s)\|\,ds\right)dt\,\sup_{t\in[u,b]}\|g'(t)\|, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^u \left(\int_t^u \|f(s)\|\,ds\right)\|g'(t)\|\,dt \\
&\leq \begin{cases} \sup_{t\in[a,u]}\left(\int_t^u \|f(s)\|\,ds\right)\int_a^u \|g'(t)\|\,dt \\ \\ \left[\int_a^u\left(\int_t^u \|f(s)\|\,ds\right)^p dt\right]^{1/p}\left(\int_a^u \|g'(t)\|^q\,dt\right)^{1/q} \\ \\ \int_a^u\left(\int_t^u \|f(s)\|\,ds\right)dt\,\sup_{t\in[a,u]}\|g'(t)\|, \end{cases} \\
&= \begin{cases} \left(\int_a^u \|f(s)\|\,ds\right)\int_a^u \|g'(t)\|\,dt, \\ \\ \left[\int_a^u\left(\int_t^u \|f(s)\|\,ds\right)^p dt\right]^{1/p}\left(\int_a^u \|g'(t)\|^q\,dt\right)^{1/q}, \\ \\ \int_a^u\left(\int_t^u \|f(s)\|\,ds\right)dt\,\sup_{t\in[a,u]}\|g'(t)\|. \end{cases}
\end{aligned}$$

By making use of (2.6) we deduce (2.2). □

Corollary 1. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}
&\left|\left\langle\int_u^b f(s)\,ds, g(b)\right\rangle+\left\langle\int_a^u f(s)\,ds, g(a)\right\rangle-\int_a^b\langle f(t), g(t)\rangle\,dt\right| \qquad (2.7)\\
&\leq \int_u^b \|f(s)\|\,ds\int_u^b \|g'(t)\|\,dt+\int_a^u \|f(s)\|\,ds\int_a^u \|g'(t)\|\,dt \\
&\leq \begin{cases} \max\left\{\int_u^b \|f(s)\|\,ds, \int_a^u \|f(s)\|\,ds\right\}\int_a^b \|g'(t)\|\,dt \\ \\ \int_a^b \|f(s)\|\,ds\max\left\{\int_u^b \|g'(t)\|\,dt, \int_a^u \|g'(t)\|\,dt\right\} \end{cases} \\
&\leq \int_a^b \|f(s)\|\,ds\int_a^b \|g'(t)\|\,dt,
\end{aligned}$$

for all $u \in [a,b]$.

The proof follows by the first branches in the bounds (2.2).

Remark 1. *If* $m \in (a,b)$ *is such that*

$$\int_a^u \|f(s)\|\,ds = \int_u^b \|f(s)\|\,ds = \frac{1}{2}\int_a^b \|f(s)\|\,ds, \tag{2.8}$$

then by (2.6) we get

$$\begin{aligned} &\left|\left\langle \int_m^b f(s)\,ds, g(b)\right\rangle + \left\langle \int_a^m f(s)\,ds, g(a)\right\rangle - \int_a^b \langle f(t), g(t)\rangle\,dt\right| \\ &\leq \frac{1}{2}\int_a^b \|f(s)\|\,ds \int_a^b \|g'(t)\|\,dt. \end{aligned} \tag{2.9}$$

Corollary 2. *With the assumptions of Theorem 4, we have*

$$\begin{aligned} &\left|\left\langle \int_u^b f(s)\,ds, g(b)\right\rangle + \left\langle \int_a^u f(s)\,ds, g(a)\right\rangle - \int_a^b \langle f(t), g(t)\rangle\,dt\right| \\ &\leq \left[\int_u^b (b-t)\|f(t)\|\,dt + \int_a^u (t-a)\|f(t)\|\,dt\right] \sup_{t\in[a,b]} \|g'(t)\| \end{aligned} \tag{2.10}$$

for all $u \in [a,b]$.

Proof. From the third branches in the bounds in (2.2) we have

$$\begin{aligned} &\left|\left\langle \int_u^b f(s)\,ds, g(b)\right\rangle + \left\langle \int_a^u f(s)\,ds, g(a)\right\rangle - \int_a^b \langle f(t), g(t)\rangle\,dt\right| \\ &\leq \int_u^b \left(\int_u^t \|f(s)\|\,ds\right) dt \sup_{t\in[u,b]} \|g'(t)\| \\ &+ \int_a^u \left(\int_t^u \|f(s)\|\,ds\right) dt \sup_{t\in[a,u]} \|g'(t)\| \\ &\leq \sup_{t\in[a,b]} \|g'(t)\| \left[\int_u^b \left(\int_u^t \|f(s)\|\,ds\right) dt + \int_a^u \left(\int_t^u \|f(s)\|\,ds\right) dt\right]. \end{aligned} \tag{2.11}$$

Using integration by parts, we have for $u \in [a,b]$ that

$$\begin{aligned} \int_u^b \left(\int_u^t \|f(s)\|\,ds\right) dt &= \left(\int_u^t \|f(s)\|\,ds\right) t \Big|_u^b - \int_u^b t\|f(t)\|\,dt \\ &= \left(\int_u^b \|f(s)\|\,ds\right) b - \int_u^b t\|f(t)\|\,dt \\ &= \int_u^b (b-t)\|f(t)\|\,dt \end{aligned}$$

and

$$\int_a^u \left(\int_t^u \|f(s)\| \, ds\right) dt = \left(\int_t^u \|f(s)\| \, ds\right) t \Big|_a^u + \int_a^u t \|f(t)\| \, dt$$
$$= -\left(\int_a^u \|f(s)\| \, ds\right) a + \int_a^u t \|f(t)\| \, dt$$
$$= \int_a^u (t-a) \|f(t)\| \, dt,$$

which, by (2.11) provides the desired result (2.10). □

Remark 2. *By making use of Hölder's integral inequality, we have for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that*

$$\int_u^b (b-t) \|f(t)\| \, dt \leq \begin{cases} \sup_{t\in[u,b]} (b-t) \int_u^b \|f(t)\| \, dt, \\ \left(\int_u^b (b-t)^q \, dt\right)^{1/q} \left(\int_u^b \|f(t)\|^p \, dt\right)^{1/p}, \\ \int_u^b (b-t) \, dt \sup_{t\in[u,b]} \|f(t)\|, \end{cases}$$
$$= \begin{cases} (b-u) \int_u^b \|f(t)\| \, dt, \\ \frac{(b-u)^{1+1/q}}{(q+1)^{1/q}} \left(\int_u^b \|f(t)\|^p \, dt\right)^{1/p}, \\ \frac{1}{2} (b-u)^2 \sup_{t\in[u,b]} \|f(t)\| \end{cases}$$

and

$$\int_a^u (t-a) \|f(t)\| \, dt \leq \begin{cases} \sup_{t\in[a,u]} (t-a) \int_a^u \|f(t)\| \, dt, \\ \left(\int_a^u (t-a)^q \, dt\right)^{1/q} \left(\int_a^u \|f(t)\|^p \, dt\right)^{1/p}, \\ \int_a^u (t-a) \, dt \sup_{t\in[a,u]} \|f(t)\|, \end{cases}$$
$$= \begin{cases} (u-a) \int_a^u \|f(t)\| \, dt, \\ \frac{(u-a)^{1+1/q}}{(q+1)^{1/q}} \left(\int_a^u \|f(t)\|^p \, dt\right)^{1/p}, \\ \frac{1}{2} (u-a)^2 \sup_{t\in[a,u]} \|f(t)\|. \end{cases}$$

By (2.10) we then get

$$\left|\left\langle\int_u^b f(s)\,ds, g(b)\right\rangle+\left\langle\int_a^u f(s)\,ds, g(a)\right\rangle-\int_a^b \langle f(t), g(t)\rangle\,dt\right| \tag{2.12}$$
$$\leq \sup_{t\in[a,b]}\|g'(t)\| \times\begin{cases}\left[(b-u)\int_u^b\|f(t)\|\,dt+(u-a)\int_a^u\|f(t)\|\,dt\right],\\ \frac{1}{(q+1)^{1/q}}\left[(b-u)^{1+1/q}\left(\int_u^b\|f(t)\|^p\,dt\right)^{1/p}\right.\\ \left.+(u-a)^{1+1/q}\left(\int_a^u\|f(t)\|^p\,dt\right)^{1/p}\right],\\ \frac{1}{2}\left[(b-u)^2\sup_{t\in[u,b]}\|f(t)\|+(u-a)^2\sup_{t\in[a,u]}\|f(t)\|\right]\end{cases}$$

for all $u\in[a,b]$.

Observe that

$$(b-u)\int_u^b\|f(t)\|\,dt+(u-a)\int_a^u\|f(t)\|\,dt$$
$$\leq\max\{b-u,u-a\}\left[\int_u^b\|f(t)\|\,dt+\int_a^u\|f(t)\|\,dt\right]$$
$$=\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\int_a^b\|f(t)\|\,dt.$$

By using the elementary inequality for $a,\ b,\ c,\ d\geq 0$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$(ab+cd)\leq(a^p+c^p)^{1/p}(b^q+d^q)^{1/q}$$

we get

$$(b-u)^{1+1/q}\left(\int_u^b\|f(t)\|^p\,dt\right)^{1/p}+(u-a)^{1+1/q}\left(\int_a^u\|f(t)\|^p\,dt\right)^{1/p}$$
$$\leq\left[\left((b-u)^{1+1/q}\right)^q+\left((u-a)^{1+1/q}\right)^q\right]^{1/q}$$
$$\times\left[\left(\left(\int_u^b\|f(t)\|^p\,dt\right)^{1/p}\right)^p+\left(\left(\int_a^u\|f(t)\|^p\,dt\right)^{1/p}\right)^p\right]^{1/p}$$
$$=\left[(b-u)^{q+1}+(u-a)^{q+1}\right]^{1/q}\left[\int_u^b\|f(t)\|^p\,dt+\int_a^u\|f(t)\|^p\,dt\right]^{1/p}$$
$$=\left[(b-u)^{q+1}+(u-a)^{q+1}\right]^{1/q}\left(\int_a^b\|f(t)\|^p\,dt\right)^{1/p}.$$

Also,

$$\frac{1}{2}\left[(b-u)^2\sup_{t\in[u,b]}\|f(t)\|+(u-a)^2\sup_{t\in[a,u]}\|f(t)\|\right]$$
$$\leq\frac{1}{2}\left[(b-u)^2+(u-a)^2\right]\sup_{t\in[a,b]}\|f(t)\|$$
$$=\left[\frac{1}{4}(b-a)+\left(u-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}\|f(t)\|.$$

Then by (2.12) we get for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\left|\left\langle\int_u^b f(s)\,ds,g(b)\right\rangle+\left\langle\int_a^u f(s)\,ds,g(a)\right\rangle-\int_a^b\langle f(t),g(t)\rangle\,dt\right| \tag{2.13}$$
$$\leq\sup_{t\in[a,b]}\|g'(t)\|$$
$$\times\begin{cases}\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\int_a^b\|f(t)\|\,dt,\\[2ex] \frac{1}{(q+1)^{1/q}}\left[(b-u)^{q+1}+(u-a)^{q+1}\right]^{1/q}\left(\int_a^b\|f(t)\|^p\,dt\right)^{1/p},\\[2ex] \left[\frac{1}{4}(b-a)+\left(u-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}\|f(t)\|\end{cases}$$

for all $u\in[a,b]$.

We also have:

Corollary 3. *With the assumptions of Theorem 4, we have for all $u\in[a,b]$,*

$$\left|\left\langle\int_u^b f(s)\,ds,g(b)\right\rangle+\left\langle\int_a^u f(s)\,ds,g(a)\right\rangle-\int_a^b\langle f(t),g(t)\rangle\,dt\right| \tag{2.14}$$
$$\leq\left[\left(\int_u^b\|f(s)\|\,ds\right)^p(b-u)+\left(\int_a^u\|f(s)\|\,ds\right)^p(u-a)\right]^{1/p}$$
$$\times\left(\int_a^b\|g'(t)\|^q\,dt\right)^{1/q}$$
$$\leq(b-a)^{1/p}\left[\left(\int_u^b\|f(s)\|\,ds\right)^p+\left(\int_a^u\|f(s)\|\,ds\right)^p\right]^{1/p}$$
$$\times\left(\int_a^b\|g'(t)\|^q\,dt\right)^{1/q}$$

for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

Proof. Observe that, by the elementary inequality for a, b, c, $d\geq0$ and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$(ab+cd)\leq(a^p+c^p)^{1/p}(b^q+d^q)^{1/q},$$

we have

$$\left[\int_u^b\left(\int_u^t\|f(s)\|\,ds\right)^p dt\right]^{1/p}\left(\int_u^b\|g'(t)\|^q\,dt\right)^{1/q}$$
$$+\left[\int_a^u\left(\int_t^u\|f(s)\|\,ds\right)^p dt\right]^{1/p}\left(\int_a^u\|g'(t)\|^q\,dt\right)^{1/q}$$
$$\leq\left(\int_u^b\left(\int_u^t\|f(s)\|\,ds\right)^p dt+\int_a^u\left(\int_t^u\|f(s)\|\,ds\right)^p dt\right)^{1/p}$$
$$\times\left(\int_u^b\|g'(t)\|^q\,dt+\int_a^u\|g'(t)\|^q\,dt\right)^{1/q}$$
$$=\left(\int_u^b\left(\int_u^t\|f(s)\|\,ds\right)^p dt+\int_a^u\left(\int_t^u\|f(s)\|\,ds\right)^p dt\right)^{1/p}$$
$$\times\left(\int_a^b\|g'(t)\|^q\,dt\right)^{1/q}$$

$$\leq\left(\left(\int_u^b\|f(s)\|\,ds\right)^p\int_u^b dt+\left(\int_a^u\|f(s)\|\,ds\right)^p\int_a^u dt\right)^{1/p}$$
$$\times\left(\int_a^b\|g'(t)\|^q\,dt\right)^{1/q}$$
$$=\left(\left(\int_u^b\|f(s)\|\,ds\right)^p(b-u)+\left(\int_a^u\|f(s)\|\,ds\right)^p(u-a)\right)^{1/p}$$
$$\times\left(\int_a^b\|g'(t)\|^q\,dt\right)^{1/q}$$
$$\leq(b-a)^{1/p}\left(\left(\int_u^b\|f(s)\|\,ds\right)^p+\left(\int_a^u\|f(s)\|\,ds\right)^p\right)^{1/p}$$
$$\times\left(\int_a^b\|g'(t)\|^q\,dt\right)^{1/q},$$

which proves (2.14). □

Remark 3. *If $m\in(a,b)$ is such that (2.8) is valid, then by (2.14) we get*

$$\left|\left\langle\int_m^b f(s)\,ds,g(b)\right\rangle+\left\langle\int_a^m f(s)\,ds,g(a)\right\rangle-\int_a^b\langle f(t),g(t)\rangle\,dt\right|$$
$$\leq\frac{1}{2}(b-a)^{1/p}\int_a^b\|f(s)\|\,ds\left(\int_a^b\|g'(t)\|^q\,dt\right)^{1/q}. \tag{2.15}$$

Assume that $f,\, g:[a,b]\to H$ are continuous and g is strongly differentiable on (a,b), then

$$\left|\left\langle \int_{\frac{a+b}{2}}^{b} f(s)\,ds, g(b)\right\rangle + \left\langle \int_{a}^{\frac{a+b}{2}} f(s)\,ds, g(a)\right\rangle - \int_a^b \langle f(t), g(t)\rangle\, dt\right| \le C(f,g), \tag{2.16}$$

where

$$C(f,g) := \int_{\frac{a+b}{2}}^{b}\left(\int_{\frac{a+b}{2}}^{t}\|f(s)\|\,ds\right)\|g'(t)\|\,dt + \int_a^{\frac{a+b}{2}}\left(\int_t^{\frac{a+b}{2}}\|f(s)\|\,ds\right)\|g'(t)\|\,dt.$$

We also have the bounds

$$C(f,g) \le \begin{cases} \int_{\frac{a+b}{2}}^{b}\|f(s)\|\,ds\int_{\frac{a+b}{2}}^{b}\|g'(t)\|\,dt, \\[2ex] \left[\int_u^b\left(\int_{\frac{a+b}{2}}^{t}\|f(s)\|\,ds\right)^p dt\right]^{1/p}\left(\int_{\frac{a+b}{2}}^{b}\|g'(t)\|^q\,dt\right)^{1/q}, \\[2ex] \int_{\frac{a+b}{2}}^{b}\left(\int_{\frac{a+b}{2}}^{t}\|f(s)\|\,ds\right)dt\sup_{t\in\left[\frac{a+b}{2},b\right]}\|g'(t)\|, \end{cases} + \begin{cases} \left(\int_a^{\frac{a+b}{2}}\|f(s)\|\,ds\right)\int_a^{\frac{a+b}{2}}\|g'(t)\|\,dt, \\[2ex] \left[\int_a^{\frac{a+b}{2}}\left(\int_t^{\frac{a+b}{2}}\|f(s)\|\,ds\right)^p dt\right]^{1/p}\left(\int_a^{\frac{a+b}{2}}\|g'(t)\|^q\,dt\right)^{1/q}, \\[2ex] \int_a^{\frac{a+b}{2}}\left(\int_t^{\frac{a+b}{2}}\|f(s)\|\,ds\right)dt\sup_{t\in\left[a,\frac{a+b}{2}\right]}\|g'(t)\|, \end{cases} \tag{2.17}$$

where $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

From (2.7) we get

$$\begin{aligned} &\left|\left\langle \int_{\frac{a+b}{2}}^{b} f(s)\,ds, g(b)\right\rangle + \left\langle \int_{a}^{\frac{a+b}{2}} f(s)\,ds, g(a)\right\rangle - \int_a^b \langle f(t), g(t)\rangle\, dt\right| \\ &\le \int_{\frac{a+b}{2}}^{b}\|f(s)\|\,ds\int_{\frac{a+b}{2}}^{b}\|g'(t)\|\,dt + \int_a^{\frac{a+b}{2}}\|f(s)\|\,ds\int_a^{\frac{a+b}{2}}\|g'(t)\|\,dt \\ &\le \begin{cases} \max\left\{\int_{\frac{a+b}{2}}^{b}\|f(s)\|\,ds, \int_a^{\frac{a+b}{2}}\|f(s)\|\,ds\right\}\int_a^b\|g'(t)\|\,dt \\[2ex] \int_a^b\|f(s)\|\,ds\max\left\{\int_{\frac{a+b}{2}}^{b}\|g'(t)\|\,dt, \int_a^{\frac{a+b}{2}}\|g'(t)\|\,dt\right\} \end{cases} \\ &\le \int_a^b\|f(s)\|\,ds\int_a^b\|g'(t)\|\,dt, \end{aligned} \tag{2.18}$$

while from (2.13) we get

$$\left|\left\langle \int_{\frac{a+b}{2}}^{b} f(s)\,ds, g(b)\right\rangle + \left\langle \int_{a}^{\frac{a+b}{2}} f(s)\,ds, g(a)\right\rangle - \int_a^b \langle f(t), g(t)\rangle\, dt\right| \tag{2.19}$$
$$\leq \sup_{t\in[a,b]} \|g'(t)\| \times \begin{cases} \frac{1}{2}(b-a)\int_a^b \|f(t)\|\,dt, \\ \frac{1}{2(q+1)^{1/q}}(b-a)^{1+1/q}\left(\int_a^b \|f(t)\|^p\,dt\right)^{1/p}, \\ \frac{1}{4}(b-a)\sup_{t\in[a,b]}\|f(t)\|. \end{cases}$$

From (2.14) we also get

$$\left|\left\langle \int_{\frac{a+b}{2}}^{b} f(s)\,ds, g(b)\right\rangle + \left\langle \int_{a}^{\frac{a+b}{2}} f(s)\,ds, g(a)\right\rangle - \int_a^b \langle f(t), g(t)\rangle\, dt\right| \tag{2.20}$$
$$\leq \frac{(b-a)^{1/p}}{2^{1/p}}\left[\left(\int_{\frac{a+b}{2}}^{b}\|f(s)\|\,ds\right)^p + \left(\int_a^{\frac{a+b}{2}}\|f(s)\|\,ds\right)^p\right]^{1/p}$$
$$\times\left(\int_a^b \|g'(t)\|^q\,dt\right)^{1/q}.$$

3. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A) \geq h(B)$ holds for any $A \geq B \geq 0$.

We have the following representation of operator monotone functions, see for instance [2, p. 144-145]:

Theorem 5. *A function $h : [0,\infty) \to \mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$h(t) = h(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda}\,d\mu(\lambda), \tag{3.1}$$

where $b \geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}\,d\mu(\lambda) < \infty. \tag{3.2}$$

Lemma 1. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq 0$, then for all selfadjoint operators V we have*

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda). \tag{3.3}$$

Proof. From (3.1) we get

$$h(t) = h(0) + bt + \int_0^\infty \left(\lambda - \frac{\lambda^2}{t+\lambda}\right)d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
&h(U+tV)-h(U)\\
&=btV+\int_0^\infty\left(\lambda-\lambda^2(U+tV+\lambda)^{-1}\right)d\mu(\lambda)-\int_0^\infty\left(\lambda-\lambda^2(U+\lambda)^{-1}\right)d\mu(\lambda)\\
&=btV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}-(\lambda+U+tV)^{-1}\right]d\mu(\lambda)\\
&=btV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}(\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right]d\mu(\lambda)\\
&=btV+t\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U+tV)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{h(U+tV)-h(U)}{t}=bV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U+tV)^{-1}\right]d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Dh(U)(V)=bV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Theorem 6. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators V we have*

$$\|Dh(U)(V)\| \leq h'(u)\|V\|. \tag{3.4}$$

Proof. From (3.3) we get

$$\begin{aligned}
\|Dh(U)(V)-bV\| &\leq \int_0^\infty\lambda^2\left\|(\lambda+U)^{-1}V(\lambda+U)^{-1}\right\|d\mu(\lambda) \qquad (3.5)\\
&\leq \|V\|\int_0^\infty\lambda^2\left\|(\lambda+U)^{-1}\right\|^2d\mu(\lambda).
\end{aligned}$$

Observe that $\lambda+U \geq \lambda+u > 0$ for $\lambda\in[0,\infty)$. Then $0<(\lambda+U)^{-1}\leq(\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\|\leq(\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2\leq(\lambda+u)^{-2}$ for $\lambda\in[0,\infty)$.

Therefore by (3.5) we get

$$\|Dh(U)(V)-bV\|\leq\|V\|\int_0^\infty\lambda^2(\lambda+u)^{-2}d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$h'(t)=b+\int_0^\infty\frac{\lambda(t+\lambda)-\lambda t}{(t+\lambda)^2}d\mu(\lambda)=b+\int_0^\infty\frac{\lambda^2}{(t+\lambda)^2}d\mu(\lambda) \tag{3.7}$$

for $t>0$.

From (3.7) we get

$$\int_0^\infty\lambda^2(\lambda+u)^{-2}d\mu(\lambda)=h'(u)-b,$$

and by (3.6) we derive

$$\|Dh(U)(V)-bV\|\leq\|V\|h'(u)-b\|V\|.$$

Finally, by the triangle inequality and by the fact that $b\geq 0$, we obtain that

$$\|Dh(U)(V)\|-b\|V\|\leq\|Dh(U)(V)-bV\|,$$

which proves the desired result (3.4). □

For a continuous function h on $(0,\infty)$ and $A,\ B>0$ we consider the auxiliary function $h_{A,B}:[0,1]\to\mathbb{R}$ defined by

$$h_{A,B}(t):=h((1-t)A+tB),\ t\in[0,1].$$

We have the following representations of the derivatives:

Lemma 2. *Assume that the operator function generated by h is Fréchet differentiable in each $A\geq 0$, then for $B\geq 0$ we have that $h_{A,B}$ is differentiable on $[0,1]$ and*

$$h'_{A,B}(t)=D(h)((1-t)A+tB)(B-A) \tag{3.8}$$

for $t\in[0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t\in(0,1)$. Let h be in a small interval around 0 such that $t+h\in(0,1)$. Then for $h\neq 0$,

$$\begin{aligned}&\frac{h_{A,B}(t+h)-h(t)}{h}\\&=\frac{h((1-(t+h))A+(t+h)B)-h((1-t)A+tB)}{h}\\&=\frac{h((1-t)A+tB+h(B-A))-h((1-t)A+tB)}{h}\end{aligned}$$

and by taking the limit over $h\to 0$, we get

$$\begin{aligned}h'_{A,B}(t)&=\lim_{h\to 0}\frac{h_{A,B}(t+h)-h(t)}{h}\\&=\lim_{h\to 0}\left[\frac{h((1-t)A+tB+h(B-A))-h((1-t)A+tB)}{h}\right]\\&=D(h)((1-t)A+tB)(B-A),\end{aligned}$$

which proves (3.8). □

Corollary 4. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A\geq a>0,\ B\geq b>0$ we have*

$$\begin{aligned}\left\|h'_{A,B}(t)\right\|&=\|D(h)((1-t)A+tB)(B-A)\|\\&\leq h'((1-t)a+tb)\|B-A\|\end{aligned} \tag{3.9}$$

for all $t\in[0,1]$.

The proof follows by Theorem 6 and Lemma 2.

One can observe that the inequality (3.9) remains valid for operator monotone functions on $(0,\infty)$. This follows by considering the function $h_\varepsilon(t):=h(t+\varepsilon)$ for $\varepsilon>0$, which is operator monotone on $[0,\infty)$ and then by letting $\varepsilon\to 0+$ and using the continuity of h and h'.

We define the generalized trapezoid functional

$$T(f,g,A,B,x,y;u) := \left\langle \int_u^1 f((1-s)A+sB)\,xds, g(b)\,y \right\rangle + \left\langle \int_0^u f((1-s)A+sB)\,xds, g(a)\,y \right\rangle - \int_0^1 \langle f((1-t)A+tB)\,x, g((1-t)A+tB)\,y\rangle\,dt, \tag{3.10}$$

where f and g are continuous on $[0,\infty)$, A, $B \geq 0$ and x, $y \in H$.

We have the following result:

Theorem 7. *Let f be continuous on $[0,\infty)$ and g be operator monotone in $[0,\infty)$. Then for all $A \geq a > 0$, $B \geq b > 0$, we have for all x, $y \in H$ that*

$$\begin{aligned} &|T(f,g,A,B,x,y;u)| \\ &\leq \|B-A\|\,\|x\|\,\|y\| \left[\int_u^1 \|f((1-t)A+tB)\|\,dt \int_u^1 g'((1-t)a+tb)\,dt \right. \\ &\left. + \int_0^u \|f((1-t)A+tB)\|\,dt \int_0^u g'((1-t)a+tb)\,dt\right] \\ &\leq \|B-A\|\,\|x\|\,\|y\| \\ &\times \max\left\{\int_u^1 \|f((1-t)A+tB)\|\,dt, \int_0^u \|f((1-t)A+tB)\|\,dt\right\} \\ &\times \begin{cases} \frac{g(b)-g(a)}{b-a} & \text{if } b \neq a, \\ g'(a) & \text{if } b = a, \end{cases} \end{aligned} \tag{3.11}$$

$$\begin{aligned} &|T(f,g,A,B,x,y;u)| \\ &\leq \|B-A\|\,\|x\|\,\|y\| \sup_{t\in[a,b]} g'((1-t)a+tb) \\ &\times \left[\int_u^1 (1-t)\,\|f((1-t)A+tB)\|\,dt + \int_0^u t\,\|f((1-t)A+tB)\|\,dt\right] \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} &|T(f,g,A,B,x,y;u)| \\ &\leq \|B-A\|\,\|x\|\,\|y\| \left(\int_0^1 [g'((1-t)a+tb)]^q\,dt\right)^{1/q} \\ &\times \left[(1-u)\left(\int_u^1 \|f((1-t)A+tB)\|\,dt\right)^p + u\left(\int_0^u \|f((1-t)A+tB)\|\,dt\right)^p\right]^{1/p} \\ &\leq \|B-A\|\,\|x\|\,\|y\| \left(\int_0^1 [g'((1-t)a+tb)]^q\,dt\right)^{1/q} \\ &\times \left[\left(\int_u^1 \|f((1-t)A+tB)\|\,dt\right)^p + \left(\int_0^u \|f((1-t)A+tB)\|\,dt\right)^p\right]^{1/p} \end{aligned} \tag{3.13}$$

for p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$

The proof follows by Corollaries 1-3 for $f_{A,B}x$ and $g_{A,B}y$. The details are omitted.

From (2.13) we also have

$$|T(f,g,A,B,x,y;u)| \tag{3.14}$$
$$\leq \|B-A\| \|x\| \|y\| \sup_{t\in[a,b]} g'((1-t)a+tb)$$
$$\times \begin{cases} \left[\frac{1}{2}+\left|u-\frac{1}{2}\right|\right]\int_0^1 \|f((1-t)A+tB)\|\,dt, \\ \frac{1}{(q+1)^{1/q}}\left[(1-u)^{q+1}+u^{q+1}\right]^{1/q}\left(\int_0^1 \|f((1-t)A+tB)\|^p\,dt\right)^{1/p}, \\ \left[\frac{1}{4}+\left(u-\frac{1}{2}\right)^2\right]\sup_{t\in[0,1]}\|f((1-t)A+tB)\| \end{cases}$$

provided that f is continuous on $[0,\infty)$, g is operator monotone in $[0,\infty)$ and $A\geq a>0$, $B\geq b>0$, while x, $y\in H$.

In particular, we have

$$|M(f,g,A,B,x,y)| \leq \frac{1}{2}\|B-A\| \|x\| \|y\| \sup_{t\in[a,b]} g'((1-t)a+tb) \tag{3.15}$$
$$\times \begin{cases} \int_0^1 \|f((1-t)A+tB)\|\,dt, \\ \frac{1}{(q+1)^{1/q}}\left(\int_0^1 \|f((1-t)A+tB)\|^p\,dt\right)^{1/p}, \\ \frac{1}{2}\sup_{t\in[0,1]}\|f((1-t)A+tB)\|, \end{cases}$$

where

$$\begin{aligned} M(f,g,A,B,x,y) &:= \left\langle \int_{1/2}^1 f((1-s)A+sB)\,xds, g(b)\,y \right\rangle \\ &+ \left\langle \int_0^{1/2} f((1-s)A+sB)\,xds, g(a)\,y \right\rangle \\ &- \int_a^b \langle f((1-t)A+tB)\,x, g((1-t)A+tB)\,y\rangle\,dt. \end{aligned}$$

4. Some Examples

We consider the function $f(t)=\ell^r(t)=t^r$ for $r\in(0,1)$. Then for A, $B\geq 0$ and $t\in[0,1]$ we have

$$\|((1-t)A+tB)^r\| \leq \|(1-t)A+tB\|^r \leq [(1-t)\|A\|+t\|B\|]^r.$$

Therefore

$$\begin{aligned} \int_0^1 \|((1-t)A+tB)^r\|\,dt &\leq \int_0^1 [(1-t)\|A\|+t\|B\|]^r \\ &= \begin{cases} \frac{\|B\|^{r+1}-\|A\|^{r+1}}{(r+1)(\|B\|-\|A\|)} \text{ if } \|B\|\neq\|A\| \\ \|A\|^r \text{ if } \|B\|=\|A\|. \end{cases} \end{aligned}$$

Also

$$\int_0^1 \|((1-t)A+tB)^r\|^p\,dt \le \int_0^1 [(1-t)\|A\|+t\|B\|]^{pr} = \begin{cases} \frac{\|B\|^{pr+1}-\|A\|^{pr+1}}{(pr+1)(\|B\|-\|A\|)} \text{ if } \|B\|\neq\|A\|, \\ \\ \|A\|^{pr} \text{ if } \|B\|=\|A\| \end{cases}$$

and

$$\sup_{t\in[0,1]}\|((1-t)A+tB)^r\| \le \sup_{t\in[0,1]}[(1-t)\|A\|+t\|B\|]^r = \max\{\|A\|^r,\|B\|^r\}.$$

From the inequality (3.14) we obtain

$$\begin{aligned} &|T(\ell^r,g,A,B,x,y;u)| \\ &\le \|B-A\|\,\|x\|\,\|y\|\sup_{t\in[a,b]} g'((1-t)a+tb) \\ &\times\begin{cases} \left[\frac{1}{2}+\left|u-\frac{1}{2}\right|\right]\times\begin{cases}\frac{\|B\|^{r+1}-\|A\|^{r+1}}{(r+1)(\|B\|-\|A\|)} \text{ if } \|B\|\neq\|A\|, \\ \\ \|A\|^r \text{ if } \|B\|=\|A\|,\end{cases} \\ \\ \frac{1}{(q+1)^{1/q}}\left[(1-u)^{q+1}+u^{q+1}\right]^{1/q} \\ \times\begin{cases}\left[\frac{\|B\|^{pr+1}-\|A\|^{pr+1}}{(pr+1)(\|B\|-\|A\|)}\right]^{1/p} \text{ if } \|B\|\neq\|A\|, \\ \\ \|A\|^r \text{ if } \|B\|=\|A\|,\end{cases} \\ \\ \left[\frac{1}{4}+\left(u-\frac{1}{2}\right)^2\right]\max\{\|A\|^r,\|B\|^r\}.\end{cases}\end{aligned} \tag{4.1}$$

if g is operator monotone in $[0,\infty)$ and $A\ge a>0$, $B\ge b>0$, while x, $y\in H$.

In particular, we have

$$\begin{aligned} &|M(\ell^r,g,A,B,x,y)| \\ &\le \frac{1}{2}\|B-A\|\,\|x\|\,\|y\|\sup_{t\in[a,b]} g'((1-t)a+tb) \\ &\times\begin{cases} \begin{cases}\frac{\|B\|^{r+1}-\|A\|^{r+1}}{(r+1)(\|B\|-\|A\|)} \text{ if } \|B\|\neq\|A\|, \\ \\ \|A\|^r \text{ if } \|B\|=\|A\|,\end{cases} \\ \\ \frac{1}{(q+1)^{1/q}}\times\begin{cases}\left[\frac{\|B\|^{pr+1}-\|A\|^{pr+1}}{(pr+1)(\|B\|-\|A\|)}\right]^{1/p} \text{ if } \|B\|\neq\|A\|, \\ \\ \|A\|^r \text{ if } \|B\|=\|A\|,\end{cases} \\ \\ \frac{1}{2}\max\{\|A\|^r,\|B\|^r\}.\end{cases},\end{aligned} \tag{4.2}$$

If in (4.1) we take $g(t) = \ln t$, then for $A \geq a > 0$, $B \geq b > 0$ and $x, y \in H$ we derive

$$
\begin{aligned}
&|T(\ell^r, \ln, A, B, x, y; u)| \\
&\leq \frac{1}{\min\{a,b\}} \|B - A\| \|x\| \|y\| \\
&\times \begin{cases}
\left[\frac{1}{2} + \left|u - \frac{1}{2}\right|\right] \times \begin{cases} \frac{\|B\|^{r+1} - \|A\|^{r+1}}{(r+1)(\|B\| - \|A\|)} \text{ if } \|B\| \neq \|A\| \\ \|A\|^r \text{ if } \|B\| = \|A\|. \end{cases} \\
\frac{1}{(q+1)^{1/q}} \left[(1-u)^{q+1} + u^{q+1}\right]^{1/q} \\
\quad \times \begin{cases} \left[\frac{\|B\|^{pr+1} - \|A\|^{pr+1}}{(pr+1)(\|B\| - \|A\|)}\right]^{1/p} \text{ if } \|B\| \neq \|A\| \\ \|A\|^r \text{ if } \|B\| = \|A\| \end{cases} \\
\left[\frac{1}{4} + \left(u - \frac{1}{2}\right)^2\right] \max\{\|A\|^r, \|B\|^r\}.
\end{cases}
\end{aligned} \tag{4.3}
$$

and

$$
\begin{aligned}
&|M(\ell^r, \ln, A, B, x, y)| \\
&\leq \frac{1}{2\min\{a,b\}} \|B - A\| \|x\| \|y\| \\
&\times \begin{cases}
\begin{cases} \frac{\|B\|^{r+1} - \|A\|^{r+1}}{(r+1)(\|B\| - \|A\|)} \text{ if } \|B\| \neq \|A\|, \\ \|A\|^r \text{ if } \|B\| = \|A\|, \end{cases} \\
\frac{1}{(q+1)^{1/q}} \times \begin{cases} \left[\frac{\|B\|^{pr+1} - \|A\|^{pr+1}}{(pr+1)(\|B\| - \|A\|)}\right]^{1/p} \text{ if } \|B\| \neq \|A\|, \\ \|A\|^r \text{ if } \|B\| = \|A\|, \end{cases} \\
\frac{1}{2} \max\{\|A\|^r, \|B\|^r\}.
\end{cases}
\end{aligned} \tag{4.4}
$$

References

[1] M. W. Alomari, New upper and lower bounds for the trapezoid inequality of absolutely continuous functions and applications. *Konuralp J. Math.* **7** (2019), no. 2, 319–323.

[2] R. Bhatia, *Matrix Analysis*. Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[3] P. Cerone and S. S. Dragomir, Trapezoidal-type rules from an inequalities point of view. *Handbook of analytic-computational methods in applied mathematics*, 65–134, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[4] Cerone, Pietro; Dragomir, Sever S.; Kikianty, Eder Ostrowski and trapezoid type inequalities related to Pompeiu's mean value theorem with complex exponential weight. J. Math. Inequal. 11 (2017), no. 4, 947–964.

[5] Cerone, Pietro; Dragomir, Sever S.; Kikianty, Eder Ostrowski and trapezoid type inequalities related to Pompeiu's mean value theorem. J. Math. Inequal. 9 (2015), no. 3, 739–762.

[6] S. S. Dragomir, C. Buşe, M. V. Boldea and L. Braescu, A generalization of the trapezoidal rule for the Riemann-Stieltjes integral and applications. *Nonlinear Anal. Forum* **6** (2001), no. 2, 337–351.

[7] S. S. Dragomir, Vector and operator trapezoidal type inequalities for continuous functions of selfadjoint operators in Hilbert spaces. *Electron. J. Linear Algebra* **22** (2011), 161–178.
[8] S. S. Dragomir, Some trapezoidal vector inequalities for continuous functions of selfadjoint operators in Hilbert spaces. *Abstr. Appl. Anal.* **2011**, Art. ID 941286, 13 pp.
[9] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* SpringerBriefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1.
[10] S. S. Dragomir, *Riemann–Stieltjes Integral Inequalities for Complex Functions Defined on Unit Circle with Applications to Unitary Operators in Hilbert Spaces*, 2019 by CRC Press 160 Pages, ISBN 9780367337100.
[11] S. S. Dragomir, Generalised trapezoid-type inequalities for complex functions defined on unit circle with applications for unitary operators in Hilbert spaces. *Mediterr. J. Math.* **12** (2015), no. 3, 573–591.
[12] S. S. Dragomir, Trapezoid type inequalities for complex functions defined on the unit circle with applications for unitary operators in Hilbert spaces. *Georgian Math. J.* **23** (2016), no. 2, 199–210.
[13] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. (ONLINE: `http://ajmaa.org/RGMIA/monographs.php`).
[14] A. Kashuri and R. Liko, Generalized trapezoidal type integral inequalities and their applications. *J. Anal.* **28** (2020), no. 4, 1023–1043.
[15] W. Liu and J. Park, Some perturbed versions of the generalized trapezoid inequality for functions of bounded variation. *J. Comput. Anal. Appl.* **22** (2017), no. 1, 11–18.
[16] W. Liu and H. Zhang, Refinements of the weighted generalized trapezoid inequality in terms of cumulative variation and applications. *Georgian Math. J.* **25** (2018), no. 1, 47–64.
[17] K. L. Tseng and S. R. Hwang, Some extended trapezoid-type inequalities and applications. *Hacet. J. Math. Stat.* **45** (2016), no. 3, 827–850.
[18] W. Yang, A companion for the generalized Ostrowski and the generalized trapezoid type inequalities. *Tamsui Oxf. J. Inf. Math. Sci.* **29** (2013), no. 2, 113–127

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa