OSTROWSKI TYPE INEQUALITIES FOR FUNCTIONS WITH VALUES IN BANACH SPACES

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Abstract. Let E be a complex Banach space. In this paper we show among others that, if $\alpha : [a,b] \to \mathbb{C}$ is continuous and $Y : [a,b] \to E$ is strongly differentiable on the interval (a,b), then for all $u \in [a,b]$,

$$\left\| \int_a^b \alpha(t) Y(t) \, dt - \left(\int_a^b \alpha(s) \, ds \right) Y(u) \right\| \leq \begin{cases} \int_u^b |\alpha(s)| \, ds \int_u^b \|Y'(t)\| \, dt, \\ \left[\int_u^b \left(\int_t^b |\alpha(s)| \, ds \right)^p dt \right]^{1/p} \left(\int_u^b \|Y'(t)\|^q \, dt \right)^{1/q}, \\ \int_u^b \left(\int_t^b |\alpha(s)| \, ds \right) dt \sup_{t \in [u,b]} \|Y'(t)\|, \end{cases} + \begin{cases} \int_a^u |\alpha(s)| \, ds \int_a^u \|Y'(t)\| \, dt, \\ \left[\int_a^u \left(\int_a^t |\alpha(s)| \, ds \right)^p dt \right]^{1/p} \left(\int_a^u \|Y'(t)\|^q \, dt \right)^{1/q}, \\ \int_a^u \left(\int_a^t |\alpha(s)| \, ds \right) dt \sup_{t \in [a,u]} \|Y'(t)\|, \end{cases}$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Applications for operator monotone functions in Hilbert spaces with examples for power and logarithmic functions are also given.

1. Introduction

In 1938, A. Ostrowski [17], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a} \int_a^b f(t) \, dt$ and the value $f(x)$, $x \in [a,b]$.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t \in (a,b)} |f'(t)| < \infty$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \|f'\|_\infty (b-a), \tag{1.1}$$

for all $x \in [a,b]$ and the constant $\frac{1}{4}$ is the best possible.

The following result, which is an improvement on Ostrowski's inequality, holds.

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Theorem 2 (Dragomir, 2002 [5])**.** *Let $f:[a,b]\rightarrow\mathbb{C}$ be an absolutely continuous function on $[a,b]$ whose derivative $f'\in L_{\infty}[a,b]$. Then*

$$\begin{aligned} (1.2)\qquad & \left| f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \\ & \leq \frac{1}{2(b-a)}\left[\|f'\|_{[a,x],\infty}(x-a)^2+\|f'\|_{[x,b],\infty}(b-x)^2\right] \\ & \leq \begin{cases} \|f'\|_{[a,b],\infty}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a); \\ \\ \frac{1}{2}\left[\|f'\|_{[a,x],\infty}^{\alpha}+\|f'\|_{[x,b],\infty}^{\alpha}\right]^{\frac{1}{\alpha}}\left[\left(\frac{x-a}{b-a}\right)^{2\beta}+\left(\frac{b-x}{b-a}\right)^{2\beta}\right]^{\frac{1}{\beta}}(b-a), \\ \text{where } \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \\ \frac{1}{2}\left[\|f'\|_{[a,x],\infty}+\|f'\|_{[x,b],\infty}\right]\left[\frac{1}{2}+\left|\frac{x-\frac{a+b}{2}}{b-a}\right|\right]^2(b-a) \end{cases} \end{aligned}$$

for all $x\in[a,b]$, where $\|\cdot\|_{[m,n],\infty}$ denotes the usual norm on $L_{\infty}[m,n]$, i.e., we recall that

$$\|g\|_{[m,n],\infty}=ess\sup_{t\in[m,n]}|g(t)|<\infty.$$

For recent Ostrowski type inequalities, see [1], [2], [6]-[14] and [18].

Let X be a Banach space and $-\infty<a<b<\infty$. We denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators acting on X. The norms of vectors or operators acting on X will be denoted by $\|\cdot\|$.

A function $f:[a,b]\rightarrow X$ is called *measurable* if there exists a sequence of simple functions $f_n:[a,b]\rightarrow X$ which converges punctually almost everywhere on $[a,b]$ at f. We recall also that a measurable function $f:[a,b]\rightarrow X$ is *Bochner integrable* if and only if its norm function (i.e. the function $t\mapsto\|f(t)\|:[a,b]\rightarrow\mathbb{R}_+$) is Lebesgue integrable on $[a,b]$.

The following generalization of Ostrowski scalar inequality holds [3].

Theorem 3. *Assume that $B:[a,b]\rightarrow\mathcal{L}(X)$ is Hölder continuous on $[a,b]$, i.e.,*

$$(1.3)\qquad \|B(t)-B(s)\|\leq H|t-s|^{\alpha}\quad \text{for all}\ \ t,\ s\in[a,b],$$

where $H>0$ and $\alpha\in(0,1]$.

If $f:[a,b]\rightarrow X$ is Bochner integrable on $[a,b]$, then we have the inequality:

$$\left\|\int_a^b B(s)f(s)\,ds-B(t)\int_a^b f(s)\,ds\right\|$$

$$\leq H \int_a^b |t-s|^{\alpha} \|f(s)\| \, ds$$

$$\leq H \times \begin{cases} \dfrac{(b-t)^{\alpha+1}+(t-a)^{\alpha+1}}{\alpha+1} \operatorname{esssup}_{t\in[a,b]} \|f(t)\|, \\ \\ \left[\dfrac{(b-t)^{q\alpha+1}+(t-a)^{q\alpha+1}}{q\alpha+1}\right]^{\frac{1}{q}} \left(\int_a^b \|f(t)\|^p \, dt\right)^{1/p} \\ ,\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left[\dfrac{1}{2}(b-a)+\left|t-\dfrac{a+b}{2}\right|\right]^{\alpha} \int_a^b \|f(t)\| \, dt \end{cases}$$

for any $t \in [a,b]$, provided the integrals and $\operatorname{esssup}_{t\in[a,b]}$ *from the right hand side are finite.*

Let E be a complex Banach space. Motivated by the above results, in this paper we provide new upper bounds for the quantity

$$\left\| \int_a^b \alpha(t) Y(t) \, dt - \left(\int_a^b \alpha(s) \, ds \right) Y(u) \right\|, \ u \in [a,b]$$

provided that $\alpha : [a,b] \to \mathbb{C}$ and $Y : [a,b] \to E$ are continuous and Y is strongly differentiable on (a,b). Applications for operator monotone functions in Hilbert spaces with examples for power and logarithmic functions are also given.

2. General Ostrowski Inequalities

We have the following weighted version of Ostrowski's inequality for two functions with values in Banach spaces:

Theorem 4. *Assume that $\alpha : [a,b] \to \mathbb{C}$ and $Y : [a,b] \to E$ are continuous and Y is strongly differentiable on (a,b), then for all $u \in [a,b]$*

$$\left\| \int_a^b \alpha(t) Y(t) \, dt - \left(\int_a^b \alpha(s) \, ds \right) Y(u) \right\| \leq B(\alpha, Y, u), \tag{2.1}$$

where

$$B(\alpha, Y, u) := \int_u^b \left(\int_t^b |\alpha(s)| \, ds \right) \|Y'(t)\| \, dt + \int_a^u \left(\int_a^t |\alpha(s)| \, ds \right) \|Y'(t)\| \, dt.$$

We also have the bounds

$$B(\alpha,Y,u) \leq \begin{cases} \int_u^b |\alpha(s)|\,ds \int_u^b \|Y'(t)\|\,dt, \\ \left[\int_u^b \left(\int_t^b |\alpha(s)|\,ds\right)^p dt\right]^{1/p} \left(\int_u^b \|Y'(t)\|^q\,dt\right)^{1/q}, \\ \int_u^b \left(\int_t^b |\alpha(s)|\,ds\right) dt \sup_{t\in[u,b]} \|Y'(t)\|, \end{cases} + \begin{cases} \int_a^u |\alpha(s)|\,ds \int_a^u \|Y'(t)\|\,dt, \\ \left[\int_a^u \left(\int_a^t |\alpha(s)|\,ds\right)^p dt\right]^{1/p} \left(\int_a^u \|Y'(t)\|^q\,dt\right)^{1/q}, \\ \int_a^u \left(\int_a^t |\alpha(s)|\,ds\right) dt \sup_{t\in[a,u]} \|Y'(t)\|, \end{cases} \tag{2.2}$$

where $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Let $u\in[a,b]$. Using the integration by parts formula for Bochner integral [15], we have

$$\begin{aligned} &\int_u^b \left(\int_t^b \alpha(s)\,ds\right) Y'(t)\,dt \\ &= \left(\int_t^b \alpha(s)\,ds\right) Y(t)\Big]_u^b + \int_u^b \alpha(t)\,Y(t)\,dt \\ &= -\left(\int_u^b \alpha(s)\,ds\right) Y(u) + \int_u^b \alpha(t)\,Y(t)\,dt \end{aligned}$$

and

$$\begin{aligned} &\int_a^u \left(\int_a^t \alpha(s)\,ds\right) Y'(t)\,dt \\ &= \left(\int_a^t \alpha(s)\,ds\right) Y(t)\Big]_a^u - \int_a^u \alpha(t)\,Y(t)\,dt \\ &= \left(\int_a^u \alpha(s)\,ds\right) Y(u) - \int_a^u \alpha(t)\,Y(t)\,dt. \end{aligned}$$

By subtracting the second identity from the first, we get

$$\begin{aligned}
&\int_u^b \left(\int_t^b \alpha(s)\,ds\right) Y'(t)\,dt - \int_a^u \left(\int_a^t \alpha(s)\,ds\right) Y'(t)\,dt \\
&= \int_u^b \alpha(t)\,Y(t)\,dt + \int_a^u \alpha(t)\,Y(t)\,dt \\
&- \left(\int_u^b \alpha(s)\,ds\right) Y(u) - \left(\int_a^u \alpha(s)\,ds\right) Y(u) \\
&= \int_a^b \alpha(t)\,Y(t)\,dt - \left(\int_a^b \alpha(s)\,ds\right) Y(u).
\end{aligned}$$

Therefore, we get the following identity of interest

$$\begin{aligned}
&\int_a^b \alpha(t)\,Y(t)\,dt - \left(\int_a^b \alpha(s)\,ds\right) Y(u) \\
&= \int_u^b \left(\int_t^b \alpha(s)\,ds\right) Y'(t)\,dt - \int_a^u \left(\int_a^t \alpha(s)\,ds\right) Y'(t)\,dt
\end{aligned} \tag{2.3}$$

for all $u \in [a, b]$.

If we take the norm in (2.3), then we get

$$\begin{aligned}
&\left\| \int_a^b \alpha(t)\,Y(t)\,dt - \left(\int_a^b \alpha(s)\,ds\right) Y(u) \right\| \\
&\leq \left\| \int_u^b \left(\int_t^b \alpha(s)\,ds\right) Y'(t)\,dt \right\| + \left\| \int_a^u \left(\int_a^t \alpha(s)\,ds\right) Y'(t)\,dt \right\| \\
&\leq \int_u^b \left| \int_t^b \alpha(s)\,ds \right| \|Y'(t)\|\,dt + \int_a^u \left| \int_a^t \alpha(s)\,ds \right| \|Y'(t)\|\,dt \\
&\leq \int_u^b \left(\int_t^b |\alpha(s)|\,ds\right) \|Y'(t)\|\,dt + \int_a^u \left(\int_a^t |\alpha(s)|\,ds\right) \|Y'(t)\|\,dt \\
&= B(\alpha, Y, u),
\end{aligned} \tag{2.4}$$

which proves (2.1).

Using Hölder's inequality, we get for p, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, that

$$\begin{aligned}
&\int_u^b\left(\int_t^b|\alpha(s)|\,ds\right)\|Y'(t)\|\,dt\\
&\leq\begin{cases}\sup_{t\in[u,b]}\left(\int_t^b|\alpha(s)|\,ds\right)\int_u^b\|Y'(t)\|\,dt,\\ \\ \left[\int_u^b\left(\int_t^b|\alpha(s)|\,ds\right)^p dt\right]^{1/p}\left(\int_u^b\|Y'(t)\|^q\,dt\right)^{1/q},\\ \\ \int_u^b\left(\int_t^b|\alpha(s)|\,ds\right)dt\sup_{t\in[u,b]}\|Y'(t)\|,\end{cases}\\
&=\begin{cases}\left(\int_u^b|\alpha(s)|\,ds\right)\int_u^b\|Y'(t)\|\,dt,\\ \\ \left[\int_u^b\left(\int_t^b|\alpha(s)|\,ds\right)^p dt\right]^{1/p}\left(\int_u^b\|Y'(t)\|^q\,dt\right)^{1/q},\\ \\ \int_u^b\left(\int_t^b|\alpha(s)|\,ds\right)dt\sup_{t\in[u,b]}\|Y'(t)\|\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^u\left(\int_a^t|\alpha(s)|\,ds\right)\|Y'(t)\|\,dt\\
&\leq\begin{cases}\sup_{t\in[a,u]}\left(\int_a^t|\alpha(s)|\,ds\right)\int_a^u\|Y'(t)\|\,dt,\\ \\ \left[\int_a^u\left(\int_a^t|\alpha(s)|\,ds\right)^p dt\right]^{1/p}\left(\int_a^u\|Y'(t)\|^q\,dt\right)^{1/q},\\ \\ \int_a^u\left(\int_a^t|\alpha(s)|\,ds\right)dt\sup_{t\in[a,u]}\|Y'(t)\|,\end{cases}
\end{aligned}$$

$$=\begin{cases}\left(\int_a^u|\alpha(s)|\,ds\right)\int_a^u\|Y'(t)\|\,dt,\\ \\ \left[\int_a^u\left(\int_a^t|\alpha(s)|\,ds\right)^p dt\right]^{1/p}\left(\int_a^u\|Y'(t)\|^q\,dt\right)^{1/q},\\ \\ \int_a^u\left(\int_a^t|\alpha(s)|\,ds\right)dt\sup_{t\in[a,u]}\|Y'(t)\|.\end{cases}$$

By making use of (2.4), we derive (2.2). □

Corollary 1. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}
(2.5) \qquad & \left\| \int_a^b \alpha(t) Y(t)\, dt - \left(\int_a^b \alpha(s)\, ds \right) Y(u) \right\| \\
& \leq \int_u^b |\alpha(s)|\, ds \int_u^b \|Y'(t)\|\, dt + \int_a^u |\alpha(s)|\, ds \int_a^u \|Y'(t)\|\, dt \\
& \leq \begin{cases} \max\left\{ \int_u^b |\alpha(s)|\, ds, \int_a^u |\alpha(s)|\, ds \right\} \int_a^b \|Y'(t)\|\, dt \\ \\ \int_a^b |\alpha(s)|\, ds \max\left\{ \int_u^b \|Y'(t)\|\, dt, \int_a^u \|Y'(t)\|\, dt \right\} \end{cases} \\
& \leq \int_a^b |\alpha(s)|\, ds \int_a^b \|Y'(t)\|\, dt,
\end{aligned}$$

for all $u \in [a,b]$.

The proof follows by the first branches in the bounds (2.2).

Remark 1. *If* $m \in (a,b)$ *is such that*

$$(2.6) \qquad \int_a^u |\alpha(s)|\, ds = \int_u^b |\alpha(s)|\, ds = \frac{1}{2} \int_a^b |\alpha(s)|\, ds,$$

then by (2.5) we get

$$(2.7) \quad \left\| \int_a^b \alpha(t) Y(t)\, dt - \left(\int_a^b \alpha(s)\, ds \right) Y(m) \right\| \leq \frac{1}{2} \int_a^b |\alpha(s)|\, ds \int_a^b \|Y'(t)\|\, dt.$$

Corollary 2. *With the assumptions of Theorem 4, we have*

$$\begin{aligned}
(2.8) \qquad & \left\| \int_a^b \alpha(t) Y(t)\, dt - \left(\int_a^b \alpha(s)\, ds \right) Y(u) \right\| \\
& \leq \int_u^b \left(\int_t^b |\alpha(s)|\, ds \right) dt \sup_{t \in [u,b]} \|Y'(t)\| \\
& + \int_a^u \left(\int_a^t |\alpha(s)|\, ds \right) dt \sup_{t \in [a,u]} \|Y'(t)\| \\
& \leq \sup_{t \in [a,b]} \|Y'(t)\| \int_a^b |t-u|\, |\alpha(s)|\, dt,
\end{aligned}$$

for all $u \in [a,b]$.

Proof. From the third branches in the bounds (2.2) we have

$$\begin{aligned}
&\left\| \int_a^b \alpha(t) Y(t)\, dt - \left(\int_a^b \alpha(s)\, ds \right) Y(u) \right\| \qquad (2.9)\\
&\leq \int_u^b \left(\int_t^b |\alpha(s)|\, ds \right) dt \sup_{t\in[u,b]} \|Y'(t)\| \\
&+ \int_a^u \left(\int_a^t |\alpha(s)|\, ds \right) dt \sup_{t\in[a,u]} \|Y'(t)\| \\
&\leq \sup_{t\in[a,b]} \|Y'(t)\| \left[\int_u^b \left(\int_t^b |\alpha(s)|\, ds \right) dt + \int_a^u \left(\int_a^t |\alpha(s)|\, ds \right) dt \right].
\end{aligned}$$

Using integration by parts, we have for $u \in [a,b]$ that

$$\begin{aligned}
\int_u^b \left(\int_t^b |\alpha(s)|\, ds \right) dt &= \left(\int_t^b |\alpha(s)|\, ds \right) t \Big|_u^b + \int_u^b t\, |\alpha(t)|\, dt \\
&= \int_u^b t\, |\alpha(t)|\, dt - \left(\int_u^b |\alpha(s)|\, ds \right) u \\
&= \int_u^b (t-u)\, |\alpha(t)|\, dt = \int_u^b |t-u|\, |\alpha(t)|\, dt
\end{aligned}$$

and

$$\begin{aligned}
\int_a^u \left(\int_a^t |\alpha(s)|\, ds \right) dt &= \left(\int_a^t |\alpha(s)|\, ds \right) t \Big|_a^u - \int_a^u t\, |\alpha(t)|\, dt \\
&= \left(\int_a^u |\alpha(s)|\, ds \right) u - \int_a^u t\, |\alpha(t)|\, dt \\
&= \int_a^u (u-t)\, |\alpha(t)|\, dt = \int_a^u |t-u|\, |\alpha(t)|\, dt,
\end{aligned}$$

which gives that

$$\begin{aligned}
&\int_u^b \left(\int_t^b |\alpha(s)|\, ds \right) dt + \int_a^u \left(\int_a^t |\alpha(s)|\, ds \right) dt \\
&= \int_u^b |t-u|\, |\alpha(t)|\, dt + \int_a^u |t-u|\, |\alpha(t)|\, dt = \int_a^b |t-u|\, |\alpha(t)|\, dt.
\end{aligned}$$

By making use of (2.9) we derive (2.8). □

Remark 2. *By making use of Hölder's integral inequality, we have for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ that*

$$\int_a^b |t-u|\, \|\alpha(t)\|\, dt \leq \begin{cases} \sup_{t\in[a,b]} |t-u| \int_a^b |\alpha(t)|\, dt, \\[2ex] \left(\int_a^b |t-u|^q\, dt \right)^{1/q} \left(\int_a^b |\alpha(t)|^p\, dt \right)^{1/p}, \\[2ex] \int_a^b |t-u|\, dt \sup_{t\in[a,b]} |\alpha(t)|. \end{cases}$$

Since

$$\sup_{t\in[a,b]}|t-u| = \max\{u-a,b-u\} = \frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|,$$

$$\left(\int_a^b |t-u|^q\,dt\right)^{1/q} = \left[\frac{(u-a)^{q+1}+(b-u)^{q+1}}{q+1}\right]^{1/q}$$

and

$$\int_a^b |t-u|\,dt = \frac{(u-a)^2+(b-u)^2}{2} = \frac{1}{4}(b-a)^2+\left(u-\frac{a+b}{2}\right)^2.$$

Then by (2.8) we derive the non-commutative Ostrowsky type inequalities for functions in Banach spaces

$$\begin{aligned}(2.10)\qquad &\left\|\int_a^b \alpha(t)\,Y(t)\,dt-\left(\int_a^b \alpha(s)\,ds\right)Y(u)\right\| \\ &\le \sup_{t\in[a,b]}\|Y'(t)\|\begin{cases}\left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\int_a^b|\alpha(t)|\,dt, \\ \left[\frac{(u-a)^{q+1}+(b-u)^{q+1}}{q+1}\right]^{1/q}\left(\int_a^b|\alpha(t)|^p\,dt\right)^{1/p}, \\ \left[\frac{1}{4}(b-a)^2+\left(u-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}|\alpha(t)|,\end{cases}\end{aligned}$$

for all $u\in[a,b]$.

We also have:

Corollary 3. *With the assumptions of Theorem 4, we have for all* $u\in[a,b]$,

$$\begin{aligned}(2.11)\qquad &\left\|\int_a^b \alpha(t)\,Y(t)\,dt-\left(\int_a^b \alpha(s)\,ds\right)Y(u)\right\| \\ &\le \left[\left(\int_u^b|\alpha(s)|\,ds\right)^p(b-u)+\left(\int_a^u|\alpha(s)|\,ds\right)^p(u-a)\right]^{1/p} \\ &\times\left(\int_a^b\|Y'(t)\|^q\,dt\right)^{1/q} \\ &\le (b-a)^{1/p}\left[\left(\int_u^b|\alpha(s)|\,ds\right)^p+\left(\int_a^u|\alpha(s)|\,ds\right)^p\right]^{1/p} \\ &\times\left(\int_a^b\|Y'(t)\|^q\,dt\right)^{1/q}\end{aligned}$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Observe that, by the elementary inequality for $a,\ b,\ c,\ d\ge 0$ and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$(ab+cd)\le(a^p+c^p)^{1/p}(b^q+d^q)^{1/q}$$

we have

$$\begin{aligned}
(2.12)\qquad & \left[\int_u^b \left(\int_t^b |\alpha(s)|\,ds\right)^p dt\right]^{1/p} \left(\int_u^b \|Y'(t)\|^q\,dt\right)^{1/q} \\
&+ \left[\int_a^u \left(\int_a^t |\alpha(s)|\,ds\right)^p dt\right]^{1/p} \left(\int_a^u \|Y'(t)\|^q\,dt\right)^{1/q} \\
&\le \left[\int_u^b \left(\int_t^b |\alpha(s)|\,ds\right)^p dt + \int_a^u \left(\int_a^t |\alpha(s)|\,ds\right)^p dt\right]^{1/p} \\
&\times \left[\int_u^b \|Y'(t)\|^q\,dt + \int_a^u \|Y'(t)\|^q\,dt\right]^{1/q} \\
&= \left[\int_u^b \left(\int_t^b |\alpha(s)|\,ds\right)^p dt + \int_a^u \left(\int_a^t |\alpha(s)|\,ds\right)^p dt\right]^{1/p} \\
&\times \left(\int_a^b \|Y'(t)\|^q\,dt\right)^{1/q} \\
&\le \left[\left(\int_u^b |\alpha(s)|\,ds\right)^p \int_u^b dt + \left(\int_a^u |\alpha(s)|\,ds\right)^p \int_a^u dt\right]^{1/p} \\
&\times \left(\int_a^b \|Y'(t)\|^q\,dt\right)^{1/q} \\
&= \left[\left(\int_u^b |\alpha(s)|\,ds\right)^p (b-u) + \left(\int_a^u |\alpha(s)|\,ds\right)^p (u-a)\right]^{1/p} \\
&\times \left(\int_a^b \|Y'(t)\|^q\,dt\right)^{1/q} \\
&\le (b-a)^{1/p} \left[\left(\int_u^b |\alpha(s)|\,ds\right)^p + \left(\int_a^u |\alpha(s)|\,ds\right)^p\right]^{1/p} \\
&\times \left(\int_a^b \|Y'(t)\|^q\,dt\right)^{1/q},
\end{aligned}$$

which proves (2.11). □

Remark 3. *If $m \in (a,b)$ is such that (2.6) is valid, then by (2.11) we get*

$$\begin{aligned}
(2.13)\qquad & \left\| \int_a^b \alpha(t)\,Y(t)\,dt - \left(\int_a^b \alpha(s)\,ds\right) Y(m) \right\| \\
&\le \frac{1}{2}(b-a)^{1/p} \int_a^b |\alpha(s)|\,ds \left(\int_a^b \|Y'(t)\|^q\,dt\right)^{1/q}.
\end{aligned}$$

Remark 4. *With the assumptions of Theorem 4 we have the mid-point inequality*

$$\left\| \int_a^b \alpha(t) Y(t) \, dt - \left(\int_a^b \alpha(s) \, ds \right) Y\left(\frac{a+b}{2} \right) \right\| \leq M(\alpha, Y), \tag{2.14}$$

where

$$M(\alpha, Y) := \int_{\frac{a+b}{2}}^b \left(\int_t^b |\alpha(s)| \, ds \right) \|Y'(t)\| \, dt + \int_a^{\frac{a+b}{2}} \left(\int_a^t |\alpha(s)| \, ds \right) \|Y'(t)\| \, dt.$$

We also have the bounds

$$M(\alpha, Y) \leq \begin{cases} \left(\int_{\frac{a+b}{2}}^b |\alpha(s)| \, ds \right) \int_{\frac{a+b}{2}}^b \|Y'(t)\| \, dt, \\ \left[\int_{\frac{a+b}{2}}^b \left(\int_t^b |\alpha(s)| \, ds \right)^p dt \right]^{1/p} \left(\int_{\frac{a+b}{2}}^b \|Y'(t)\|^q \, dt \right)^{1/q}, \\ \int_{\frac{a+b}{2}}^b \left(\int_t^b |\alpha(s)| \, ds \right) dt \sup_{t \in \left[\frac{a+b}{2}, b\right]} \|Y'(t)\|, \end{cases} \tag{2.15}$$

$$+ \begin{cases} \left(\int_a^{\frac{a+b}{2}} |\alpha(s)| \, ds \right) \int_a^{\frac{a+b}{2}} \|Y'(t)\| \, dt, \\ \left[\int_a^{\frac{a+b}{2}} \left(\int_a^t |\alpha(s)| \, ds \right)^p dt \right]^{1/p} \left(\int_a^{\frac{a+b}{2}} \|Y'(t)\|^q \, dt \right)^{1/q}, \\ \int_a^{\frac{a+b}{2}} \left(\int_a^t |\alpha(s)| \, ds \right) dt \sup_{t \in \left[a, \frac{a+b}{2}\right]} \|Y'(t)\|. \end{cases}$$

Making use of (2.5), we get

$$\begin{aligned} &\left\| \int_a^b \alpha(t) Y(t) \, dt - \left(\int_a^b \alpha(s) \, ds \right) Y\left(\frac{a+b}{2} \right) \right\| \\ &\leq \left(\int_{\frac{a+b}{2}}^b |\alpha(s)| \, ds \right) \int_{\frac{a+b}{2}}^b \|Y'(t)\| \, dt \\ &+ \left(\int_a^{\frac{a+b}{2}} |\alpha(s)| \, ds \right) \int_a^{\frac{a+b}{2}} \|Y'(t)\| \, dt \\ &\leq \begin{cases} \max\left\{ \int_{\frac{a+b}{2}}^b |\alpha(s)| \, ds, \int_a^{\frac{a+b}{2}} |\alpha(s)| \, ds \right\} \int_a^b \|Y'(t)\| \, dt \\ \int_a^b |\alpha(s)| \, ds \max\left\{ \int_{\frac{a+b}{2}}^b \|Y'(t)\| \, dt, \int_a^{\frac{a+b}{2}} \|Y'(t)\| \, dt \right\} \end{cases} \\ &\leq \int_a^b |\alpha(s)| \, ds \int_a^b \|Y'(t)\| \, dt \end{aligned} \tag{2.16}$$

and by (2.8),

$$\left\| \int_a^b \alpha(t) Y(t) \, dt - \left(\int_a^b \alpha(s) \, ds \right) Y\left(\frac{a+b}{2} \right) \right\|$$

$$\leq \int_{\frac{a+b}{2}}^{b}\left(\int_{t}^{b}|\alpha(s)|\,ds\right)dt \sup_{t\in\left[\frac{a+b}{2},b\right]}\|Y'(t)\|$$
$$+\int_{a}^{\frac{a+b}{2}}\left(\int_{a}^{t}|\alpha(s)|\,ds\right)dt \sup_{t\in\left[a,\frac{a+b}{2}\right]}\|Y'(t)\|$$
$$\leq \sup_{t\in[a,b]}\|Y'(t)\|\int_{a}^{b}\left|t-\frac{a+b}{2}\right||\alpha(t)|\,dt.$$

From (2.10) we derive the non-commutative mid-point type inequalities for functions in Banach spaces

$$\left\|\int_{a}^{b}\alpha(t)\,Y(t)\,dt-\left(\int_{a}^{b}\alpha(s)\,ds\right)Y\left(\frac{a+b}{2}\right)\right\| \tag{2.17}$$
$$\leq \sup_{t\in[a,b]}\|Y'(t)\|\begin{cases}\frac{1}{2}(b-a)\int_{a}^{b}|\alpha(t)|\,dt,\\[2mm] \frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}}\left(\int_{a}^{b}|\alpha(t)|^{p}\,dt\right)^{1/p},\\[2mm] \frac{1}{4}(b-a)^{2}\sup_{t\in[a,b]}|\alpha(t)|.\end{cases}$$

By (2.11) we obtain that

$$\left\|\int_{a}^{b}\alpha(t)\,Y(t)\,dt-\left(\int_{a}^{b}\alpha(s)\,ds\right)Y\left(\frac{a+b}{2}\right)\right\| \tag{2.18}$$
$$\leq \frac{(b-a)^{1/p}}{2^{1/p}}\left[\left(\int_{\frac{a+b}{2}}^{b}|\alpha(s)|\,ds\right)^{p}+\left(\int_{a}^{\frac{a+b}{2}}|\alpha(s)|\,ds\right)^{p}\right]^{1/p}$$
$$\times\left(\int_{a}^{b}\|Y'(t)\|^{q}\,dt\right)^{1/q}.$$

If we consider the case when $\alpha(t)=1,\ t\in[a,b]$, then by (2.1) we get

$$\left\|\int_{a}^{b}Y(t)\,dt-(b-a)\,Y(u)\right\|\leq B(Y,u), \tag{2.19}$$

where

$$B(Y,u):=\int_{u}^{b}(b-t)\,\|Y'(t)\|\,dt+\int_{a}^{u}(t-a)\,\|Y'(t)\|\,dt.$$

Subsequently, by (2.2) we also have the bounds

$$B(\alpha, Y, u) \leq \begin{cases} (b-u)\int_u^b \|Y'(t)\|\,dt, \\ \\ \frac{1}{(p+1)^{1/p}}(b-u)^{1+1/p}\left(\int_u^b \|Y'(t)\|^q\,dt\right)^{1/q}, \\ \\ \frac{1}{2}(b-u)^2 \sup_{t\in[u,b]} \|Y'(t)\|, \end{cases} + \begin{cases} (u-a)\int_a^u \|Y'(t)\|\,dt, \\ \\ \frac{1}{(p+1)^{1/p}}(u-a)^{1+1/p}\left(\int_a^u \|Y'(t)\|^q\,dt\right)^{1/q}, \\ \\ \frac{1}{2}(u-a)^2 \sup_{t\in[a,u]} \|Y'(t)\|. \end{cases} \tag{2.20}$$

From (2.5) we get

$$\begin{aligned} &\left\|\int_a^b Y(t)\,dt-(b-a)Y(u)\right\| \\ &\leq (b-u)\int_u^b \|Y'(t)\|\,dt+(u-a)\int_a^u \|Y'(t)\|\,dt \\ &\leq \begin{cases} \left[\frac{1}{2}(b-a)+\left|u-\frac{a+b}{2}\right|\right]\int_a^b \|Y'(t)\|\,dt, \\ \\ (b-a)\max\left\{\int_u^b \|Y'(t)\|\,dt, \int_a^u \|Y'(t)\|\,dt\right\}. \end{cases} \end{aligned} \tag{2.21}$$

By (2.9) we also have the Ostrowski's inequality

$$\begin{aligned} &\left\|\int_a^b Y(t)\,dt-(b-a)Y(u)\right\| \\ &\leq \left[\frac{1}{4}(b-a)^2+\left(u-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]} \|Y'(t)\| \end{aligned} \tag{2.22}$$

for $u \in [a, b]$.

3. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0, \infty)$ is said to be operator monotone if $h(A) \geq h(B)$ holds for any $A \geq B \geq 0$ operators on the Hilbert space H.

We have the following representation of operator monotone functions, see for instance [4, p. 144-145]:

Theorem 5. *A function $h : [0, \infty) \to \mathbb{R}$ is operator monotone in $[0, \infty)$ if and only if it has the representation*

$$h(t) = h(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda}\,d\mu(\lambda), \tag{3.1}$$

where $b \geq 0$ and a positive measure μ on $[0, \infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}\,d\mu(\lambda) < \infty. \tag{3.2}$$

We have the following representation result:

Lemma 1. *Let* $h:[0,\infty)\to\mathbb{R}$ *be operator monotone in* $[0,\infty)$*. Assume that* $U\geq 0$*, then for all selfadjoint operators* V *we have*

$$Dh(U)(V)=bV+\int_0^\infty \lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda). \tag{3.3}$$

Proof. From (3.1) we get

$$h(t)=h(0)+bt+\int_0^\infty\left(\lambda-\frac{\lambda^2}{t+\lambda}\right)d\mu(\lambda).$$

Assume that $U\geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
&h(U+tV)-h(U)\\
&=btV+\int_0^\infty\left(\lambda-\lambda^2(U+tV+\lambda)^{-1}\right)d\mu(\lambda)-\int_0^\infty\left(\lambda-\lambda^2(U+\lambda)^{-1}\right)d\mu(\lambda)\\
&=btV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}-(\lambda+U+tV)^{-1}\right]d\mu(\lambda)\\
&=btV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}(\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right]d\mu(\lambda)\\
&=btV+t\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U+tV)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Dividing by $t\neq 0$, we get

$$\frac{h(U+tV)-h(U)}{t}=bV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U+tV)^{-1}\right]d\mu(\lambda)$$

and by taking the limit over $t\to 0$, we get

$$Dh(U)(V)=bV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Lemma 2. *Let* $h:[0,\infty)\to\mathbb{R}$ *be operator monotone in* $[0,\infty)$*. Assume that* $U\geq u>0$*, then for all selfadjoint operators* V *we have*

$$\|Dh(U)(V)\|\leq h'(u)\|V\|. \tag{3.4}$$

Proof. From (3.3) we get

$$\begin{aligned}
\|Dh(U)(V)-bV\| &\leq \int_0^\infty\lambda^2\left\|(\lambda+U)^{-1}V(\lambda+U)^{-1}\right\|d\mu(\lambda) \\
&\leq \|V\|\int_0^\infty\lambda^2\left\|(\lambda+U)^{-1}\right\|^2d\mu(\lambda).
\end{aligned}\tag{3.5}$$

Observe that $\lambda+U\geq\lambda+u>0$ for $\lambda\in[0,\infty)$. Then $0<(\lambda+U)^{-1}\leq(\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\|\leq(\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2\leq(\lambda+u)^{-2}$ for $\lambda\in[0,\infty)$.

Therefore by (3.5) we get

$$\|Dh(U)(V)-bV\|\leq\|V\|\int_0^\infty\lambda^2(\lambda+u)^{-2}d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$h'(t) = b + \int_0^\infty \frac{\lambda(t+\lambda) - \lambda t}{(t+\lambda)^2} d\mu(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) \tag{3.7}$$

for $t > 0$.

From (3.7) we get

$$\int_0^\infty \lambda^2 (\lambda + u)^{-2} d\mu(\lambda) = h'(u) - b,$$

and by (3.6) we derive

$$\|Dh(U)(V) - bV\| \le \|V\| h'(u) - b\|V\|.$$

Finally, by the triangle inequality and by the fact that $b \ge 0$, we obtain that

$$\|Dh(U)(V)\| - b\|V\| \le \|Dh(U)(V) - bV\|,$$

which proves the desired result (3.4). □

For a continuous function h on $(0, \infty)$ and A, $B > 0$ we consider the auxiliary function $h_{A,B} : [0,1] \to \mathbb{R}$ defined by

$$h_{A,B}(t) := h((1-t)A + tB), \ t \in [0,1].$$

We have the following representations of the derivatives:

Lemma 3. *Assume that the operator function generated by h is Fréchet differentiable in each $A \ge 0$, then for $B \ge 0$ we have that $h_{A,B}$ is differentiable on $[0,1]$ and*

$$h'_{A,B}(t) = D(h)((1-t)A + tB)(B - A) \tag{3.8}$$

for $t \in [0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned}
&\frac{h_{A,B}(t+h) - h(t)}{h} \\
&= \frac{h((1-(t+h))A + (t+h)B) - h((1-t)A + tB)}{h} \\
&= \frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}
h'_{A,B}(t) &= \lim_{h\to 0} \frac{h_{A,B}(t+h) - h(t)}{h} \\
&= \lim_{h\to 0} \left[\frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h}\right] \\
&= D(h)((1-t)A + tB)(B - A),
\end{aligned}$$

which proves (3.8). □

Corollary 4. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A \geq a > 0$, $B \geq b > 0$ we have*

$$\begin{aligned} \left\| h'_{A,B}(t) \right\| &= \left\| D(h)((1-t)A + tB)(B-A) \right\| \\ &\leq h'((1-t)a + tb) \left\| B - A \right\| \end{aligned} \tag{3.9}$$

for all $t \in [0,1]$.

The proof follows by Lemma 2 and Lemma 3.

One can observe that the inequality (3.9) remains valid for operator monotone functions on $(0,\infty)$. This follows by considering the function $h_\varepsilon(t) := h(t+\varepsilon)$ for $\varepsilon > 0$, which is operator monotone on $[0,\infty)$ and then by letting $\varepsilon \to 0+$ and using the continuity of h and h'.

Theorem 6. *Let $f : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$ and $\alpha : [0,1] \to \mathbb{C}$ a continuous function on $[0,1]$. Then for all $A \geq a > 0$, $B \geq b > 0$ we have for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\begin{aligned} &\left\| \int_0^1 \alpha(t) f((1-t)A + tB)\,dt - \left(\int_0^1 \alpha(s)\,ds \right) f((1-u)A + uB) \right\| \\ &\leq \left\| B - A \right\| \\ &\times \begin{cases} \max\left\{ \int_u^1 |\alpha(s)|\,ds, \int_0^u |\alpha(s)|\,ds \right\} \times \int_0^1 h'((1-t)a + tb) \\ \\ \sup_{t\in[0,1]} h'((1-t)a + tb) \\ \quad \times \begin{cases} \left[\frac{1}{2} + \left| u - \frac{1}{2} \right| \right] \int_0^1 |\alpha(t)|\,dt, \\ \\ \left[\frac{u^{q+1} + (1-u)^{q+1}}{q+1} \right]^{1/q} \left(\int_0^1 |\alpha(t)|^p\,dt \right)^{1/p}, \\ \\ \left[\frac{1}{4} + \left(u - \frac{1}{2} \right)^2 \right] \sup_{t\in[0,1]} |\alpha(t)|, \end{cases} \\ \\ \left[\left(\int_u^1 |\alpha(s)|\,ds \right)^p + \left(\int_0^u |\alpha(s)|\,ds \right)^p \right]^{1/p} \\ \quad \times \left(\int_0^1 [h'((1-t)a + tb)]^q\,dt \right)^{1/q} \end{cases} \end{aligned} \tag{3.10}$$

Proof. We use inequality (2.5) for α and $f_{A,B}$ on $[0,1]$ to get

$$\begin{aligned} &\left\| \int_0^1 \alpha(t) f((1-t)A + tB)\,dt - \left(\int_0^1 \alpha(s)\,ds \right) f((1-u)A + uB) \right\| \\ &\leq \max\left\{ \int_u^1 |\alpha(s)|\,ds, \int_0^u |\alpha(s)|\,ds \right\} \int_a^b \left\| f'_{A,B}(t) \right\| dt, \end{aligned} \tag{3.11}$$

and since

$$\left\| f'_{A,B}(t) \right\| \leq f'((1-t)a + tb) \left\| B - A \right\|,$$

hence by (3.11) we obtain the first inequality in (3.10).

The second branch follows by (2.10) and the third branch follows by (2.11). $\square$

Corollary 5. *With the assumptions of Theorem 6 we have the midpoint inequalities*

$$\begin{aligned}&\left\|\int_0^1 \alpha(t) f((1-t)A+tB)\,dt - \left(\int_0^1 \alpha(s)\,ds\right) f\left(\frac{A+B}{2}\right)\right\| \\ &\leq \|B-A\| \\ &\times\begin{cases} \max\left\{\int_{1/2}^1 |\alpha(s)|\,ds, \int_0^{1/2} |\alpha(s)|\,ds\right\} \times \int_0^1 h'((1-t)a+tb) \\ \\ \frac{1}{2}\sup_{t\in[0,1]} h'((1-t)a+tb) \\ \times\begin{cases} \int_0^1 |\alpha(t)|\,dt, \\ \\ \frac{1}{(1+q)^{1/q}}\left(\int_0^1 |\alpha(t)|^p\,dt\right)^{1/p}, \\ \\ \frac{1}{2}\sup_{t\in[0,1]}|\alpha(t)|, \end{cases} \\ \\ \left[\left(\int_{1/2}^1 |\alpha(s)|\,ds\right)^p + \left(\int_0^{1/2} |\alpha(s)|\,ds\right)^p\right]^{1/p} \\ \times\left(\int_0^1 [h'((1-t)a+tb)]^q\,dt\right)^{1/q} \end{cases}\end{aligned} \tag{3.12}$$

for all $A \geq a > 0$, $B \geq b > 0$ *and* $p, q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

If we take $f(t) = t^r$, $r \in (0,1)$, which is operator monotone on $[0,\infty)$, then

$$\begin{aligned}&\left\|\int_0^1 \alpha(t)((1-t)A+tB)^r\,dt - \left(\int_0^1 \alpha(s)\,ds\right)\left(\frac{A+B}{2}\right)^r\right\| \\ &\leq r\|B-A\| \\ &\times\begin{cases} \max\left\{\int_{1/2}^1 |\alpha(s)|\,ds, \int_0^{1/2} |\alpha(s)|\,ds\right\} \times \begin{cases} \frac{b^r-a^r}{r(b-a)} \text{ if } b\neq a, \\ \\ a^{r-1} \text{ if } b=a, \end{cases} \\ \\ \frac{1}{2}\max\left\{a^{r-1}, b^{r-1}\right\} \times \begin{cases} \int_0^1 |\alpha(t)|\,dt, \\ \\ \frac{1}{(1+q)^{1/q}}\left(\int_0^1 |\alpha(t)|^p\,dt\right)^{1/p}, \\ \\ \frac{1}{2}\sup_{t\in[0,1]}|\alpha(t)|, \end{cases} \\ \\ \left[\left(\int_{1/2}^1 |\alpha(s)|\,ds\right)^p + \left(\int_0^{1/2} |\alpha(s)|\,ds\right)^p\right]^{1/p} \\ \times\begin{cases} \left(\frac{b^{q(r-1)+1}-a^{q(r-1)+1}}{[q(r-1)+1](b-a)}\right)^{1/q} \text{ if } b\neq a,\ q(r-1)+1\neq 0,\ q>1, \\ \\ a^{r-1} \text{ if } b=a, \end{cases} \end{cases}\end{aligned} \tag{3.13}$$

for all $A \geq a > 0$, $B \geq b > 0$ and $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$.

If we take $f(t) = \ln t$, which is operator monotone on $(0,\infty)$, then

$$\left\| \int_0^1 \alpha(t) \ln((1-t)A + tB)\,dt - \left(\int_0^1 \alpha(s)\,ds\right) \ln\left(\frac{A+B}{2}\right)\right\| \tag{3.14}$$
$$\leq \|B-A\| \times \begin{cases} \max\left\{\int_{1/2}^1 |\alpha(s)|\,ds, \int_0^{1/2} |\alpha(s)|\,ds\right\} \times \begin{cases} \frac{\ln b - \ln a}{b-a} \text{ if } b \neq a, \\ \frac{1}{a} \text{ if } b = a, \end{cases} \\ \frac{1}{2\min\{a,b\}} \times \begin{cases} \int_0^1 |\alpha(t)|\,dt, \\ \frac{1}{(1+q)^{1/q}} \left(\int_0^1 |\alpha(t)|^p\,dt\right)^{1/p}, \\ \frac{1}{2}\sup_{t\in[0,1]} |\alpha(t)|, \end{cases} \\ \left[\left(\int_{1/2}^1 |\alpha(s)|\,ds\right)^p + \left(\int_0^{1/2} |\alpha(s)|\,ds\right)^p\right]^{1/p} \\ \times \begin{cases} \left(\frac{b^{1-q}-a^{1-q}}{(1-q)(b-a)}\right)^{1/q} \text{ if } b \neq a, \text{ for all } q > 1, \\ \frac{1}{a} \text{ if } b = a. \end{cases} \end{cases}$$

References

[1] M. W. Alomari, A generalization of weighted companion of Ostrowski integral inequality for mappings of bounded variation. *Int. J. Nonlinear Sci. Numer. Simul.* **21** (2020), no. 7-8, 667–673.

[2] M. W. Alomari, A weighted companion of Ostrowski-midpoint inequality for mappings of bounded variation. *Konuralp J. Math.* **7** (2019), no. 2, 337–343.

[3] N. S. Barnett, C. Buşe, P. Cerone and S. S. Dragomir, On weighted Ostrowski type inequalities for operators and vector-valued functions. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 1, Article 12, 21 pp.

[4] R. Bhatia, *Matrix Analysis.* Graduate Texts in Mathematics, 169. Springer-Verlag, New York, 1997. xii+347 pp. ISBN: 0-387-94846-5.

[5] S. S. Dragomir, A refinement of Ostrowski's inequality for absolutely continuous functions whose derivatives belong to L_∞ and applications. *Libertas Math.* **22** (2002), 49–63.

[6] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31.[Online https://www.emis.de/journals/JIPAM/article183.html?sid=183]

[7] S. S. Dragomir, Ostrowski type inequalities for Lebesgue integral: a survey of recent results. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 1, 283 pp.

[8] S. S. Dragomir, Reverses of Féjer's inequalities for convex functions, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art. 88, 11pp. [Online https://rgmia.org/papers/v22/v22a88.pdf].

[9] S. S. Dragomir, Bounds for the difference between weighted and integral means of convex function, Preprint *RGMIA Res. Rep. Coll.* **22** (2019), Art. 95, 10 pp. [Online shttps://rgmia.org/papers/v22/v22a95.pdf].

[10] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. (ONLINE: http://ajmaa.org/RGMIA/monographs.php).

[11] L. Féjer, Über die Fourierreihen, II, (In Hungarian). *Math. Naturwiss, Anz. Ungar. Akad. Wiss.*, **24** (1906), 369-390.

[12] H. Hong, A new companion of Ostrowski's inequality and its applications. *Kragujevac J. Math.* **43** (2019), no. 3, 443–449.
[13] N. Irshad and A. R. Khan, Some applications of quadrature rules for mappings on Lp[u,v] space via Ostrowski-type inequality. *J. Numer. Anal. Approx. Theory* **46** (2017), no. 2, 141–149.
[14] S. Kermausuor, A generalization of Ostrowski's inequality for functions of bounded variation via a parameter. *Aust. J. Math. Anal. Appl.* **16** (2019), no. 1, Art. 16, 12 pp.
[15] J. Mikusiński, *The Bochner Integral*, Birkhäuser Verlag, 1978.
[16] D. S. Mitrinović, J. E. Pečarić and A .M. Fink, *Inequalities for Functions and Their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht, 1994.
[17] A. Ostrowski, Über die Absolutabweichung einer differentiierbaren Funktion von ihrem Integralmittelwert, *Comment. Math. Helv.,* **10** (1938), 226-227.
[18] S. Obeidat, M. A. Latif and A. Qayyum, A weighted companion of Ostrowski's inequality using three step weighted kernel. *Miskolc Math. Notes* **20** (2019), no. 2, 1101–1118.

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