TRAPEZOID TYPE INEQUALITIES FOR FUNCTIONS WITH VALUES IN BANACH SPACES

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Abstract. Let E be a complex Banach space. In this paper we show among others that, if $\alpha : [a,b] \to \mathbb{C}$ is continuous and $Y : [a,b] \to E$ is strongly differentiable on the interval (a,b), then

$$\begin{aligned}
&\left\| \left(\int_a^b \alpha(s)\,ds \right) \frac{Y(a)+Y(b)}{2} - \int_a^b \alpha(t)\,Y(t)\,dt \right\| \\
&\le \frac{1}{2} \int_a^b \left| \int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds \right| \left\| Y'(t) \right\| dt \\
&\le \frac{1}{2} \int_a^b |\alpha(s)|\,dt \times \begin{cases} \int_a^b \|Y'(t)\|\,dt, \\ (b-a)^{1/p} \left(\int_a^b \|Y'(t)\|^q\,dt \right)^{1/q}, \\ (b-a) \sup_{t\in[a,b]} \|Y'(t)\|. \end{cases}
\end{aligned}$$

Applications for operator monotone functions in Hilbert spaces with examples for power and logarithmic functions are also given.

1. Introduction

Assume that $p : [a,b] \to [0,\infty)$ is integrable on $[a,b]$ and $f : [a,b] \to \mathbb{R}$ is of bounded variation, then the following weighted trapezoid inequality holds [16]:

$$\left| \frac{f(a)+f(b)}{2} \int_a^b p(t)\,dt - \int_a^b p(t)\,f(t)\,dt \right| \le \frac{1}{2} \bigvee_a^b (f) \int_a^b p(t)\,dt, \tag{1.1}$$

where $\bigvee_a^b (f)$ denotes the total variation of f on $[a,b]$.

We have the following refinement and reverse of the weighted trapezoid inequality [5]:

Theorem 1. *Let f be a convex function on I and $a,\ b \in I$, with $a < b$. If $p : [a,b] \to [0,\infty)$ is Lebesgue integrable and symmetric, namely $p(b+a-t) = p(t)$ for all $t \in [a,b]$, then*

$$\begin{aligned}
0 &\le \frac{1}{2} \int_a^b \left(\frac{1}{2} - \left| t - \frac{a+b}{2} \right| \right) p(t)\,dt \left[f'_+\left(\frac{a+b}{2}\right) - f'_-\left(\frac{a+b}{2}\right) \right] \\
&\le \left(\int_a^b p(t)\,dt \right) \frac{f(a)+f(b)}{2} - \int_a^b p(t)\,f(t)\,dt \\
&\le \frac{1}{2} \int_a^b \left(\frac{1}{2} - \left| t - \frac{a+b}{2} \right| \right) p(t)\,dt \left[f'_-(b) - f'_+(a) \right].
\end{aligned} \tag{1.2}$$

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For $p \equiv 1$ we have the unweighted inequalities

$$0 \leq \frac{1}{8}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right](b-a)^{2} \tag{1.3}$$
$$\leq \frac{f(a)+f(b)}{2}(b-a)-\int_{a}^{b} f(t)\,dt$$
$$\leq \frac{1}{8}\left[f'_{-}(b)-f'_{+}(a)\right](b-a)^{2},$$

that was obtained by the author in 2002, see [4].

For $p(t)=\left|t-\frac{a+b}{2}\right|$, $t\in[a,b]$ we get

$$0 \leq \frac{1}{48}(b-a)^{3}\left[f'_{+}\left(\frac{a+b}{2}\right)-f'_{-}\left(\frac{a+b}{2}\right)\right] \tag{1.4}$$
$$\leq \frac{f(a)+f(b)}{8}(b-a)^{2}-\int_{a}^{b}\left|t-\frac{a+b}{2}\right| f(t)\,dt$$
$$\leq \frac{1}{48}(b-a)^{3}\left[f'_{-}(b)-f'_{+}(a)\right].$$

For some recent results related to the trapezoid type inequalities, see [1] and [11]-[15].

Let $(H,\langle\cdot,\cdot\rangle)$ be a Hilbert space. The continuous function f is operator monotone on the interval I if for any selfadjoint operators A and B with spectra $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset I$,

$$f((1-t)A+tB) \leq (1-t)f(A)+tf(B)$$

for all $t\in[0,1]$.

In the recent paper [10] we obtained the following result:

Theorem 2. *Let f be an operator convex function on I and A, B, $A\neq B$, selfadjoint operators with $\operatorname{Sp}(A)$, $\operatorname{Sp}(B)\subset I$. If f is Gâteaux differentiable on $[A,B]:=\{(1-t)A+tB,\ t\in[0,1]\}$ and $p:[0,1]\to[0,\infty)$ is Lebesgue integrable and symmetric, namely $p(1-t)=p(t)$ for all $t\in[0,1]$, then*

$$0 \leq \left(\int_{0}^{1} p(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_{0}^{1} p(t) f((1-t)A+tB)\,dt \tag{1.5}$$
$$\leq \frac{1}{2}\int_{0}^{1}\left(\frac{1}{2}-\left|t-\frac{1}{2}\right|\right)p(t)\,dt\left[\nabla f_{B}(B-A)-\nabla f_{A}(B-A)\right],$$

where $\nabla f_{C}(V)$ is the Gâteaux derivative in C over the direction V.

In particular, for $p\equiv 1$ we get

$$0 \leq \frac{f(A)+f(B)}{2}-\int_{0}^{1} f((1-t)A+tB)\,dt \tag{1.6}$$
$$\leq \frac{1}{8}\left[\nabla f_{B}(B-A)-\nabla f_{A}(B-A)\right].$$

For some trapezoid operator inequalities, see [8], [9], [6] and [7].

Let E be a complex Banach space. In this paper we show among others that, if $\alpha:[a,b]\to\mathbb{C}$ is continuous and $Y:[a,b]\to E$ is strongly differentiable on the interval (a,b) , then

$$\begin{aligned}&\left\|\left(\int_a^b \alpha(s)\,ds\right)\frac{Y(a)+Y(b)}{2}-\int_a^b \alpha(t)\,Y(t)\,dt\right\|\\&\leq \frac{1}{2}\int_a^b\left|\int_a^t \alpha(s)\,ds-\int_t^b \alpha(s)\,ds\right|\left\|Y'(t)\right\|dt\\&\leq \frac{1}{2}\int_a^b|\alpha(s)|\,dt\times\begin{cases}\int_a^b\left\|Y'(t)\right\|dt,\\[2ex](b-a)^{1/p}\left(\int_a^b\left\|Y'(t)\right\|^q dt\right)^{1/q},\\[2ex](b-a)\sup_{t\in[a,b]}\left\|Y'(t)\right\|.\end{cases}\end{aligned}$$

Applications for operator monotone functions in Hilbert spaces with examples for power and logarithmic functions are also given.

2. Main Results

We have the following weighted version of generalized trapezoid inequality for two functions with values in Banach algebras:

Theorem 3. *Let $\alpha:[a,b]\to\mathbb{C}$ be continuous and $Y:[a,b]\to E$ be strongly differentiable on the interval (a,b), then*

$$\begin{aligned}&\left\|\left(\int_a^b \alpha(s)\,ds\right)\frac{Y(a)+Y(b)}{2}-\int_a^b \alpha(t)\,Y(t)\,dt\right\|\\&\leq \frac{1}{2}\int_a^b\left|\int_a^t \alpha(s)\,ds-\int_t^b \alpha(s)\,ds\right|\left\|Y'(t)\right\|dt=:D(\alpha,Y)\\&\leq \frac{1}{2}\int_a^b|\alpha(s)|\,ds\int_a^b\left\|Y'(t)\right\|dt.\end{aligned}\tag{2.1}$$

We also have the bounds,

$$\begin{aligned}&D(\alpha,Y)\\&\leq\frac{1}{2}\times\begin{cases}\sup_{t\in[a,b]}\left|\int_a^t \alpha(s)\,ds-\int_t^b \alpha(s)\,ds\right|\int_a^b\left\|Y'(t)\right\|dt,\\[2ex]\left(\int_a^b\left|\int_a^t \alpha(s)\,ds-\int_t^b \alpha(s)\,ds\right|^p dt\right)^{1/p}\left(\int_a^b\left\|Y'(t)\right\|^q dt\right)^{1/q},\\[2ex]\int_a^b\left|\int_a^t \alpha(s)\,ds-\int_t^b \alpha(s)\,ds\right|dt\sup_{t\in[a,b]}\left\|Y'(t)\right\|,\end{cases}\\&\leq \frac{1}{2}\int_a^b|\alpha(s)|\,dt\times\begin{cases}\int_a^b\left\|Y'(t)\right\|dt,\\[2ex](b-a)^{1/p}\left(\int_a^b\left\|Y'(t)\right\|^q dt\right)^{1/q},\\[2ex](b-a)\sup_{t\in[a,b]}\left\|Y'(t)\right\|\end{cases}\end{aligned}\tag{2.2}$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using the integration by parts formula for Bochner integral, we have

$$\begin{aligned}
&\int_a^b \left[\int_a^t \alpha(s)\,ds - \frac{1}{2}\int_a^b \alpha(s)\,ds\right] Y'(t)\,dt \\
&= \left[\int_a^t \alpha(s)\,ds - \frac{1}{2}\int_a^b \alpha(s)\,ds\right] Y(t)\Big|_a^b \\
&- \int_a^b \left[\int_a^t \alpha(s)\,ds - \frac{1}{2}\int_a^b \alpha(s)\,ds\right]' Y(t)\,dt \\
&= \left[\int_a^b \alpha(s)\,ds - \frac{1}{2}\int_a^b \alpha(s)\,ds\right] Y(b) \\
&- \left[\int_a^a \alpha(s)\,ds - \frac{1}{2}\int_a^b \alpha(s)\,ds\right] Y(a) - \int_a^b \alpha(t)\,Y(t)\,dt \\
&= \left(\int_a^b \alpha(s)\,ds\right) \frac{Y(a)+Y(b)}{2} - \int_a^b \alpha(t)\,Y(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left[\int_a^t \alpha(s)\,ds - \frac{1}{2}\int_a^b \alpha(s)\,ds\right] Y'(t)\,dt \\
&= \frac{1}{2}\int_a^b \left(\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right) Y'(t)\,dt.
\end{aligned}$$

Therefore we have the following identity of interest

$$\begin{aligned}
&\left(\int_a^b \alpha(s)\,ds\right) \frac{Y(a)+Y(b)}{2} - \int_a^b \alpha(t)\,Y(t)\,dt \\
&= \frac{1}{2}\int_a^b \left(\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right) Y'(t)\,dt.
\end{aligned} \tag{2.3}$$

Taking the norm in (2.3) and using the integral's properties, we get

$$\begin{aligned}
&\left\|\left(\int_a^b \alpha(s)\,ds\right) \frac{Y(a)+Y(b)}{2} - \int_a^b \alpha(t)\,Y(t)\,dt\right\| \\
&\leq \frac{1}{2}\int_a^b \left\|\left(\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right) Y'(t)\right\| dt
\end{aligned}$$

$$\leq \frac{1}{2}\int_a^b \left|\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right| \|Y'(t)\|\,dt$$
$$\leq \frac{1}{2}\int_a^b \left(\left|\int_a^t \alpha(s)\,ds\right| + \left|\int_t^b \alpha(s)\,ds\right|\right) \|Y'(t)\|\,dt$$
$$\leq \frac{1}{2}\int_a^b \left(\int_a^t |\alpha(s)|\,ds + \int_t^b |\alpha(s)|\,ds\right) \|Y'(t)\|\,dt$$
$$= \frac{1}{2}\int_a^b |\alpha(s)|\,ds \int_a^b \|Y'(t)\|\,dt.$$

The first inequality in (2.2) follows by Hölder's inequality applied to the integral of the product

$$\int_a^b \left|\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right| \|Y'(t)\|\,dt.$$

The last part follows by the fact that

$$\left|\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right| \leq \int_a^b |\alpha(s)|\,ds$$

for all $t \in [a,b]$.

This implies that

$$\sup_{t\in[a,b]} \left|\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right| \leq \int_a^b |\alpha(s)|\,ds,$$

$$\int_a^b \left|\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right|^p dt \leq \int_a^b \left(\int_a^b |\alpha(s)|\,ds\right)^p dt = (b-a)\left(\int_a^b |\alpha(s)|\,ds\right)^p$$

and

$$\int_a^b \left|\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right| dt \leq (b-a)\int_a^b |\alpha(s)|\,ds,$$

and the second part of (2.2) is thus proved. □

Corollary 1. *Assume that $\alpha : [a,b] \to \mathbb{C}$, $Z : [a,b] \to E$ are continuous and Z is strongly differentiable on (a,b) and such that*

$$\|Z'(t) - v\| \leq M \text{ for all } t \in (a,b)$$

for some vector $v \in E$ *and* $M > 0,$ *then*

$$\begin{aligned}(2.4)\qquad & \left\| \left(\int_a^b \alpha(s)\,ds \right) \frac{Z(a)+Z(b)}{2} \right. \\ & \left. + \int_a^b \left(t - \frac{a+b}{2} \right) \alpha(t)\,dtv - \int_a^b \alpha(t) Z(t)\,dt \right\| \\ & \leq \frac{1}{2} M \int_a^b \left| \int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds \right| dt \\ & \leq \frac{1}{2} M (b-a) \int_a^b |\alpha(s)|\,ds.\end{aligned}$$

Moreover, if α *is symmetric on* $[a,b]$, *namely* $\alpha(a+b-t) = \alpha(t)$ *for all* $t \in [a,b]$, *then*

$$\begin{aligned}(2.5)\qquad & \left\| \left(\int_a^b \alpha(s)\,ds \right) \frac{Z(a)+Z(b)}{2} - \int_a^b \alpha(t) Z(t)\,dt \right\| \\ & \leq \frac{1}{2} M \int_a^b \left| \int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds \right| dt \leq \frac{1}{2} M (b-a) \int_a^b |\alpha(s)|\,ds.\end{aligned}$$

Proof. Put $Y(t) = Z(t) - tv$, $t \in [0,1]$, then

$$\begin{aligned}& \left(\int_a^b \alpha(s)\,ds \right) \frac{Y(a)+Y(b)}{2} - \int_a^b \alpha(t) Y(t)\,dt \\ & = \left(\int_a^b \alpha(s)\,ds \right) \frac{Z(a)-av+Z(b)-bv}{2} - \int_a^b \alpha(t)[Z(t)-tv]\,dt \\ & = \left(\int_a^b \alpha(s)\,ds \right) \frac{Z(a)+Z(b)}{2} - \frac{a+b}{2} \left(\int_a^b \alpha(s)\,ds \right) v \\ & - \int_a^b \alpha(t) Z(t)\,dt + \int_a^b t\alpha(t)\,vdt \\ & = \left(\int_a^b \alpha(s)\,ds \right) \frac{Z(a)+Z(b)}{2} + \int_a^b t\alpha(t)\,vdt \\ & - \frac{a+b}{2} \left(\int_a^b \alpha(s)\,ds \right) v - \int_a^b \alpha(t) Z(t)\,dt \\ & = \left(\int_a^b \alpha(s)\,ds \right) \frac{Z(a)+Z(b)}{2} + \int_a^b \left(t - \frac{a+b}{2} \right) \alpha(t)\,vdt \\ & - \int_a^b \alpha(t) Z(t)\,dt.\end{aligned}$$

Also

$$\begin{aligned}
&\int_a^b \left|\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right| \|Y'(t)\|\,dt \\
&= \int_a^b \left|\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right| \|Z'(t) - v\|\,dt \\
&\leq M \int_a^b \left|\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right| dt \leq M(b-a)\int_a^b |\alpha(s)|\,ds.
\end{aligned}$$

By utilising (2.1) we derive the desired result (2.4).

Now, if α is symmetric on $[a,b]$, then $\left(\cdot - \frac{a+b}{2}\right)\alpha$ is antisymmetric on $[a,b]$, which implies that

$$\int_a^b \left(t - \frac{a+b}{2}\right)\alpha(t)\,dt = 0,$$

and the inequality (2.5) is thus proved. □

Remark 1. *If there exists u, $w \in E$ such that*

$$\left\|Z'(t) - \frac{u+w}{2}\right\| \leq \frac{1}{2}\|w-u\| \text{ for all } t \in (a,b), \tag{2.6}$$

then by (2.4) we get

$$\begin{aligned}
&\left\|\left(\int_a^b \alpha(s)\,ds\right)\frac{Z(a)+Z(b)}{2}\right. \\
&\left.+\int_a^b \left(t - \frac{a+b}{2}\right)\alpha(t)\,dt\left(\frac{u+w}{2}\right) - \int_a^b \alpha(t)\,Z(t)\,dt\right\| \\
&\leq \frac{1}{4}\|w-u\| \int_a^b \left|\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right| dt \\
&\leq \frac{1}{4}\|w-u\| (b-a)\int_a^b |\alpha(s)|\,ds.
\end{aligned} \tag{2.7}$$

Moreover, if α is symmetric on $[a,b]$, then

$$\begin{aligned}
&\left\|\left(\int_a^b \alpha(s)\,ds\right)\frac{Z(a)+Z(b)}{2} - \int_a^b \alpha(t)\,Z(t)\,dt\right\| \\
&\leq \frac{1}{4}\|w-u\| \int_a^b \left|\int_a^t \alpha(s)\,ds - \int_t^b \alpha(s)\,ds\right| dt \\
&\leq \frac{1}{4}\|w-u\|(b-a)\int_a^b |\alpha(s)|\,ds.
\end{aligned} \tag{2.8}$$

We also observe that, if $E = H$ a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$, then the condition (2.6) is equivalent to

$$\operatorname{Re}\langle w - Z'(t), Z'(t) - u\rangle \geq 0 \text{ for all } t \in (a,b). \tag{2.9}$$

This follows by the simple identity

$$\operatorname{Re}\langle w-z,z-u\rangle = \frac{1}{4}\|w-u\|^2 - \left\|z-\frac{u+w}{2}\right\|^2$$

that holds for all u, w, z *in* H.

Remark 2. *If we take* $\alpha(s)=1$, $s\in[a,b]$, *then by (2.1) we get*

$$\left\|(b-a)\frac{Y(a)+Y(b)}{2}-\int_a^b Y(t)\,dt\right\| \le \int_a^b\left|t-\frac{a+b}{2}\right|\|Y'(t)\|\,dt \le \frac{1}{4}(b-a)^2\sup_{t\in[a,b]}\|Y'(t)\|. \tag{2.10}$$

If Z *is strongly differentiable on* (a,b) *and*

$$\|Z'(t)-v\|\le M \text{ for all } t\in(a,b)$$

for some element $v\in E$ *and* $M>0$, *then*

$$\left\|(b-a)\frac{Z(a)+Z(b)}{2}-\int_a^b Z(t)\,dt\right\|\le\frac{1}{4}(b-a)^2 M. \tag{2.11}$$

If there exists u, $w\in E$ *such that (2.6) holds, then*

$$\left\|(b-a)\frac{Z(a)+Z(b)}{2}-\int_a^b Z(t)\,dt\right\|\le\frac{1}{8}(b-a)^2\|w-u\|. \tag{2.12}$$

We also have the dual result:

Theorem 4. *Let* $\alpha:[a,b]\to\mathbb{C}$ *be differentiable on* (a,b) *and* $Y:[a,b]\to E$ *be continuous on the interval* $[a,b]$, *then*

$$\left\|\frac{\alpha(a)+\alpha(b)}{2}\left(\int_a^b Y(s)\,ds\right)-\int_a^b\alpha(t)Y(t)\,dt\right\| \le \frac{1}{2}\int_a^b|\alpha'(t)|\left\|\int_a^t Y(s)\,ds-\int_t^b Y(s)\,ds\right\|dt =: \tilde{D}(\alpha,Y) \le \frac{1}{2}\int_a^b|\alpha'(s)|\,ds\int_a^b\|Y(t)\|\,dt. \tag{2.13}$$

We also have the bounds,

$$\tilde{D}(\alpha,Y) \tag{2.14}$$

$$\leq \frac{1}{2}\times\begin{cases}\sup_{t\in[a,b]}\left\|\int_a^t Y(s)\,ds-\int_t^b Y(s)\,ds\right\|\int_a^b|\alpha'(t)|\,dt,\\[2ex] \left(\int_a^b\left\|\int_a^t Y(s)\,ds-\int_t^b Y(s)\,ds\right\|^p dt\right)^{1/p}\left(\int_a^b|\alpha'(t)|^q\,dt\right)^{1/q},\\[2ex] \int_a^b\left|\left\|\int_a^t Y(s)\,ds-\int_t^b Y(s)\,ds\right\|\right|dt\sup_{t\in[a,b]}|\alpha'(t)|,\end{cases}$$

$$\leq \frac{1}{2}\int_a^b\|Y(t)\|\,dt\times\begin{cases}\int_a^b|\alpha'(t)|\,dt,\\[2ex] (b-a)^{1/p}\,t\left(\int_a^b|\alpha'(t)|^q\,dt\right)^{1/q},\\[2ex] (b-a)\sup_{t\in[a,b]}|\alpha'(t)|\end{cases}$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using integration by parts we also have

$$\begin{aligned}&\int_a^b\alpha'(t)\left(\int_a^t Y(s)\,ds-\int_t^b Y(s)\,ds\right)dt\\ &=\alpha(t)\left(\int_a^t Y(s)\,ds-\int_t^b Y(s)\,ds\right)\Bigg|_a^b\\ &-\int_a^b\alpha(t)\,d\left(\int_a^t Y(s)\,ds-\int_t^b Y(s)\,ds\right)\\ &=\alpha(b)\left(\int_a^b Y(s)\,ds\right)-\alpha(a)\left(-\int_a^b Y(s)\,ds\right)\\ &-\int_a^b\alpha(t)\,(Y(t)+Y(t))\,dt\\ &=[\alpha(a)+\alpha(b)]\int_a^b Y(s)\,ds-2\int_a^b\alpha(t)\,Y(t)\,dt\end{aligned}$$

and dividing by 2, we obtain the dual identity of (2.3)

$$\begin{aligned}&\frac{\alpha(a)+\alpha(b)}{2}\left(\int_a^b Y(s)\,ds\right)-\int_a^b\alpha(t)\,Y(t)\,dt\\ &=\frac{1}{2}\int_a^b\alpha'(t)\left(\int_a^t Y(s)\,ds-\int_t^b Y(s)\,ds\right)dt.\end{aligned}\tag{2.15}$$

By utilising now a similar argument to the one in the proof of Theorem 3 we obtain the desired results. □

Corollary 2. *Let $\alpha:[a,b]\to\mathbb{C}$ be differentiable on (a,b) and $Z:[a,b]\to E$ be continuous on the interval $[a,b]$. If there exists the constant $\beta\in\mathbb{C}$, $L>0$ such that*

$|\alpha'(t)-\beta|\leq L$ *for all* $t\in(a,b)$, *then*

$$\left\|\frac{\alpha(a)+\beta(b)}{2}\left(\int_a^b Z(s)\,ds\right)+\beta\int_a^b\left(t-\frac{a+b}{2}\right)Z(t)\,dt-\int_a^b\alpha(t)\,Z(t)\,dt\right\| \leq\frac{1}{2}L\int_a^b\left\|\int_a^t Z(s)\,ds-\int_t^b Z(s)\,ds\right\|dt \leq\frac{1}{2}L(b-a)\int_a^b\|Z(t)\|\,dt. \tag{2.16}$$

If Z *is symmetric on* $[a,b]$, *then*

$$\left\|\frac{\alpha(a)+\beta(b)}{2}\int_a^b Z(s)\,ds-\int_a^b\alpha(t)\,Z(t)\,dt\right\| \leq\frac{1}{2}L\int_a^b\left\|\int_a^t Z(s)\,ds-\int_t^b Z(s)\,ds\right\|dt\leq\frac{1}{2}L(b-a)\int_a^b\|Z(t)\|\,dts. \tag{2.17}$$

Remark 3. *If there exists* $\gamma,\ \Gamma\in\mathbb{C}$ *such that*

$$\left|\alpha'(t)-\frac{\gamma+\Gamma}{2}\right|\leq\frac{1}{2}|\Gamma-\gamma|\ \text{ for all } t\in(a,b),$$

or, equivalently

$$\operatorname{Re}(\Gamma-\alpha'(t))\left(\overline{\alpha'(t)}-\overline{\gamma}\right)\geq 0 \text{ for all } t\in(a,b),$$

then by (2.16)

$$\left\|\frac{\alpha(a)+\beta(b)}{2}\left(\int_a^b Z(s)\,ds\right)+\frac{\gamma+\Gamma}{2}\int_a^b\left(t-\frac{a+b}{2}\right)Z(t)\,dt-\int_a^b\alpha(t)\,Z(t)\,dt\right\| \leq\frac{1}{4}|\Gamma-\gamma|\int_a^b\left\|\int_a^t Z(s)\,ds-\int_t^b Z(s)\,ds\right\|dt \leq\frac{1}{4}|\Gamma-\gamma|(b-a)\int_a^b\|Z(t)\|\,dt \tag{2.18}$$

and if Z *is symmetric on* $[a,b]$, *then by (2.17) we get*

$$\left\|\frac{\alpha(a)+\beta(b)}{2}\int_a^b Z(s)\,ds-\int_a^b\alpha(t)\,Z(t)\,dt\right\| \leq\frac{1}{4}|\Gamma-\gamma|\int_a^b\left\|\int_a^t Z(s)\,ds-\int_t^b Z(s)\,ds\right\|dt \leq\frac{1}{4}|\Gamma-\gamma|(b-a)\int_a^b\|Z(t)\|\,dt. \tag{2.19}$$

If $\alpha:[a,b]\to\mathbb{R}$ and $k\le\alpha'(t)\le K$ for $t\in(a,b)$, then by (2.18) we get

$$\begin{aligned}\tag{2.20}&\left\|\frac{\alpha(a)+\beta(b)}{2}\left(\int_a^b Z(s)\,ds\right)\right.\\&\left.+\frac{k+K}{2}\int_a^b\left(t-\frac{a+b}{2}\right)Z(t)\,dt-\int_a^b\alpha(t)\,Z(t)\,dt\right\|\\&\le\frac{1}{4}(K-k)\int_a^b\left\|\int_a^t Z(s)\,ds-\int_t^b Z(s)\,ds\right\|dt\\&\le\frac{1}{4}(K-k)(b-a)\int_a^b\|Z(t)\|\,dt,\end{aligned}$$

while by (2.19) we get

$$\begin{aligned}\tag{2.21}&\left\|\frac{\alpha(a)+\beta(b)}{2}\int_a^b Z(s)\,ds-\int_a^b\alpha(t)\,Z(t)\,dt\right\|\\&\le\frac{1}{4}(K-k)\int_a^b\left\|\int_a^t Z(s)\,ds-\int_t^b Z(s)\,ds\right\|dt\\&\le\frac{1}{4}(K-k)(b-a)\int_a^b\|Z(t)\|\,dt.\end{aligned}$$

3. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A)\ge h(B)$ holds for any $A\ge B\ge 0$ operators on the Hilbert space H.

We have the following representation of operator monotone functions, see for instance [2, p. 144-145]:

Theorem 5. *A function $h:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$h(t)=h(0)+bt+\int_0^\infty\frac{t\lambda}{t+\lambda}d\mu(\lambda),\tag{3.1}$$

where $b\ge 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty\frac{\lambda}{1+\lambda}d\mu(\lambda)<\infty.\tag{3.2}$$

We have the following representation result:

Lemma 1. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U\ge 0$, then for all selfadjoint operators V we have*

$$Dh(U)(V)=bV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda).\tag{3.3}$$

Proof. From (3.1) we get

$$h(t)=h(0)+bt+\int_0^\infty\left(\lambda-\frac{\lambda^2}{t+\lambda}\right)d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
& h(U+tV) - h(U) \\
&= btV + \int_0^\infty \left(\lambda - \lambda^2 (U+tV+\lambda)^{-1}\right) d\mu(\lambda) - \int_0^\infty \left(\lambda - \lambda^2 (U+\lambda)^{-1}\right) d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} - (\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} (\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\
&= btV + t\int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda).
\end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{h(U+tV) - h(U)}{t} = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1}\right] d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Lemma 2. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators V we have*

$$\|Dh(U)(V)\| \leq h'(u)\|V\|. \tag{3.4}$$

Proof. From (3.3) we get

$$\begin{aligned}
\|Dh(U)(V) - bV\| &\leq \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1} V (\lambda+U)^{-1}\right\| d\mu(\lambda) \\
&\leq \|V\| \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1}\right\|^2 d\mu(\lambda).
\end{aligned} \tag{3.5}$$

Observe that $\lambda + U \geq \lambda + u > 0$ for $\lambda \in [0,\infty)$. Then $0 < (\lambda+U)^{-1} \leq (\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\| \leq (\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2 \leq (\lambda+u)^{-2}$ for $\lambda \in [0,\infty)$.

Therefore by (3.5) we get

$$\|Dh(U)(V) - bV\| \leq \|V\| \int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$h'(t) = b + \int_0^\infty \frac{\lambda(t+\lambda) - \lambda t}{(t+\lambda)^2} d\mu(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) \tag{3.7}$$

for $t > 0$.

From (3.7) we get

$$\int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda) = h'(u) - b,$$

and by (3.6) we derive

$$\|Dh(U)(V) - bV\| \leq \|V\| h'(u) - b\|V\|.$$

Finally, by the triangle inequality and by the fact that $b \geq 0$, we obtain that

$$\|Dh(U)(V)\| - b\|V\| \leq \|Dh(U)(V) - bV\|,$$

which proves the desired result (3.4). □

For a continuous function h on $(0, \infty)$ and $A, B > 0$ we consider the auxiliary function $h_{A,B} : [0, 1] \to \mathbb{R}$ defined by

$$h_{A,B}(t) := h((1-t)A + tB), \ t \in [0, 1].$$

We have the following representations of the derivatives:

Lemma 3. *Assume that the operator function generated by h is Fréchet differentiable in each $A \geq 0$, then for $B \geq 0$ we have that $h_{A,B}$ is differentiable on $[0, 1]$ and*

$$h'_{A,B}(t) = D(h)((1-t)A + tB)(B - A) \tag{3.8}$$

for $t \in [0, 1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0, 1)$. Let h be in a small interval around 0 such that $t + h \in (0, 1)$. Then for $h \neq 0$,

$$\begin{aligned} &\frac{h_{A,B}(t+h) - h(t)}{h} \\ &= \frac{h((1-(t+h))A + (t+h)B) - h((1-t)A + tB)}{h} \\ &= \frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} h'_{A,B}(t) &= \lim_{h \to 0} \frac{h_{A,B}(t+h) - h(t)}{h} \\ &= \lim_{h \to 0} \left[\frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h} \right] \\ &= D(h)((1-t)A + tB)(B - A), \end{aligned}$$

which proves (3.8). □

Corollary 3. *Let $h : [0, \infty) \to \mathbb{R}$ be operator monotone in $[0, \infty)$. Then for all $A \geq a > 0$, $B \geq b > 0$ we have*

$$\begin{aligned} \left\| h'_{A,B}(t) \right\| &= \|D(h)((1-t)A + tB)(B - A)\| \\ &\leq h'((1-t)a + tb)\|B - A\| \end{aligned} \tag{3.9}$$

for all $t \in [0, 1]$.

The proof follows by Lemma 2 and Lemma 3.

One can observe that the inequality (3.9) remains valid for operator monotone functions on $(0, \infty)$. This follows by considering the function $h_\varepsilon(t) := h(t + \varepsilon)$ for $\varepsilon > 0$, which is operator monotone on $[0, \infty)$ and then by letting $\varepsilon \to 0+$ and using the continuity of h and h'.

Theorem 6. *Let $f:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$ and $\alpha:[0,1]\to\mathbb{C}$ a continuous function on $[0,1]$. Then for all $A\geq a>0$, $B\geq b>0$ we have for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,*

$$\begin{aligned}(3.10)\quad &\left\|\left(\int_0^1\alpha(t)\,dt\right)\frac{f(A)+f(B)}{2}-\int_0^1\alpha(t)\,f((1-t)A+tB)\,dt\right\|\\ &\leq\frac{1}{2}\|B-A\|\int_0^1\left|\int_0^t\alpha(s)\,ds-\int_t^1\alpha(s)\,ds\right|h'((1-t)a+tb)\,dt\\ &\leq\frac{1}{2}\|B-A\|\int_0^1|\alpha(s)|\,dt\times\begin{cases}\int_0^1 h'((1-t)a+tb)\,dt,\\ \left(\int_0^1[h'((1-t)a+tb)]^q\,dt\right)^{1/q},\\ \sup_{t\in[0,1]}h'((1-t)a+tb).\end{cases}\end{aligned}$$

Proof. We use Theorem 3 for α and $f_{A,B}$ on $[0,1]$ to get

$$\begin{aligned}&\left\|\left(\int_0^1\alpha(s)\,ds\right)\frac{f_{A,B}(0)+f_{A,B}(1)}{2}-\int_0^1\alpha(t)\,f_{A,B}(t)\,dt\right\|\\ &\leq\frac{1}{2}\int_0^1\left|\int_0^t\alpha(s)\,ds-\int_t^1\alpha(s)\,ds\right|\left\|f'_{A,B}(t)\right\|dt\\ &\leq\frac{1}{2}\times\begin{cases}\int_0^1|\alpha(s)|\,dt\int_0^1\left\|f'_{A,B}(t)\right\|dt,\\ \int_0^1|\alpha(s)|\,dt\left(\int_0^1\left\|f'_{A,B}(t)\right\|^q dt\right)^{1/q},\\ \int_0^1|\alpha(s)|\,dt\sup_{t\in[0,1]}\left\|f'_{A,B}(t)\right\|.\end{cases}\end{aligned}$$

By making use of (3.9) we derive (3.10). □

If we take in (3.10) $f(t)=t^r$, $r\in(0,1)$, which is operator monotone on $[0,\infty)$, then for all $A\geq a>0$, $B\geq b>0$ we have for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$

$$\begin{aligned}(3.11)\quad &\left\|\left(\int_0^1\alpha(t)\,dt\right)\frac{A^r+B^r}{2}-\int_0^1\alpha(t)\,((1-t)A+tB)^r\,dt\right\|\\ &\leq\frac{1}{2}r\|B-A\|\int_0^1\left|\int_0^t\alpha(s)\,ds-\int_t^1\alpha(s)\,ds\right|h((1-t)a+tb)^{r-1}\,dt\\ &\leq\frac{1}{2}r\|B-A\|\int_0^1|\alpha(s)|\,dt\times\begin{cases}\begin{cases}\frac{b^r-a^r}{r(b-a)}\text{ if }b\neq a,\\ a^{r-1}\text{ if }b=a,\end{cases}\\ \begin{cases}\left(\frac{b^{q(r-1)+1}-a^{q(r-1)+1}}{[q(r-1)+1](b-a)}\right)^{1/q}\text{ if }b\neq a,\\ q(r-1)+1\neq 0,\ q>1\\ a^{r-1}\text{ if }b=a,\end{cases}\\ \max\left\{a^{r-1},b^{r-1}\right\}.\end{cases},\end{aligned}$$

If we take in (3.10) $f(t) = \ln t$, which is operator monotone on $(0,\infty)$, then for all $A \geq a > 0$, $B \geq b > 0$,

$$
\begin{aligned}
&\left\| \left(\int_0^1 \alpha(t)\,dt \right) \frac{\ln A + \ln B}{2} - \int_0^1 \alpha(t) \ln((1-t)A + tB)\,dt \right\| \quad (3.12)\\
&\leq \frac{1}{2} \|B-A\| \int_0^1 \left| \int_0^t \alpha(s)\,ds - \int_t^1 \alpha(s)\,ds \right| ((1-t)a + tb)^{-1}\,dt \\
&\leq \frac{1}{2} \|B-A\| \int_0^1 |\alpha(s)|\,dt \times \begin{cases} \begin{cases} \frac{\ln b - \ln a}{b-a} \text{ if } b \neq a, \\ \frac{1}{a} \text{ if } b = a, \end{cases} \\ \begin{cases} \left(\frac{b^{1-q} - a^{1-q}}{(1-q)(b-a)} \right)^{1/q} \text{ if } b \neq a, \\ \text{for all } q > 1, \\ \frac{1}{a} \text{ if } b = a, \end{cases} \\ \frac{1}{\min\{a,b\}}. \end{cases}
\end{aligned}
$$

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