GRÜSS TYPE UNIVARIATE INTEGRAL INEQUALITIES FOR DIFFERENTIABLE FUNCTIONS IN BANACH SPACES

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Abstract. Assume that $(E, \|\cdot\|)$ is a complex Banach space. In this paper we show among others that, if $Y : [a, b] \to E$ is a strongly differentiable function on the interval (a, b) and $\alpha : [a, b] \to \mathbb{C}$ is a continuous function, then for all $\beta \in \mathbb{C}$,

$$\left\|(b-a)\int_a^b \alpha(t) Y(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b Y(t)\,dt\right\|$$
$$\le \int_a^b |\alpha(t)-\beta| \left[\int_a^t (s-a)\left\|Y'(s)\right\| ds + \int_t^b (b-s)\left\|Y'(s)\right\| ds\right] dt$$
$$\le \begin{cases} \int_a^b |\alpha(t)-\beta| \max\{t-a, b-t\}\,dt \int_a^b \|Y'(s)\|\,ds, \\ \frac{1}{(q+1)^{1/q}} \int_a^b |\alpha(t)-\beta| \left[(t-a)^{q+1} + (b-t)^{q+1}\right]^{1/q} dt \\ \times \left(\int_a^b \|Y'(s)\|^p\,ds\right)^{1/p},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}\int_a^b |\alpha(t)-\beta| \left[(t-a)^2 + (b-t)^2\right] dt \sup_{s\in[a,b]} \|Y'(s)\|. \end{cases}$$

Applications for operator monotone functions in Hilbert spaces with examples for the logarithmic function are also given.

1. Introduction

In [3] the authors obtained the following Grüss' type inequalities for functions with values in Banach spaces.

Theorem 1. *Let F be a Banach space over the real or complex number field $\mathbb{K}$, $\Omega \in \mathbb{R}^n$ a measurable set and $\rho : \Omega \to [0, \infty)$ a Lebesgue integrable function with $\int_\Omega \rho(x)\,dx = 1$. If $\alpha : \Omega \to \mathbb{K}$ is a Lebesgue integrable function such that there exists γ, $\Gamma \in \mathbb{K}$ with*

$$\left|\alpha(x) - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma| \tag{1.1}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma-\alpha(x))\left(\overline{\alpha(x)}-\overline{\gamma}\right)\right] \ge 0 \tag{1.2}$$

for a.e. $x \in \Omega$, and $g : \Omega \to F$ is a Bochner measurable function such that $\rho\alpha g$ and ρg are Bochner integrable on Ω, then,

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$$\left\| \int_{\Omega} \rho(x) \alpha(x) g(x) \, dx - \int_{\Omega} \rho(x) \alpha(x) \, dx \int_{\Omega} \rho(x) g(x) \, dx \right\| \leq \frac{1}{2} |\Gamma - \gamma| \int_{\Omega} \rho(x) \left\| g(x) - \int_{\Omega} \rho(y) g(y) \, dy \right\| dx. \tag{1.3}$$

The constant $\frac{1}{2}$ in (1.3) is the best possible.

The following dual result also holds:

Theorem 2. *Let F and Ω, ρ be as above. If $g : \Omega \to X$ is Bochner measurable on Ω and there exist vector $v \in X$ and $r > 0$ such that*

$$\|g(x) - v\| \leq r \text{ for a.e. } x \in \Omega$$

and $\alpha : \Omega \to \mathbb{K}$ is a Lebesgue integrable function with $\rho\alpha g$, ρg Bochner integrable functions on Ω, then we have the sharp inequalities

$$\begin{aligned} &\left\| \int_{\Omega} \rho(x) \alpha(x) g(x) \, dx - \int_{\Omega} \rho(x) \alpha(x) \, dx \int_{\Omega} \rho(x) g(x) \, dx \right\| \\ &\leq r \int_{\Omega} \rho(x) \left| \alpha(x) - \int_{\Omega} \rho(y) \alpha(y) \, dy \right| dx \\ &\leq r \left[\int_{\Omega} \rho(x) |\alpha(x)|^2 \, dx - \left| \int_{\Omega} \rho(x) \alpha(x) \, dx \right|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{1.4}$$

Now, consider the function f defined on the open and convex subset C of the Banach space E with values in the Banach space F and $\Omega = [0,1]$. Also let $\rho(t) = 1$ and $g(t) = f((1-t)x + ty)$ for $t \in [0,1]$ and x, $y \in C$. Then we can state the following particular case of interest:

Corollary 1. *Assume that $f : C \subset E \to F$ is continuous on C and x, $y \in C$, $x \neq y$. If $p : [0,1] \to \mathbb{K}$ is a Lebesgue integrable function such that there exists γ, $\Gamma \in \mathbb{K}$ with*

$$\left| p(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \tag{1.5}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma - p(t)) \left(\overline{p(t)} - \overline{\gamma} \right) \right] \geq 0 \tag{1.6}$$

for a.e. $t \in [0,1]$, then,

$$\begin{aligned} &\left\| \int_0^1 p(t) f((1-t)x + ty) \, dt - \int_0^1 p(t) \, dt \int_0^1 f((1-t)x + ty) \, dt \right\| \\ &\leq \frac{1}{2} |\Gamma - \gamma| \int_0^1 \left\| f((1-t)x + ty) - \int_0^1 f((1-s)x + sy) \, ds \right\| dt. \end{aligned} \tag{1.7}$$

The constant $\frac{1}{2}$ in (1.7) is the best possible.

If there exists a vector v and $r > 0$ such that

$$\|f((1-t)x + ty) - v\| \leq r \text{ for a.e. } t \in [0,1], \tag{1.8}$$

then for $q:[0,1]\to\mathbb{C}$ Lebesgue integrable, we have the sharp inequalities

$$\left\|\int_0^1 q(t)\,f((1-t)x+ty)\,dt-\int_0^1 q(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \tag{1.9}$$
$$\leq r\int_0^1\left|q(t)-\int_0^1 q(s)\,ds\right|dt\leq r\left[\int_0^1|q(t)|^2\,dt-\left|\int_\Omega q(t)\,dt\right|^2\right]^{\frac{1}{2}}.$$

We observe that, if there exists two vectors $z,\ w\in F$ such that

$$\left\|f((1-t)x+ty)-\frac{z+w}{2}\right\|\leq\frac{1}{2}\|w-z\|\ \text{ for a.e. }\ t\in[0,1], \tag{1.10}$$

then for $q:[0,1]\to\mathbb{C}$ Lebesgue integrable, we have the sharp inequalities

$$\left\|\int_0^1 q(t)\,f((1-t)x+ty)\,dt-\int_0^1 q(t)\,dt\int_0^1 f((1-t)x+ty)\,dt\right\| \tag{1.11}$$
$$\leq\frac{1}{2}\|w-z\|\int_0^1\left|q(t)-\int_0^1 q(s)\,ds\right|dt$$
$$\leq\frac{1}{2}\|w-z\|\left[\int_0^1|q(t)|^2\,dt-\left|\int_\Omega q(t)\,dt\right|^2\right]^{\frac{1}{2}}.$$

For some recent results on Grüss' type inequalities, see [1]-[6] and [8]-[17].

In this paper we show among others that, if $Y:[a,b]\to E$ is a strongly differentiable function on the interval (a,b) and $\alpha:[a,b]\to\mathbb{C}$ is a continuous function, then for all $\beta\in\mathbb{C}$

$$\left\|(b-a)\int_a^b\alpha(t)\,Y(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^b Y(t)\,dt\right\|$$
$$\leq\int_a^b|\alpha(t)-\beta|\left[\int_a^t(s-a)\,\|Y'(s)\|\,ds+\int_t^b(b-s)\,\|Y'(s)\|\,ds\right]dt$$
$$\leq\begin{cases}\int_a^b|\alpha(t)-\beta|\max\{t-a,b-t\}\,dt\int_a^b\|Y'(s)\|\,ds,\\[2ex] \frac{1}{(q+1)^{1/q}}\int_a^b|\alpha(t)-\beta|\left[(t-a)^{q+1}+(b-t)^{q+1}\right]^{1/q}dt\\ \times\left(\int_a^b\|Y'(s)\|^p\,ds\right)^{1/p},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \frac{1}{2}\int_a^b|\alpha(t)-\beta|\left[(t-a)^2+(b-t)^2\right]dt\sup_{s\in[a,b]}\|Y'(s)\|.\end{cases}$$

Applications for operator monotone functions in Hilbert spaces with examples for the logarithmic function are also given.

2. Main Results

We have the following equality:

Lemma 1. *Let $Y:[a,b]\to E$ be a strongly differentiable function on the interval (a,b) and $\alpha:[a,b]\to\mathbb{C}$ a continuous function, then for all $\beta\in\mathbb{C}$*

$$\begin{aligned}(2.1)\qquad &(b-a)\int_a^b \alpha(t)\, Y(t)\, dt - \int_a^b \alpha(t)\, dt \int_a^b Y(t)\, dt \\ &= \int_a^b (\alpha(t)-\beta)\left(\int_a^t (s-a)\, Y'(s)\, ds + \int_t^b (s-b)\, Y'(s)\, ds\right) dt.\end{aligned}$$

Proof. We start to the Montgomery identity for a strongly differentiable function $Y:[a,b]\to E$

$$Y(t)(b-a) - \int_a^b Y(s)\, ds = \int_a^t (s-a)\, Y'(s)\, ds + \int_t^b (s-b)\, Y'(s)\, ds \tag{2.2}$$

that holds for all $t\in[a,b]$.

Indeed, integrating by parts in Bochner's integral [14], we have

$$\int_a^t (s-a)\, Y'(s)\, ds = (t-a)\, Y(t) - \int_a^t Y(s)\, ds$$

and

$$\int_t^b (s-b)\, Y'(s)\, ds = (b-t)\, Y(t) - \int_t^b Y(s)\, ds$$

which by addition gives (2.2).

If we multiply this identity by $\alpha(t)$ and integrate over t in $[a,b]$, then we get

$$\begin{aligned}(2.3)\qquad &(b-a)\int_a^b \alpha(t)\, Y(t)\, dt - \int_a^b \alpha(t)\, dt \int_a^b Y(s)\, ds \\ &= \int_a^b \alpha(t)\left(\int_a^t (s-a)\, Y'(s)\, ds + \int_t^b (s-b)\, Y'(s)\, ds\right) dt.\end{aligned}$$

Now, if we replace $\alpha(t)$ by $\alpha(t)-\beta$, then we get

$$\begin{aligned}&(b-a)\int_a^b [\alpha(t)-\beta]\, Y(t)\, dt - \int_a^b [\alpha(t)-\beta]\, dt \int_a^b Y(t)\, dt \\ &= (b-a)\int_a^b \alpha(t)\, Y(t)\, dt - (b-a)\,\beta\int_a^b Y(t)\, dt \\ &- \int_a^b \alpha(t)\, dt \int_a^b Y(t)\, dt + (b-a)\,\beta\int_a^b Y(t)\, dt \\ &= (b-a)\int_a^b \alpha(t)\, Y(t)\, dt - \int_a^b \alpha(t)\, dt \int_a^b Y(s)\, ds\end{aligned}$$

and by (2.3) we derive (2.1). □

Lemma 2. *With the assumptions of Lemma 2 we have*

$$
\begin{aligned}
(2.4)\qquad & (b-a)\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt \\
& -(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)w \\
& =\int_a^b(\alpha(t)-\beta) \\
& \times\left(\int_a^t(s-a)\left[Y'(s)-w\right]ds+\int_t^b(s-b)\left[Y'(s)-w\right]ds\right)dt
\end{aligned}
$$

for all $w\in E$.

Proof. If we replace $Y(t)$ with $Y(t)-tw$ in (2.1), then we get

$$
\begin{aligned}
(2.5)\qquad & (b-a)\int_a^b \alpha(t)\left[Y(t)-tw\right]dt-\int_a^b \alpha(t)\,dt\int_a^b (Y(t)-tw)\,dt \\
& =\int_a^b(\alpha(t)-\beta) \\
& \times\left(\int_a^t(s-a)\left[Y'(s)-w\right]ds+\int_t^b(s-b)\left[Y'(s)-w\right]ds\right)dt.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
(2.6)\qquad & (b-a)\int_a^b \alpha(t)\left[Y(t)-tw\right]dt-\int_a^b \alpha(t)\,dt\int_a^b (Y(t)-tw)\,dt \\
& =(b-a)\left[\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b t\alpha(t)\,dtw\right] \\
& -\int_a^b \alpha(t)\,dt\left[\int_a^b Y(t)\,dt-\frac{1}{2}\left(b^2-a^2\right)w\right] \\
& =(b-a)\int_a^b \alpha(t)\,Y(t)\,dt-(b-a)\int_a^b t\alpha(t)\,dtw \\
& -\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt+\frac{1}{2}\left(b^2-a^2\right)\int_a^b \alpha(t)\,dtw \\
& =(b-a)\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt \\
& -(b-a)\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dtw
\end{aligned}
$$

and by (2.5), we get (2.4). □

Remark 1. *If* α *is symmetric on* $[a,b]$, *namely* $\alpha(a+b-t)=\alpha(t)$ *for all* $t\in[a,b]$, *then* $h(t):=\left(t-\frac{a+b}{2}\right)\alpha(t)$ *is antisymmetric, which gives that*

$$
\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt=0,
$$

and by (2.4) we derive

$$(2.7)\qquad (b-a)\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt$$
$$=\int_a^b (\alpha(t)-\beta)$$
$$\times\left(\int_a^t (s-a)\left[Y'(s)-w\right]ds+\int_t^b (s-b)\left[Y'(s)-w\right]ds\right)dt$$

for all $\beta\in\mathbb{C}$, $w\in E$.

We have the inequalities:

Theorem 3. *Let* $Y:[a,b]\to E$ *be a strongly differentiable function on the interval* (a,b) *and* $\alpha:[a,b]\to\mathbb{C}$ *a continuous function, then for all* $\beta\in\mathbb{C}$

$$(2.8)\qquad \left\|(b-a)\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt\right\|$$
$$\le\int_a^b |\alpha(t)-\beta|\left[\int_a^t (s-a)\left\|Y'(s)\right\|ds+\int_t^b (b-s)\left\|Y'(s)\right\|ds\right]dt$$
$$=:B(\alpha,Y,\beta).$$

We have the bounds

$$(2.9)\qquad B(\alpha,Y,\beta)$$
$$\le\begin{cases}\int_a^b |\alpha(t)-\beta|\max\{t-a,b-t\}\,dt\int_a^b \|Y'(s)\|\,ds,\\[2ex] \frac{1}{(q+1)^{1/q}}\int_a^b|\alpha(t)-\beta|\left[(t-a)^{q+1}+(b-t)^{q+1}\right]^{1/q}dt\\ \times\left(\int_a^b\|Y'(s)\|^p ds\right)^{1/p},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \frac{1}{2}\int_a^b|\alpha(t)-\beta|\left[(t-a)^2+(b-t)^2\right]dt\sup_{s\in[a,b]}\|Y'(s)\|.\end{cases}$$

Proof. If we take the norm in (2.1), then we get

$$(2.10)\qquad \left\|(b-a)\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt\right\|$$
$$\le\int_a^b\left\|(\alpha(t)-\beta)\left(\int_a^t (s-a)\,Y'(s)\,ds+\int_t^b (s-b)\,Y'(s)\,ds\right)\right\|dt$$
$$\le\int_a^b|\alpha(t)-\beta|\left\|\int_a^t (s-a)\,Y'(s)\,ds+\int_t^b (s-b)\,Y'(s)\,ds\right\|dt$$
$$\le\int_a^b|\alpha(t)-\beta|\left[\left\|\int_a^t (s-a)\,Y'(s)\,ds\right\|+\left\|\int_t^b (s-b)\,Y'(s)\,ds\right\|\right]dt$$
$$\le\int_a^b|\alpha(t)-\beta|\left[\int_a^t (s-a)\left\|Y'(s)\right\|ds+\int_t^b (b-s)\left\|Y'(s)\right\|ds\right]dt,$$

which proves the inequality (2.8).

Using Hölder's inequality, we have for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\int_a^t (s-a) \|Y'(s)\| \, ds \leq \begin{cases} \sup_{s\in[a,t]} (s-a) \int_a^t \|Y'(s)\| \, ds, \\ \left(\int_a^t (s-a)^q \, ds\right)^{1/q} \left(\int_a^t \|Y'(s)\|^p \, ds\right)^{1/p}, \\ \sup_{s\in[a,t]} \|Y'(s)\| \int_a^t (s-a) \, ds, \end{cases}$$

$$= \begin{cases} (t-a) \int_a^t \|Y'(s)\| \, ds, \\ \frac{(t-a)^{1+1/q}}{(q+1)^{1/q}} \left(\int_a^t \|Y'(s)\|^p \, ds\right)^{1/p}, \\ \frac{(t-a)^2}{2} \sup_{s\in[a,t]} \|Y'(s)\| \end{cases}$$

and

$$\int_t^b (b-s) \|Y'(s)\| \, ds \leq \begin{cases} \sup_{s\in[t,b]} (b-s) \int_t^b \|Y'(s)\| \, ds, \\ \left(\int_t^b (b-s)^q \, ds\right)^{1/q} \left(\int_t^b \|Y'(s)\|^p \, ds\right)^{1/p}, \\ \sup_{s\in[t,b]} \|Y'(s)\| \int_t^b (b-s) \, ds, \end{cases}$$

$$= \begin{cases} (b-t) \int_t^b \|Y'(s)\| \, ds, \\ \frac{(b-t)^{1+1/q}}{(q+1)^{1/q}} \left(\int_t^b \|Y'(s)\|^p \, ds\right)^{1/p}, \\ \frac{(b-t)^2}{2} \sup_{s\in[t,b]} \|Y'(s)\|. \end{cases}$$

Therefore

$$\int_a^t (s-a) \|Y'(s)\| \, ds + \int_t^b (b-s) \|Y'(s)\| \, ds$$

$$\leq \begin{cases} (t-a) \int_a^t \|Y'(s)\| \, ds + (b-t) \int_t^b \|Y'(s)\| \, ds, \\ \frac{1}{(q+1)^{1/q}} \left[(t-a)^{1+1/q} \left(\int_a^t \|Y'(s)\|^p \, ds\right)^{1/p} \right. \\ \left. + (b-t)^{1+1/q} \left(\int_t^b \|Y'(s)\|^p \, ds\right)^{1/p}\right]. \\ \frac{1}{2} \left[(t-a)^2 \sup_{s\in[a,t]} \|Y'(s)\| + (b-t)^2 \sup_{s\in[t,b]} \|Y'(s)\|\right], \end{cases}$$

$$\leq \begin{cases} \max\{t-a, b-t\} \int_a^b \|Y'(s)\| \, ds, \\ \frac{1}{(q+1)^{1/q}} \left[(t-a)^{q+1} + (b-t)^{q+1}\right]^{1/q} \left(\int_a^b \|Y'(s)\|^p \, ds\right)^{1/p}, \\ \frac{1}{2} \left[(t-a)^2 + (b-t)^2\right] \sup_{s\in[a,b]} \|Y'(s)\| \end{cases}$$

and by (2.10) we get the desired result (2.8). $\square$

Remark 2. *We have the inequalities*

$$\left\|(b-a)\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt\right\| \tag{2.11}$$
$$\leq \int_a^b |\alpha(t)|\left[\int_a^t (s-a)\,\|Y'(s)\|\,ds+\int_t^b (b-s)\,\|Y'(s)\|\,ds\right]dt$$
$$\leq \begin{cases} \int_a^b |\alpha(t)|\max\{t-a,b-t\}\,dt\int_a^b \|Y'(s)\|\,ds, \\ \\ \frac{1}{(q+1)^{1/q}}\int_a^b |\alpha(t)|\left[(t-a)^{q+1}+(b-t)^{q+1}\right]^{1/q}dt \\ \times\left(\int_a^b \|Y'(s)\|^p\,ds\right)^{1/p},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{1}{2}\int_a^b |\alpha(t)|\left[(t-a)^2+(b-t)^2\right]dt\,\sup_{s\in[a,b]}\|Y'(s)\| \end{cases}$$

provided that $Y:[a,b]\to E$ *is a strongly differentiable function on the interval* (a,b) *and* $\alpha:[a,b]\to\mathbb{C}$ *is a continuous function on* $[a,b]$.

Corollary 2. *With the assumptions of Theorem 3 and if there exists* $\beta\in\mathbb{C}$ *and* $M>0$ *such that*

$$|\alpha(t)-\beta|\leq M \text{ for all } t\in[a,b], \tag{2.12}$$

then

$$\left\|(b-a)\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt\right\| \tag{2.13}$$
$$\leq 2M\int_a^b (b-t)(t-a)\,\|Y'(t)\|\,dt$$
$$\leq M\times\begin{cases} \frac{1}{2}(b-a)^2\int_a^b \|Y'(t)\|\,dt, \\ \\ 2(b-a)^{2+1/q}\left[B(q+1,q+1)\right]^{1/q}\left(\int_a^b \|Y'(t)\|^p\,dt\right)^{1/p}, \\ \\ \frac{1}{3}(b-a)^3\sup_{t\in[a,b]}\|Y'(t)\| \end{cases}$$

for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *where* $B(\cdot,\cdot)$ *is the Beta function.*

Proof. By (2.12) we get

$$\int_a^b |\alpha(t)-\beta|\left[\int_a^t (s-a)\,\|Y'(s)\|\,ds+\int_t^b (b-s)\,\|Y'(s)\|\,ds\right]dt \tag{2.14}$$
$$\leq M\left[\int_a^b\left(\int_a^t (s-a)\,\|Y'(s)\|\,ds\right)dt+\int_a^b\left(\int_t^b (b-s)\,\|Y'(s)\|\,ds\right)dt\right].$$

Using integration by parts, we can state that

$$\begin{aligned}
&\int_a^b \left(\int_a^t (s-a)\left\|Y'(s)\right\| ds\right) dt \\
&= \left(\int_a^t (s-a)\left\|Y'(s)\right\| ds\right) t\Big|_a^b - \int_a^b (t-a)\, t\left\|Y'(t)\right\| dt \\
&= b\int_a^b (s-a)\left\|Y'(s)\right\| ds - \int_a^b (t-a)\, t\left\|Y'(t)\right\| dt \\
&= \int_a^b (b-t)(t-a)\left\|Y'(t)\right\| dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_a^b \left(\int_t^b (b-s)\left\|Y'(s)\right\| ds\right) dt \\
&= \left(\int_t^b (b-s)\left\|Y'(s)\right\| ds\right) t\Big|_a^b + \int_a^b (b-t)\, t\left\|Y'(t)\right\| dt \\
&= -a\int_a^b (b-s)\left\|Y'(s)\right\| ds + \int_a^b (b-t)\, t\left\|Y'(t)\right\| dt \\
&= \int_a^b (b-t)(t-a)\left\|Y'(t)\right\| dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_a^b \left(\int_a^t (s-a)\left\|Y'(s)\right\| ds\right) dt + \int_a^b \left(\int_t^b (b-s)\left\|Y'(s)\right\| ds\right) dt \\
&= 2\int_a^b (b-t)(t-a)\left\|Y'(t)\right\| dt
\end{aligned}$$

and by (2.14) and (2.8) we get the first inequality in (2.13).

Using Hölder's inequality for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then we get

$$\begin{aligned}
&\int_a^b (b-t)(t-a)\left\|Y'(t)\right\| dt \\
&\leq \begin{cases}
\sup_{t\in[a,b]}\left[(b-t)(t-a)\right]\int_a^b \left\|Y'(t)\right\| dt, \\[2ex]
\left(\int_a^b \left[(b-t)(t-a)\right]^q dt\right)^{1/q}\left(\int_a^b \left\|Y'(t)\right\|^p dt\right)^{1/p}, \\[2ex]
\int_a^b (b-t)(t-a)\, dt \sup_{t\in[a,b]}\left\|Y'(t)\right\|,
\end{cases} \\
&= \begin{cases}
\frac{1}{4}(b-a)^2\int_a^b \left\|Y'(t)\right\| dt, \\[2ex]
(b-a)^{2+1/q}\left[B(q+1,q+1)\right]^{1/q}\left(\int_a^b \left\|Y'(t)\right\|^p dt\right)^{1/p}, \\[2ex]
\frac{1}{6}(b-a)^3\sup_{t\in[a,b]}\left\|Y'(t)\right\|
\end{cases}
\end{aligned}$$

and the last part of (2.13). □

Remark 3. *Assume that* $\alpha : [a,b] \to \mathbb{C}$ *is a continuous function and there exists* $\gamma, \Gamma \in \mathbb{C}$ *such that*

$$\operatorname{Re}\left[(\Gamma - \alpha(t))\left(\overline{\alpha(t)} - \overline{\gamma}\right)\right] \geq 0 \text{ for all } t \in [a,b]$$

or, equivalently

$$\left|\alpha(t) - \frac{\gamma + \Gamma}{2}\right| \leq \frac{1}{2}|\Gamma - \gamma| \text{ for all } t \in [a,b] \tag{2.15}$$

then by (2.13) we get

$$\begin{aligned} &\left\|(b-a)\int_a^b \alpha(t) Y(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b Y(t)\,dt\right\| \\ &\leq |\Gamma - \gamma| \int_a^b (b-t)(t-a)\|Y'(t)\|\,dt \\ &\leq \frac{1}{2}|\Gamma - \gamma| \times \begin{cases} \frac{1}{2}(b-a)^2 \int_a^b \|Y'(t)\|\,dt, \\ \\ 2(b-a)^{2+1/q}[B(q+1,q+1)]^{1/q}\left(\int_a^b \|Y'(t)\|^p\,dt\right)^{1/p}, \\ \\ \frac{1}{3}(b-a)^3 \sup_{t\in[a,b]}\|Y'(t)\|, \end{cases} \end{aligned} \tag{2.16}$$

where $Y : [a,b] \to E$ *is a strongly differentiable function on the interval* (a,b)*.*

The perturbed Grüss' type inequality also holds:

Theorem 4. *Let* $Y : [a,b] \to E$ *be a strongly differentiable function on the interval* (a,b) *and* $\alpha : [a,b] \to \mathbb{C}$ *a continuous function, then for all* $\beta \in \mathbb{C}$, $w \in E$

$$\begin{aligned} &\left\|(b-a)\int_a^b \alpha(t) Y(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b Y(t)\,dt \right. \\ &\left. - (b-a)\left(\int_a^b \left(t - \frac{a+b}{2}\right)\alpha(t)\,dt\right) w\right\| \\ &\leq \int_a^b |\alpha(t) - \beta| \\ &\times \left[\int_a^t (s-a)\|Y'(s) - w\|\,ds + \int_t^b (b-s)\|Y'(s) - w\|\,ds\right] dt \\ &= B(\alpha, Y, \beta, w). \end{aligned} \tag{2.17}$$

We have the bounds for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$,

$$B(\alpha,Y,\beta,w) \leq \begin{cases} \int_a^b|\alpha(t)-\beta|\max\{t-a,b-t\}\,dt\int_a^b\|Y'(s)-w\|\,ds, \\ \\ \frac{1}{(q+1)^{1/q}}\int_a^b|\alpha(t)-\beta|\left[(t-a)^{q+1}+(b-t)^{q+1}\right]^{1/q}dt \\ \times\left(\int_a^b\|Y'(s)-w\|^p\,ds\right)^{1/p}, \\ \\ \frac{1}{2}\int_a^b|\alpha(t)-\beta|\left[(t-a)^2+(b-t)^2\right]dt\sup_{s\in[a,b]}\|Y'(s)-w\|. \end{cases} \tag{2.18}$$

Moreover, if α *is symmetric on* $[a,b]$, *then*

$$\left\|(b-a)\int_a^b\alpha(t)Y(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^bY(t)\,dt\right\|\leq B(\alpha,Y,\beta,w) \tag{2.19}$$

for all $\beta\in\mathbb{C}$, $w\in E$.

The proof is similar to the one of Theorem 3 and we omit the details.

Corollary 3. *With the assumptions of Theorem 4 and if there exists* $w\in E$ *and* $L>0$ *such that*

$$\|Y'(s)-w\|\leq L \text{ for all } t\in(a,b), \tag{2.20}$$

then for $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned} &\left\|(b-a)\int_a^b\alpha(t)Y(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^bY(t)\,dt\right. \\ &\left.-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)w\right\| \\ &\leq\frac{1}{2}L\int_a^b|\alpha(t)-\beta|\left[(t-a)^2+(b-t)^2\right]dt \\ &\leq\frac{1}{2}L\times\begin{cases}(b-a)^2\int_a^b|\alpha(t)-\beta|\,dt, \\ \\ (b-a)^{2+1/q}\left(\int_a^b|\alpha(t)-\beta|^p\,dt\right)^{1/p} \\ \times\left(\int_0^1\left[t^2+(1-t)^2\right]^q dt\right)^{1/q}, \\ \\ \frac{2}{3}(b-a)^3\sup_{t\in[a,b]}|\alpha(t)-\beta| \end{cases} \end{aligned} \tag{2.21}$$

for all $\beta\in\mathbb{C}$.

Moreover, if α *is symmetric on* $[a,b]$*, then for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1,$

$$\left\Vert (b-a)\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt\right\Vert \tag{2.22}$$
$$\le \frac{1}{2}L\int_a^b |\alpha(t)-\beta|\left[(t-a)^2+(b-t)^2\right]dt$$
$$\le \frac{1}{2}L\times\begin{cases}(b-a)^2\int_a^b|\alpha(t)-\beta|\,dt,\\ \\ (b-a)^{2+1/q}\left(\int_a^b|\alpha(t)-\beta|^p\,dt\right)^{1/p}\\ \times\left(\int_0^1\left[t^2+(1-t)^2\right]^q dt\right)^{1/q},\\ \\ \frac{2}{3}(b-a)^3\sup_{t\in[a,b]}|\alpha(t)-\beta|\end{cases}$$

for all $\beta\in\mathbb{C}$*.*

Remark 4. *If there exists* $v,\ z\in E$ *such that*

$$\left\Vert Y(t)-\frac{v+z}{2}\right\Vert \le \frac{1}{2}\Vert z-v\Vert \ \text{ for all } t\in[a,b], \tag{2.23}$$

then by (2.21) we get

$$\left\Vert (b-a)\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt\right. \tag{2.24}$$
$$\left.-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)\frac{v+z}{2}\right\Vert$$
$$\le \frac{1}{4}\Vert z-v\Vert\int_a^b|\alpha(t)-\beta|\left[(t-a)^2+(b-t)^2\right]dt$$
$$\le \frac{1}{4}\Vert z-v\Vert\times\begin{cases}(b-a)^2\int_a^b|\alpha(t)-\beta|\,dt,\\ \\ (b-a)^{2+1/q}\left(\int_a^b|\alpha(t)-\beta|^p\,dt\right)^{1/p}\\ \times\left(\int_0^1\left[t^2+(1-t)^2\right]^q dt\right)^{1/q},\\ \ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\ \\ \frac{2}{3}(b-a)^3\sup_{t\in[a,b]}|\alpha(t)-\beta|\end{cases}$$

for all $\beta\in\mathbb{C}$*.*

Moreover, if α *is symmetric on* $[a,b]$*, then*

$$\left\Vert (b-a)\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt\right\Vert \tag{2.25}$$

$$\leq \frac{1}{4}\|z-v\|\int_a^b |\alpha(t)-\beta|\left[(t-a)^2+(b-t)^2\right]dt$$

$$\leq \frac{1}{4}\|z-v\|\times\begin{cases}(b-a)^2\int_a^b|\alpha(t)-\beta|\,dt,\\[2ex] (b-a)^{2+1/q}\left(\int_a^b|\alpha(t)-\beta|^p\,dt\right)^{1/p}\\ \times\left(\int_0^1\left[t^2+(1-t)^2\right]^q dt\right)^{1/q},\\[2ex] \frac{2}{3}(b-a)^3\sup_{t\in[a,b]}|\alpha(t)-\beta|\,.\end{cases}$$

We observe that if E is a Hilbert space $(H,\langle\cdot,\cdot\rangle)$ then the condition (2.23) is equivalent to

$$\operatorname{Re}\langle z-Y(t),Y(t)-v\rangle\geq 0 \text{ for all } t\in[a,b]\,.$$

This fact follows by the identity

$$\operatorname{Re}\langle z-Y,Y-v\rangle=\frac{1}{4}\|z-v\|^2-\left\|Y-\frac{v+z}{2}\right\|^2$$

that holds for all z, v, $Y\in H$.

Remark 5. *If there exists $\beta\in\mathbb{C}$, $w\in E$, $M>0$ and $L>0$ such that the conditions (2.12) and (2.20) are satisfied, then*

$$\begin{aligned}&\left\|(b-a)\int_a^b\alpha(t)Y(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^b Y(t)\,dt\right.\\&\left.-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)w\right\|\\&\leq\frac{1}{3}LM(b-a)^3\,.\end{aligned}\tag{2.26}$$

Moreover, if α is symmetric on $[a,b]$, then

$$\left\|(b-a)\int_a^b\alpha(t)Y(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^b Y(t)\,dt\right\|\leq\frac{1}{3}LM(b-a)^3\,.\tag{2.27}$$

If α satisfies condition (2.15) while Y satisfies assumption (2.23), then by (2.26) we derive

$$\begin{aligned}&\left\|(b-a)\int_a^b\alpha(t)Y(t)\,dt-\int_a^b\alpha(t)\,dt\int_a^b Y(t)\,dt\right.\\&\left.-(b-a)\left(\int_a^b\left(t-\frac{a+b}{2}\right)\alpha(t)\,dt\right)\frac{v+z}{2}\right\|\\&\leq\frac{1}{12}|\Gamma-\gamma|\,\|z-v\|\,(b-a)^3\,.\end{aligned}\tag{2.28}$$

In addition, if α is symmetric on $[a,b]$, then

$$\left\| (b-a)\int_a^b \alpha(t)\,Y(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b Y(t)\,dt \right\| \leq \frac{1}{12}\left|\Gamma-\gamma\right| \left\|z-v\right\| (b-a)^3 . \tag{2.29}$$

3. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A) \geq h(B)$ holds for any $A \geq B \geq 0$ operators on the Hilbert space H.

We have the following representation of operator monotone functions, see for instance [4, p. 144-145]:

Theorem 5. *A function $h:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$h(t) = h(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda}\,d\mu(\lambda), \tag{3.1}$$

where $b \geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda}\,d\mu(\lambda) < \infty. \tag{3.2}$$

We have the following representation result:

Lemma 3. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq 0$, then for all selfadjoint operators V we have*

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1}\right] d\mu(\lambda). \tag{3.3}$$

Proof. From (3.1) we get

$$h(t) = h(0) + bt + \int_0^\infty \left(\lambda - \frac{\lambda^2}{t+\lambda}\right) d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
& h(U+tV) - h(U) \\
&= btV + \int_0^\infty \left(\lambda - \lambda^2 (U+tV+\lambda)^{-1}\right) d\mu(\lambda) - \int_0^\infty \left(\lambda - \lambda^2 (U+\lambda)^{-1}\right) d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} - (\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1}(\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\
&= btV + t\int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda).
\end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{h(U+tV)-h(U)}{t} = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda + U)^{-1} V (\lambda + U)^{-1} \right] d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Theorem 6. *Let $h : [0, \infty) \to \mathbb{R}$ be operator monotone in $[0, \infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators V we have*

$$\|Dh(U)(V)\| \leq h'(u) \|V\|. \tag{3.4}$$

Proof. From (3.3) we get

$$\begin{aligned} \|Dh(U)(V) - bV\| &\leq \int_0^\infty \lambda^2 \left\| (\lambda + U)^{-1} V (\lambda + U)^{-1} \right\| d\mu(\lambda) \\ &\leq \|V\| \int_0^\infty \lambda^2 \left\| (\lambda + U)^{-1} \right\|^2 d\mu(\lambda). \end{aligned} \tag{3.5}$$

Observe that $\lambda + U \geq \lambda + u > 0$ for $\lambda \in [0, \infty)$. Then $0 < (\lambda + U)^{-1} \leq (\lambda + u)^{-1}$, which implies that $\left\| (\lambda + U)^{-1} \right\| \leq (\lambda + u)^{-1}$, namely $\left\| (\lambda + U)^{-1} \right\|^2 \leq (\lambda + u)^{-2}$ for $\lambda \in [0, \infty)$.

Therefore by (3.5) we get

$$\|Dh(U)(V) - bV\| \leq \|V\| \int_0^\infty \lambda^2 (\lambda + u)^{-2} d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$h'(t) = b + \int_0^\infty \frac{\lambda(t + \lambda) - \lambda t}{(t + \lambda)^2} d\mu(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t + \lambda)^2} d\mu(\lambda) \tag{3.7}$$

for $t > 0$.

From (3.7) we get

$$\int_0^\infty \lambda^2 (\lambda + u)^{-2} d\mu(\lambda) = h'(u) - b,$$

and by (3.6) we derive

$$\|Dh(U)(V) - bV\| \leq \|V\| h'(u) - b \|V\|.$$

Finally, by the triangle inequality and by the fact that $b \geq 0$, we obtain that

$$\|Dh(U)(V)\| - b \|V\| \leq \|Dh(U)(V) - bV\|,$$

which proves the desired result (3.4). □

For a continuous function h on $(0, \infty)$ and $A, B > 0$ we consider the auxiliary function $h_{A,B} : [0, 1] \to \mathbb{R}$ defined by

$$h_{A,B}(t) := h((1 - t) A + tB), \ t \in [0, 1].$$

We have the following representations of the derivatives:

Lemma 4. *Assume that the operator function generated by h is Fréchet differentiable in each $A \geq 0$, then for $B \geq 0$ we have that $h_{A,B}$ is differentiable on $[0, 1]$ and*

$$h'_{A,B}(t) = D(h)((1 - t) A + tB)(B - A) \tag{3.8}$$

for $t \in [0, 1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t+h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned}&\frac{h_{A,B}(t+h)-h(t)}{h}\\&=\frac{h((1-(t+h))A+(t+h)B)-h((1-t)A+tB)}{h}\\&=\frac{h((1-t)A+tB+h(B-A))-h((1-t)A+tB)}{h}\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}h'_{A,B}(t)&=\lim_{h\to 0}\frac{h_{A,B}(t+h)-h(t)}{h}\\&=\lim_{h\to 0}\left[\frac{h((1-t)A+tB+h(B-A))-h((1-t)A+tB)}{h}\right]\\&=D(h)((1-t)A+tB)(B-A),\end{aligned}$$

which proves (3.8). □

Corollary 4. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A \geq a > 0$, $B \geq b > 0$ we have*

$$\begin{aligned}\left\|h'_{A,B}(t)\right\|&=\|D(h)((1-t)A+tB)(B-A)\|\\&\leq h'((1-t)a+tb)\|B-A\|\end{aligned}\tag{3.9}$$

for all $t\in[0,1]$.

The proof follows by Theorem 6 and Lemma 4.

One can observe that the inequality (3.9) remains valid for operator monotone functions on $(0,\infty)$. This follows by considering the function $h_\varepsilon(t) := h(t+\varepsilon)$ for $\varepsilon > 0$, which is operator monotone on $[0,\infty)$ and then by letting $\varepsilon \to 0+$ and using the continuity of h and h'.

Theorem 7. *Let $f:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$ and $\alpha:[0,1]\to\mathbb{C}$ a continuous function on $[0,1]$. Then for all $A\geq a>0$, $B\geq b>0$ we have*

$$\begin{aligned}&\left\|\int_0^1\alpha(t)f((1-t)A+tB)\,dt-\int_0^1\alpha(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\right\|\\&\leq\|B-A\|\int_0^1|\alpha(t)-\beta|\\&\times\left[\int_0^t sf'((1-s)a+sb)\,ds+\int_t^1(1-s)f'((1-s)a+sb)\,ds\right]dt\\&\leq\|B-A\|\\&\times\begin{cases}\int_0^1|\alpha(t)-\beta|\max\{t,1-t\}\,dt\int_0^1 f'((1-s)a+sb)\,ds,\\[2ex]\frac{1}{(q+1)^{1/q}}\int_0^1|\alpha(t)-\beta|\left[t^{q+1}+(1-t)^{q+1}\right]^{1/q}dt\\\times\left(\int_0^1[f'((1-s)a+sb)]^p\,ds\right)^{1/p},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex]\frac{1}{2}\int_0^1|\alpha(t)-\beta|\left[t^2+(1-t)^2\right]dt\sup_{s\in[0,1]}f'((1-s)a+sb)\end{cases}\end{aligned}\tag{3.10}$$

for all $\beta \in \mathbb{C}$.

Moreover, if α *satisfies condition (2.15), then*

$$\left\| \int_0^1 \alpha(t) f((1-t)A + tB)\, dt - \int_0^1 \alpha(t)\, dt \int_0^1 f((1-t)A+tB)\, dt \right\| \tag{3.11}$$
$$\leq |\Gamma - \gamma| \, \|B - A\| \int_0^1 (1-t)\, t f'((1-t)a + tb)\, dt$$
$$\leq \frac{1}{2} |\Gamma - \gamma| \, \|B - A\|$$
$$\times \begin{cases} \frac{1}{2} \int_0^1 f'((1-t)a+tb)\, dt, \\ \\ 2\left[B(q+1,q+1)\right]^{1/q} \left(\int_0^1 [f'((1-t)a+tb)]^p\, dt\right)^{1/p}, \\ \\ \frac{1}{3} \sup_{s\in[0,1]} f'((1-t)a+tb). \end{cases}$$

Proof. If we apply Theorem 3 for the function $Y(t) = f((1-t)A + tB)$, $t \in [0,1]$, then we get

$$\left\| \int_0^1 \alpha(t) f((1-t)A + tB)\, dt - \int_0^1 \alpha(t)\, dt \int_0^1 f((1-t)A+tB)\, dt \right\|$$
$$\leq \int_0^1 |\alpha(t) - \beta| \left[\int_0^t s \left\| f'_{A,B}(s) \right\| ds + \int_t^1 (1-s) \left\| f'_{A,B}(s) \right\| ds \right] dt$$
$$\leq \begin{cases} \int_0^1 |\alpha(t) - \beta| \max\{t, 1-t\}\, dt \int_a^b \left\| f'_{A,B}(s) \right\| ds, \\ \\ \frac{1}{(q+1)^{1/q}} \int_0^1 |\alpha(t)-\beta| \left[t^{q+1} + (1-t)^{q+1} \right]^{1/q} dt \\ \times \left(\int_0^1 \left\| f'_{A,B}(s) \right\|^p ds \right)^{1/p}, \\ \\ \frac{1}{2} \int_0^1 |\alpha(t) - \beta| \left[t^2 + (1-t)^2 \right] dt \sup_{s\in[0,1]} \left\| f'_{A,B}(s) \right\| \end{cases}$$

and by (3.9) for the operator monotonic function f we derive (3.10).

The inequality (3.11) follows in a similar way from (2.16). □

If we take $f(t) = \ln t$ in (3.10), then for all $A \geq a > 0$, $B \geq b > 0$ we have

$$\left\| \int_0^1 \alpha(t) \ln((1-t)A + tB)\, dt - \int_0^1 \alpha(t)\, dt \int_0^1 \ln((1-t)A+tB)\, dt \right\| \tag{3.12}$$

$$\leq \|B-A\| \times \begin{cases} \int_0^1 |\alpha(t)-\beta| \max\{t,1-t\}\,dt \times \begin{cases} \frac{\ln b-\ln a}{b-a} \text{ if } b\neq a, \\ \frac{1}{a} \text{ if } b=a, \end{cases} \\ \frac{1}{(q+1)^{1/q}} \int_0^1 |\alpha(t)-\beta| \left[t^{q+1}+(1-t)^{q+1}\right]^{1/q} dt \\ \times \begin{cases} \left(\frac{b^{1-p}-a^{1-p}}{(1-p)(b-a)}\right)^{1/p} \text{ if } b\neq a, \\ \frac{1}{a} \text{ if } b=a \end{cases}, \ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2\min\{b,a\}} \int_0^1 |\alpha(t)-\beta| \left[t^2+(1-t)^2\right] dt \end{cases}$$

for all $\beta \in \mathbb{C}$. Moreover, if α satisfies condition (2.15), then

$$\left\|\int_0^1 \alpha(t)\ln((1-t)A+tB)\,dt - \int_0^1 \alpha(t)\,dt \int_0^1 \ln((1-t)A+tB)\,dt\right\| \tag{3.13}$$
$$\leq \frac{1}{2}|\Gamma-\gamma| \|B-A\| \times \begin{cases} \frac{1}{2}\begin{cases} \frac{\ln b-\ln a}{b-a} \text{ if } b\neq a, \\ \frac{1}{a} \text{ if } b=a, \end{cases} \\ 2\left[B(q+1,q+1)\right]^{1/q} \begin{cases} \left(\frac{b^{1-p}-a^{1-p}}{(1-p)(b-a)}\right)^{1/p} \text{ if } b\neq a, \\ \frac{1}{a} \text{ if } b=a, \end{cases}, \\ \frac{1}{3\min\{b,a\}}. \end{cases}$$

If we take $\alpha(t)=e^{2\pi it}$, $t\in[0,1]$, then by (3.10) for $\beta=0$, we derive

$$\left\|\int_0^1 e^{2\pi it} f((1-t)A+tB)\,dt\right\| \tag{3.14}$$
$$\leq \|B-A\| \times \begin{cases} \int_0^1 \max\{t,1-t\}\,dt \int_0^1 f'((1-s)a+sb)\,ds, \\ \frac{1}{(q+1)^{1/q}} \int_0^1 \left[t^{q+1}+(1-t)^{q+1}\right]^{1/q} dt \\ \times \left(\int_0^1 \left[f'((1-s)a+sb)\right]^p ds\right)^{1/p},\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}\int_0^1 \left[t^2+(1-t)^2\right] dt \sup_{s\in[0,1]} f'((1-s)a+sb), \end{cases}$$

for $f:[0,\infty)\to\mathbb{R}$ operator monotone in $[0,\infty)$ and $A\geq a>0$, $B\geq b>0$.

Since

$$\int_0^1 \max\{t,1-t\}\,dt = \frac{3}{4} \text{ and } \int_0^1 \left[t^2+(1-t)^2\right] dt = \frac{2}{3}$$

hence by (3.14) we derive

$$\left\| \int_0^1 e^{2\pi i t} f\left((1-t)A + tB\right) dt \right\| \tag{3.15}$$
$$\leq \|B-A\| \begin{cases} \frac{3}{4}\int_0^1 f'\left((1-s)a+sb\right) ds, \\ \\ \frac{1}{(q+1)^{1/q}}\int_0^1 \left[t^{q+1}+(1-t)^{q+1}\right]^{1/q} dt \\ \times \left(\int_0^1 \left[f'\left((1-s)a+sb\right)\right]^p ds\right)^{1/p}, \ p, \ q>1, \\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{1}{3}\int_0^1 \left[t^2+(1-t)^2\right] dt \sup_{s\in[0,1]} f'\left((1-s)a+sb\right), \end{cases}$$

for $f:[0,\infty)\to\mathbb{R}$ operator monotone in $[0,\infty)$ and $A\geq a>0$, $B\geq b>0$.

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