BOUNDS FOR THE ČEBYŠEV FUNCTIONAL OF DIFFERENTIABLE FUNCTIONS IN BANACH SPACES

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Abstract. Assume that $(E, \|\cdot\|)$ is a complex Banach space. In this paper we show among others that, if $Y : [a,b] \to E$ is a strongly differentiable function on the interval (a,b) and $\alpha : [a,b] \to \mathbb{C}$ is a continuous function, then

$$\left\|(b-a)\int_a^b \alpha(t)\,Y(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b Y(t)\,dt\right\|$$
$$\leq \frac{1}{4}(b-a)^2 \begin{cases} \|\alpha'\|_{[a,b],1}\,\|Y'\|_{[a,b],1}\,, \\ \\ \frac{1}{3}(b-a)^2\,\|\alpha'\|_{[a,b],\infty}\,\|Y'\|_{[a,b],\infty}\,, \\ \\ \frac{1}{2}(b-a)\,\|\alpha'\|_{[a,b],1}\,\|Y'\|_{[a,b],\infty}\,, \end{cases}$$

where $\|\cdot\|_{[a,b],1}$ is L_1-norm and $\|\cdot\|_{[a,b],\infty}$ is the sup-norm. Applications for operator monotone functions in Hilbert spaces with examples for the logarithmic and power functions are also given.

1. Introduction

For two Lebesgue integrable functions $f,\ g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g) := (b-a)\int_a^b f(t)\,g(t)\,dt - \int_a^b f(t)\,dt \int_a^b g(t)\,dt.$$

In 1934, G. Grüss [19] showed that

$$|D(f,g)| \leq \frac{1}{4}(b-a)^2 (M-m)(N-n)\,, \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on} \quad [a,b]\,. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $D(f,g)$ was derived in 1882 by Čebyšev [11] under the assumption that f', g' exist and are continuous on $[a,b]$, and is given by

$$|D(f,g)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^4\,, \tag{1.3}$$

where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

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Čebyšev's inequality (1.3) also holds if f, $g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and f', $g' \in L_\infty [a,b]$.

In 1970, A. M. Ostrowski [28] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(f,g)| \leq \frac{1}{8}(b-a)^3 (M-m) \|g'\|_\infty , \tag{1.4}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_\infty [a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [22] (see also [24, p. 210]) obtained the following result as well:

$$|D(f,g)| \leq \frac{1}{\pi^2} \|f'\|_2 \|g'\|_2 (b-a)^3 , \tag{1.5}$$

provided f, g are absolutely continuous and f', $g' \in L_2 [a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [8], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(f,g)| \leq (b-a) \times \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt, \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \left(\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right|^p dt \right)^{\frac{1}{p}} \\ \qquad\qquad \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(f,g)| \leq (b-a) \|g\|_\infty \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\| g - \frac{m+M}{2} \right\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [10]

$$|D(f,g)| \leq \frac{1}{2}(b-a)(M-m) \int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [9].

The following result holds [14].

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(f,g)| \leq \frac{1}{2}(b-a) \bigvee_a^b (f) \int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| dt \tag{1.9}$$

where $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{2}$ is best possible in (1.9).

For more recent upper bounds related to the Čebyšev functional see [8], [9] and [12]-[14].

In [3] the authors obtained the following Grüss' type inequalities for functions with values in Banach spaces.

Theorem 2. *Let F be a Banach space over the real or complex number field $\mathbb{K}$, $\Omega \in \mathbb{R}^n$ a measurable set and $\rho : \Omega \to [0,\infty)$ a Lebesgue integrable function with $\int_\Omega \rho(x)\,dx = 1$. If $\alpha : \Omega \to \mathbb{K}$ is a Lebesgue integrable function such that there exists γ, $\Gamma \in \mathbb{K}$ with*

$$\left|\alpha(x) - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma| \tag{1.10}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma - \alpha(x))\left(\overline{\alpha(x)} - \overline{\gamma}\right)\right] \ge 0 \tag{1.11}$$

for a.e. $x \in \Omega$, and $g : \Omega \to F$ is a Bochner measurable function such that $\rho\alpha g$ and ρg are Bochner integrable on Ω, then,

$$\begin{aligned}(1.12)\qquad &\left\|\int_\Omega \rho(x)\alpha(x)g(x)\,dx - \int_\Omega \rho(x)\alpha(x)\,dx\int_\Omega \rho(x)g(x)\,dx\right\| \\ &\le \frac{1}{2}|\Gamma-\gamma|\int_\Omega \rho(x)\left\|g(x) - \int_\Omega \rho(y)g(y)\,dy\right\|dx.\end{aligned}$$

The constant $\frac{1}{2}$ in (1.12) is the best possible.

The following dual result also holds:

Theorem 3. *Let F and Ω, ρ be as above. If $g : \Omega \to X$ is Bochner measurable on Ω and there exist vector $v \in X$ and $r > 0$ such that*

$$\|g(x) - v\| \le r \text{ for a.e. } x \in \Omega$$

and $\alpha : \Omega \to \mathbb{K}$ is a Lebesgue integrable function with $\rho\alpha g$, ρg Bochner integrable functions on Ω, then we have the sharp inequalities

$$\begin{aligned}(1.13)\qquad &\left\|\int_\Omega \rho(x)\alpha(x)g(x)\,dx - \int_\Omega \rho(x)\alpha(x)\,dx\int_\Omega \rho(x)g(x)\,dx\right\| \\ &\le r\int_\Omega \rho(x)\left|\alpha(x) - \int_\Omega \rho(y)\alpha(y)\,dy\right|dx \\ &\le r\left[\int_\Omega \rho(x)|\alpha(x)|^2\,dx - \left|\int_\Omega \rho(x)\alpha(x)\,dx\right|^2\right]^{\frac{1}{2}}.\end{aligned}$$

Now, consider the function f defined on the open and convex subset C of the Banach space E with values in the Banach space F and $\Omega = [0,1]$. Also let $\rho(t) = 1$ and $g(t) = f((1-t)x + ty)$ for $t \in [0,1]$ and x, $y \in C$. Then we can state the following particular case of interest:

Corollary 1. *Assume that $f : C \subset E \to F$ is continuous on C and x, $y \in C$, $x \ne y$. If $p : [0,1] \to \mathbb{K}$ is a Lebesgue integrable function such that there exists γ, $\Gamma \in \mathbb{K}$ with*

$$\left|p(t) - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma| \tag{1.14}$$

or, equivalently,

$$\operatorname{Re}\left[\left(\Gamma-p\left(t\right)\right)\left(\overline{p\left(t\right)}-\overline{\gamma}\right)\right]\geq 0 \tag{1.15}$$

for a.e. $t\in[0,1]$ *, then,*

$$\begin{aligned}&\left\|\int_0^1 p\left(t\right)f\left(\left(1-t\right)x+ty\right)dt-\int_0^1 p\left(t\right)dt\int_0^1 f\left(\left(1-t\right)x+ty\right)dt\right\| \\ &\leq\frac{1}{2}\left|\Gamma-\gamma\right|\int_0^1\left\|f\left(\left(1-t\right)x+ty\right)-\int_0^1 f\left(\left(1-s\right)x+sy\right)ds\right\|dt.\end{aligned} \tag{1.16}$$

The constant $\frac{1}{2}$ *in (1.16) is the best possible.*

If there exists a vector v *and* $r>0$ *such that*

$$\left\|f\left(\left(1-t\right)x+ty\right)-v\right\|\leq r \text{ for a.e. } t\in[0,1], \tag{1.17}$$

then for $q:[0,1]\to\mathbb{C}$ *Lebesgue integrable, we have the sharp inequalities*

$$\begin{aligned}&\left\|\int_0^1 q\left(t\right)f\left(\left(1-t\right)x+ty\right)dt-\int_0^1 q\left(t\right)dt\int_0^1 f\left(\left(1-t\right)x+ty\right)dt\right\| \\ &\leq r\int_0^1\left|q\left(t\right)-\int_0^1 q\left(s\right)ds\right|dt\leq r\left[\int_0^1\left|q\left(t\right)\right|^2dt-\left|\int_\Omega q\left(t\right)dt\right|^2\right]^{\frac{1}{2}}.\end{aligned} \tag{1.18}$$

We observe that, if there exists two vectors z, $w\in F$ such that

$$\left\|f\left(\left(1-t\right)x+ty\right)-\frac{z+w}{2}\right\|\leq\frac{1}{2}\left\|w-z\right\| \text{ for a.e. } t\in[0,1], \tag{1.19}$$

then for $q:[0,1]\to\mathbb{C}$ Lebesgue integrable, we have the sharp inequalities

$$\begin{aligned}&\left\|\int_0^1 q\left(t\right)f\left(\left(1-t\right)x+ty\right)dt-\int_0^1 q\left(t\right)dt\int_0^1 f\left(\left(1-t\right)x+ty\right)dt\right\| \\ &\leq\frac{1}{2}\left\|w-z\right\|\int_0^1\left|q\left(t\right)-\int_0^1 q\left(s\right)ds\right|dt \\ &\leq\frac{1}{2}\left\|w-z\right\|\left[\int_0^1\left|q\left(t\right)\right|^2dt-\left|\int_\Omega q\left(t\right)dt\right|^2\right]^{\frac{1}{2}}.\end{aligned} \tag{1.20}$$

For some recent results on Grüss' type inequalities, see [1]-[6] and [18]-[28].

In this paper we show among others that, if $Y:[a,b]\to E$ is a strongly differentiable function on the interval (a,b) and $\alpha:[a,b]\to\mathbb{C}$ is a continuous function, then

$$\begin{aligned}&\left\|\left(b-a\right)\int_a^b\alpha\left(t\right)Y\left(t\right)dt-\int_a^b\alpha\left(t\right)dt\int_a^b Y\left(t\right)dt\right\| \\ &\leq\frac{1}{4}\left(b-a\right)^2\begin{cases}\left\|\alpha'\right\|_{[a,b],1}\left\|Y'\right\|_{[a,b],1}, \\ \frac{1}{3}\left(b-a\right)^2\left\|\alpha'\right\|_{[a,b],\infty}\left\|Y'\right\|_{[a,b],\infty}, \\ \frac{1}{2}\left(b-a\right)\left\|\alpha'\right\|_{[a,b],1}\left\|Y'\right\|_{[a,b],\infty}.\end{cases}\end{aligned}$$

Applications for operator monotone functions in Hilbert spaces with examples for the logarithmic and power functions are also given.

2. Main Results

Let E be a complex Banach space. For two continuous functions $\alpha : [a,b] \to \mathbb{C}$, $Y : [a,b] \to E$ we define the *Čebyšev functional*

$$D(\alpha, Y) := (b-a) \int_a^b \alpha(t) Y(t) \, dt - \int_a^b \alpha(t) \, dt \int_a^b Y(t) \, dt.$$

We have the following result of interest:

Theorem 4. *Let $\alpha : [a,b] \to \mathbb{C}$ be differentiable and $Y : [a,b] \to E$ be strongly differentiable on the interval (a,b). Then*

$$\|D(\alpha, Y)\| \le D\left(\int_a^{\cdot} |\alpha'(u)| \, du, \int_a^{\cdot} \|Y'(u)\| \, du \right) \le \frac{1}{4} (b-a)^2 \|\alpha'\|_{[a,b],1} \|Y'\|_{[a,b],1}, \tag{2.1}$$

where $\|\alpha'\|_{[a,b],1} := \int_a^b |\alpha'(u)| \, du$ and $\|Y'\|_{[a,b],1} := \int_a^b \|Y'(u)\| \, du$.

Proof. Observe that

$$\begin{aligned}
&\int_a^b \int_a^b [\alpha(t) - \alpha(s)] [Y(t) - Y(s)] \, dt ds \\
&= \int_a^b \int_a^b (\alpha(t) Y(t) - \alpha(s) Y(t) - \alpha(t) Y(s) + \alpha(s) Y(s)) \, dt ds \\
&= (b-a) \int_a^b \alpha(t) Y(t) \, dt - \int_a^b \alpha(s) \, ds \int_a^b Y(t) \, dt \\
&- \int_a^b \alpha(t) \, dt \int_a^b Y(s) \, ds + (b-a) \int_a^b \alpha(s) Y(s) \, ds \\
&= 2(b-a) \int_a^b \alpha(t) Y(t) \, dt - 2 \int_a^b \alpha(t) \, dt \int_a^b Y(t) \, dt = 2D(\alpha, Y),
\end{aligned}$$

which give the Korkine's identity for functions with values in Banach spaces

$$D(\alpha, Y) = \frac{1}{2} \int_a^b \int_a^b [\alpha(t) - \alpha(s)] [Y(t) - Y(s)] \, dt ds.$$

For Korkine's classical identity for real-valued functions, see [24, p. 242].

If we take the norm and use the integral's properties, we get

$$\begin{aligned}
\|D(\alpha, Y)\| &\le \frac{1}{2} \int_a^b \int_a^b \|[\alpha(t) - \alpha(s)] [Y(t) - Y(s)]\| \, dt ds \\
&\le \frac{1}{2} \int_a^b \int_a^b |\alpha(t) - \alpha(s)| \|Y(t) - Y(s)\| \, dt ds.
\end{aligned} \tag{2.2}$$

Observe that for $s, t \in [a,b]$

$$\alpha(t) - \alpha(s) = \int_s^t \alpha'(u) \, du, \; Y(t) - Y(s) = \int_s^t Y'(u) \, du,$$

which implies that

$$\begin{aligned}|\alpha(t)-\alpha(s)|\|Y(t)-Y(s)\| &= \left|\int_s^t \alpha'(u)\,du\right|\left\|\int_s^t Y'(u)\,du\right\| \\ &\le \left|\int_s^t |\alpha'(u)|\,du\right|\left|\int_s^t \|Y'(u)\|\,du\right| \\ &= \left|\int_s^t |\alpha'(u)|\,du\int_s^t \|Y'(u)\|\,du\right| \\ &= \int_s^t |\alpha'(u)|\,du\int_s^t \|Y'(u)\|\,du,\end{aligned}$$

for all $s,\, t\in[a,b]$.

By (2.2) we get

$$\|D(\alpha,Y)\| \le \frac{1}{2}\int_a^b\int_a^b\left(\int_s^t |\alpha'(u)|\,du\right)\left(\int_s^t \|Y'(u)\|\,du\right)dtds. \tag{2.3}$$

Since

$$\begin{aligned}&\int_s^t |\alpha'(u)|\,du\int_s^t \|Y'(u)\|\,du \\ &= \left(\int_a^t |\alpha'(u)|\,du-\int_a^s |\alpha'(u)|\,du\right)\left(\int_a^t \|Y'(u)\|\,du-\int_a^s \|Y'(u)\|\,du\right),\end{aligned}$$

hence by Korkine's identity for real valued functions $f(t)=\int_a^t \|\alpha'(u)\|\,du$ and $g(t)=\int_a^t \|Y'(u)\|\,du$, we have

$$\begin{aligned}&\frac{1}{2}\int_a^b\int_a^b\left(\int_a^t |\alpha'(u)|\,du-\int_a^s |\alpha'(u)|\,du\right) \\ &\times\left(\int_a^t \|Y'(u)\|\,du-\int_a^s \|Y'(u)\|\,du\right) \\ &= (b-a)\int_a^b\left(\int_a^t |\alpha'(u)|\,du\right)\left(\int_a^t \|Y'(u)\|\,du\right)dt \\ &-\int_a^b\left(\int_a^t |\alpha'(u)|\,du\right)dt\int_a^b\left(\int_a^t \|Y'(u)\|\,du\right)dt \\ &= D\left(\int_a^{\cdot} |\alpha'(u)|\,du,\int_a^{\cdot} \|Y'(u)\|\,du\right).\end{aligned} \tag{2.4}$$

By utilising (2.3) and (2.4), we deduce the first inequality in (2.1).

Observe that

$$0\le\int_a^t |\alpha'(u)|\,du\le\int_a^b |\alpha'(u)|\,du$$

and

$$0\le\int_a^t \|Y'(u)\|\,du\le\int_a^b \|Y'(u)\|\,du$$

for all $t\in[a,b]$, then by Grüss's inequality for the functions $f(t)=\int_a^t |\alpha'(u)|\,du$ and $g(t)=\int_a^t \|Y'(u)\|\,du$, $t\in[a,b]$, we get the last part of (2.1). □

We have the Čebyšev's inequality:

Corollary 2. *Let $\alpha : [a,b] \to \mathbb{C}$ be differentiable and $Y : [a,b] \to E$ be strongly differentiable on the interval (a,b) with*

$$\|\alpha'\|_{[a,b],\infty} := \sup_{u\in(a,b)} |\alpha'(u)|, \quad \|Y'\|_{[a,b],\infty} < \infty,$$

then

$$\|D(\alpha, Y)\| \leq \frac{1}{12}(b-a)^4 \|\alpha'\|_{[a,b],\infty} \|Y'\|_{[a,b],\infty}. \tag{2.5}$$

Proof. If we use Čebyšev's inequality (1.3) for $f(t) = \int_a^t |\alpha'(u)| du$ and $g(t) = \int_a^t \|Y'(u)\| du$, $t \in [a,b]$, then we get

$$\begin{aligned} 0 &\leq D\left(\int_a^{\cdot} |\alpha'(u)| du, \int_a^{\cdot} \|Y'(u)\| du\right) \\ &\leq \frac{1}{12}(b-a)^4 \|f'\|_\infty \|g'\|_\infty = \frac{1}{12}(b-a)^4 \|\alpha'\|_{[a,b],\infty} \|Y'\|_{[a,b],\infty}, \end{aligned}$$

which by the first inequality in (2.1) gives the desired result (2.5). □

By the use of Ostrowski's inequality (1.4) we derive:

Corollary 3. *Let $\alpha : [a,b] \to \mathbb{C}$ be differentiable and $Y : [a,b] \to E$ be strongly differentiable on the interval (a,b) with $\|Y'\|_{[a,b],\infty} < \infty$, then*

$$\|D(\alpha, Y)\| \leq \frac{1}{8}(b-a)^3 \|\alpha'\|_{[a,b],1} \|Y'\|_{[a,b],\infty}. \tag{2.6}$$

For a differentiable complex valued function z on (a,b), we define

$$\|z'\|_{[a,b],2} := \left(\int_a^b |z'(u)|^2 du\right)^{1/2}.$$

By the use of Lupaş inequality for $f(t) = \int_a^t |\alpha'(u)| du$ and $g(t) = \int_a^t \|Y'(u)\| du$, $t \in [a,b]$, we get:

Corollary 4. *Let $\alpha : [a,b] \to \mathbb{C}$ be differentiable and $Y : [a,b] \to E$ be strongly differentiable on the interval (a,b) with $\|\alpha'\|_{[a,b],2}, \|Y'\|_{[a,b],2} < \infty$, then*

$$\|D(\alpha, Y)\| \leq \frac{1}{\pi^2}(b-a)^3 \|\alpha'\|_{[a,b],2} \|Y'\|_{[a,b],2}. \tag{2.7}$$

Observe that for $f(t) = \int_a^t |\alpha'(u)| du$, we get integrating by parts that

$$\begin{aligned} &\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s) ds \right| dt \\ &= \int_a^b \left| \int_a^t |\alpha'(u)| du - \frac{1}{b-a}\int_a^b \left(\int_a^s |\alpha'(u)| du\right) ds \right| dt \\ &= \int_a^b \left| \int_a^t |\alpha'(u)| du - \frac{1}{b-a}\left(\left(\int_a^b |\alpha'(u)| du\right) b - \int_a^b |\alpha'(s)| s ds\right) \right| dt \end{aligned}$$

$$
\begin{aligned}
&= \int_a^b \left| \int_a^t |\alpha'(u)|\, du - \frac{1}{b-a}\left(\int_a^b (b-u)\, |\alpha'(u)|\, du \right) \right| dt \\
&= \frac{1}{b-a} \int_a^b \left| (b-a) \int_a^t |\alpha'(u)|\, du - \int_a^b (b-u)\, |\alpha'(u)|\, du \right| dt \\
&= \frac{1}{b-a} \int_a^b \left| \int_a^t (u-a)\, |\alpha'(u)|\, du - \int_t^b (b-u)\, |\alpha'(u)|\, du \right| dt.
\end{aligned}
$$

By utilising (1.8) for $f(t) = \int_a^t |\alpha'(u)|\, du$ and $g(t) = \int_a^t \|Y'(u)\|\, du$, $t \in [a,b]$, we get:

Corollary 5. *Let $\alpha : [a,b] \to \mathbb{C}$ be differentiable and $Y : [a,b] \to E$ be strongly differentiable on the interval (a,b), then*

$$
\begin{aligned}
\|D(\alpha, Y)\| \le{}& \frac{1}{2} \|Y'\|_{[a,b],1} \\
&\times \int_a^b \left| \int_a^t (u-a)\, |\alpha'(u)|\, du - \int_t^b (b-u)\, |\alpha'(u)|\, du \right| dt.
\end{aligned} \tag{2.8}
$$

Remark 1. *We observe that*

$$
\begin{aligned}
&\int_a^b \left| \int_a^t (u-a)\, |\alpha'(u)|\, du - \int_t^b (b-u)\, |\alpha'(u)|\, du \right| dt \\
&\le \int_a^b \left[\left| \int_a^t (u-a)\, |\alpha'(u)|\, du \right| + \left| \int_t^b (b-u)\, |\alpha'(u)|\, du \right| \right] dt \\
&\le \int_a^b \left[\int_a^t (u-a)\, |\alpha'(u)|\, du + \int_t^b (b-u)\, |\alpha'(u)|\, du \right] dt
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\int_a^t (u-a)\, |\alpha'(u)|\, du + \int_t^b (b-u)\, |\alpha'(u)|\, du \right] t \Big|_a^b \\
&- \int_a^b t\left((t-a)\, |\alpha'(t)| - (b-t)\, |\alpha'(t)| \right) dt
\end{aligned}
$$

$$
\begin{aligned}
&= b \int_a^b (u-a)\, |\alpha'(u)|\, du - a \int_a^b (b-u)\, |\alpha'(u)|\, du \\
&- \int_a^b t\left((t-a)\, |\alpha'(t)| - (b-t)\, |\alpha'(t)| \right) dt \\
&= 2 \int_a^b (b-t)(t-a)\, |\alpha'(t)|\, dt
\end{aligned}
$$

and by (2.8) we get

$$
\|D(\alpha, Y)\| \le \|Y'\|_{[a,b],1} \int_a^b (b-t)(t-a)\, |\alpha'(t)|\, dt. \tag{2.9}
$$

Theorem 5. *Let $\alpha : [a,b] \to \mathbb{C}$ be differentiable and $Y : [a,b] \to E$ be strongly differentiable on the interval (a,b), then*

$$\|D(\alpha, Y)\| \leq \begin{cases} \inf_{\beta\in\mathbb{C}} \|\alpha-\beta\|_{[a,b],\infty} \int_a^b \left\|(b-a)Y(t) - \int_a^b Y(s)\,ds\right\| dt, \\ \inf_{\beta\in\mathbb{C}} \|\alpha-\beta\|_{[a,b],q} \left(\int_a^b \left\|(b-a)Y(t) - \int_a^b Y(s)\,ds\right\|^p dt\right)^{1/p}, \\ \inf_{\beta\in\mathbb{C}} \|\alpha-\beta\|_{[a,b],1} \sup_{t\in[a,b]} \left\|(b-a)Y(t) - \int_a^b Y(s)\,ds\right\| \end{cases} \tag{2.10}$$

for all $p,\ q>1$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. For all $\beta \in \mathbb{C}$ we have

$$\begin{aligned} &\int_a^b [\alpha(t)-\beta]\left[(b-a)Y(t) - \int_a^b Y(s)\,ds\right] dt \\ &= \int_a^b \alpha(t)\left[(b-a)Y(t) - \int_a^b Y(s)\,ds\right] dt \\ &- \beta \int_a^b \left[(b-a)Y(t) - \int_a^b Y(s)\,ds\right] dt \\ &= (b-a)\int_a^b \alpha(t)Y(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b Y(s)\,ds \\ &- \beta\left[(b-a)\int_a^b Y(t)\,dt - (b-a)\int_a^b Y(s)\,ds\right] = D(\alpha, Y). \end{aligned}$$

Taking the norm in this equality, we get by Hölder's inequality that

$$\|D(\alpha,Y)\| \leq \int_a^b \left\|[\alpha(t)-\beta]\left[(b-a)Y(t) - \int_a^b Y(s)\,ds\right]\right\| dt$$

$$\leq \int_a^b |\alpha(t)-\beta| \left\|(b-a)Y(t) - \int_a^b Y(s)\,ds\right\| dt$$

$$\leq \begin{cases} \sup_{t\in[a,b]} |\alpha(t)-\beta| \int_a^b \left\|(b-a)Y(t) - \int_a^b Y(s)\,ds\right\| dt, \\ \left(\int_a^b |\alpha(t)-\beta|^q dt\right)^{1/q} \left(\int_a^b \left\|(b-a)Y(t) - \int_a^b Y(s)\,ds\right\|^p dt\right)^{1/p}, \\ \int_a^b |\alpha(t)-\beta| \sup_{t\in[a,b]} \left\|(b-a)Y(t) - \int_a^b Y(s)\,ds\right\| \end{cases}$$

for all $\beta \in \mathbb{C}$.

By taking the infimum over $\beta \in \mathbb{C}$, we obtain the desired result (2.10). □

Corollary 6. *With the assumptions of Theorem 5 and if there exists $\beta \in \mathbb{C}$ and $M > 0$ such that*

$$|\alpha(t)-\beta| \leq M \text{ for all } t \in [a,b],$$

then

$$\|D(\alpha,Y)\| \le M\int_a^b \left\|(b-a)Y(t)-\int_a^b Y(s)\,ds\right\| dt. \tag{2.11}$$

Remark 2. *If there exists* γ, $\Gamma\in\mathbb{C}$ *with*

$$\left|\alpha(t)-\frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma| \text{ for all } t\in[a,b]$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma-\alpha(t))\left(\overline{\alpha(t)}-\overline{\gamma}\right)\right]\ge 0, \text{ for all } t\in[a,b],$$

then by (2.11) we get

$$\|D(\alpha,Y)\| \le \frac{1}{2}|\Gamma-\gamma|\int_a^b \left\|(b-a)Y(t)-\int_a^b Y(s)\,ds\right\| dt. \tag{2.12}$$

The proof is obvious from the first branch of (2.10).

Corollary 7. *Let* $\alpha:[a,b]\to\mathbb{C}$ *be differentiable and* $Y:[a,b]\to E$ *be strongly differentiable on the interval* (a,b), *then for* p, $q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}\|D(\alpha,Y)\| &\le \sup_{t\in[a,b]}\left\|(b-a)Y(t)-\int_a^b Y(s)\,ds\right\| \\ &\quad\times\left[\int_a^{\frac{a+b}{2}}(t-a)|\alpha'(t)|\,dt+\int_{\frac{a+b}{2}}^b (b-t)|\alpha'(t)|\,dt\right] \\ &\le \sup_{t\in[a,b]}\left\|(b-a)Y(t)-\int_a^b Y(s)\,ds\right\| \\ &\quad\times\begin{cases}\frac{1}{8}(b-a)^2\left[\|\alpha'\|_{\left[a,\frac{a+b}{2}\right],\infty}+\|\alpha'\|_{\left[\frac{a+b}{2},b\right],\infty}\right], \\ \\ \frac{1}{(q+1)^{1/q}2^{1+1/q}}\left[\|\alpha'\|_{\left[a,\frac{a+b}{2}\right],p}+\|\alpha'\|_{\left[\frac{a+b}{2},b\right],p}\right], \\ \\ \frac{1}{2}(b-a)\|\alpha'\|_{[a,b],1}.\end{cases}\end{aligned} \tag{2.13}$$

Proof. We have

$$\begin{aligned}&\int_a^b\left|\alpha(t)-\alpha\left(\frac{a+b}{2}\right)\right|dt \\ &=\int_a^b\left|\int_{\frac{a+b}{2}}^t \alpha'(s)\,ds\right|dt\le\int_a^b\left|\int_{\frac{a+b}{2}}^t|\alpha'(s)|\,ds\right|dt \\ &=\int_a^{\frac{a+b}{2}}\left(\int_t^{\frac{a+b}{2}}|\alpha'(s)|\,ds\right)dt+\int_{\frac{a+b}{2}}^b\left(\int_{\frac{a+b}{2}}^t|\alpha'(s)|\,ds\right)dt\end{aligned}$$

$$\begin{aligned}
&= \left(\int_t^{\frac{a+b}{2}} |\alpha'(s)|\, ds\right) t\Big|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} t\,|\alpha'(t)|\, dt \\
&+ \left(\int_{\frac{a+b}{2}}^t |\alpha'(s)|\, ds\right) t\Big|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b t\,|\alpha'(t)|\, dt \\
&= \int_a^{\frac{a+b}{2}} t\,|\alpha'(t)|\, dt - a\int_a^{\frac{a+b}{2}} |\alpha'(s)|\, ds \\
&+ b\int_{\frac{a+b}{2}}^b |\alpha'(s)|\, ds - \int_{\frac{a+b}{2}}^b t\,|\alpha'(t)|\, dt \\
&= \int_a^{\frac{a+b}{2}} (t-a)\,|\alpha'(t)|\, dt + \int_{\frac{a+b}{2}}^b (b-t)\,|\alpha'(t)|\, dt,
\end{aligned}$$

which, by the third branch of (2.10), gives the first part of (2.13).

The last part follows by Hölder's inequality. □

3. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A) \geq h(B)$ holds for any $A \geq B \geq 0$ operators on Hilbert space H.

We have the following representation of operator monotone functions, see for instance [4, p. 144-145]:

Theorem 6. *A function $h:[0,\infty)\to\mathbb{R}$ is operator monotone in $[0,\infty)$ if and only if it has the representation*

$$h(t) = h(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{3.1}$$

where $b \geq 0$ and a positive measure μ on $[0,\infty)$ such that

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{3.2}$$

We have the following representation result:

Lemma 1. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq 0$, then for all selfadjoint operators V we have*

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1}\right] d\mu(\lambda). \tag{3.3}$$

Proof. From (3.1) we get

$$h(t) = h(0) + bt + \int_0^\infty \left(\lambda - \frac{\lambda^2}{t+\lambda}\right) d\mu(\lambda).$$

Assume that $U \geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
& h(U+tV) - h(U) \\
&= btV + \int_0^\infty \left(\lambda - \lambda^2 (U+tV+\lambda)^{-1}\right) d\mu(\lambda) - \int_0^\infty \left(\lambda - \lambda^2 (U+\lambda)^{-1}\right) d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} - (\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\
&= btV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} (\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right] d\mu(\lambda) \\
&= btV + t\int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda).
\end{aligned}$$

Dividing by $t \neq 0$, we get

$$\frac{h(U+tV) - h(U)}{t} = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U+tV)^{-1}\right] d\mu(\lambda)$$

and by taking the limit over $t \to 0$, we get

$$Dh(U)(V) = bV + \int_0^\infty \lambda^2 \left[(\lambda+U)^{-1} V (\lambda+U)^{-1}\right] d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Theorem 7. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U \geq u > 0$, then for all selfadjoint operators V we have*

$$\|Dh(U)(V)\| \leq h'(u) \|V\|. \tag{3.4}$$

Proof. From (3.3) we get

$$\begin{aligned}
\|Dh(U)(V) - bV\| &\leq \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1} V (\lambda+U)^{-1}\right\| d\mu(\lambda) \\
&\leq \|V\| \int_0^\infty \lambda^2 \left\|(\lambda+U)^{-1}\right\|^2 d\mu(\lambda).
\end{aligned} \tag{3.5}$$

Observe that $\lambda + U \geq \lambda + u > 0$ for $\lambda \in [0,\infty)$. Then $0 < (\lambda+U)^{-1} \leq (\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\| \leq (\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2 \leq (\lambda+u)^{-2}$ for $\lambda \in [0,\infty)$.

Therefore by (3.5) we get

$$\|Dh(U)(V) - bV\| \leq \|V\| \int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$h'(t) = b + \int_0^\infty \frac{\lambda(t+\lambda) - \lambda t}{(t+\lambda)^2} d\mu(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) \tag{3.7}$$

for $t > 0$.

From (3.7) we get

$$\int_0^\infty \lambda^2 (\lambda+u)^{-2} d\mu(\lambda) = h'(u) - b,$$

and by (3.6) we derive

$$\|Dh(U)(V) - bV\| \leq \|V\| h'(u) - b\|V\|.$$

Finally, by the triangle inequality and by the fact that $b \geq 0$, we obtain that

$$\|Dh(U)(V)\| - b\|V\| \leq \|Dh(U)(V) - bV\|,$$

which proves the desired result (3.4). □

For a continuous function h on $(0,\infty)$ and $A,\ B > 0$ we consider the auxiliary function $h_{A,B} : [0,1] \to \mathbb{R}$ defined by

$$h_{A,B}(t) := h((1-t)A + tB),\ t \in [0,1].$$

We have the following representations of the derivatives:

Lemma 2. *Assume that the operator function generated by h is Fréchet differentiable in each $A \geq 0$, then for $B \geq 0$ we have that $h_{A,B}$ is differentiable on $[0,1]$ and*

$$h'_{A,B}(t) = D(h)((1-t)A + tB)(B - A) \tag{3.8}$$

for $t \in [0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned}
&\frac{h_{A,B}(t+h) - h(t)}{h} \\
&= \frac{h((1-(t+h))A + (t+h)B) - h((1-t)A + tB)}{h} \\
&= \frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h}
\end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned}
h'_{A,B}(t) &= \lim_{h\to 0} \frac{h_{A,B}(t+h) - h(t)}{h} \\
&= \lim_{h\to 0} \left[\frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h}\right] \\
&= D(h)((1-t)A + tB)(B-A),
\end{aligned}$$

which proves (3.8). □

Corollary 8. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A \geq a > 0$, $B \geq b > 0$ we have*

$$\begin{aligned}
\left\|h'_{A,B}(t)\right\| &= \|D(h)((1-t)A + tB)(B-A)\| \\
&\leq h'((1-t)a + tb)\|B - A\|
\end{aligned} \tag{3.9}$$

for all $t \in [0,1]$.

The proof follows by Theorem 7 and Lemma 2.

One can observe that the inequality (3.9) remains valid for operator monotone functions on $(0,\infty)$. This follows by considering the function $h_\varepsilon(t) := h(t+\varepsilon)$ for $\varepsilon > 0$, which is operator monotone on $[0,\infty)$ and then by letting $\varepsilon \to 0+$ and using the continuity of h and h'.

Theorem 8. *Let $f:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$ and $\alpha:[0,1]\to\mathbb{C}$ a continuous function on $[0,1]$. Then for all $A\geq a>0$, $B\geq b>0$ we have*

$$\left\|\int_0^1\alpha(t)f((1-t)A+tB)\,dt-\int_0^1\alpha(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\right\| \tag{3.10}$$
$$\leq\frac{1}{4}\|B-A\|\times\begin{cases}\|\alpha'\|_{[a,b],1}\int_0^1 h'((1-t)a+tb)\,dt,\\[1ex] \frac{1}{3}\|\alpha'\|_{[a,b],\infty}\sup_{t\in[0,1]}h'((1-t)a+tb),\\[1ex] \frac{1}{2}\|\alpha'\|_{[a,b],1}\sup_{t\in[0,1]}h'((1-t)a+tb),\\[1ex] \frac{4}{\pi^2}\|\alpha'\|_{[a,b],2}\left(\int_0^1[h'((1-t)a+tb)]^2\,dt\right)^{1/2},\end{cases}$$

provided that the quantities in the right hand side are finite.

Proof. We use the inequality (2.1) for α and $f_{A,B}$ on $[0,1]$ to get

$$\left\|\int_0^1\alpha(t)f((1-t)A+tB)\,dt-\int_0^1\alpha(t)\,dt\int_0^1 f((1-t)A+tB)\,dt\right\| \tag{3.11}$$
$$\leq\frac{1}{4}\|\alpha'\|_{[0,1],1}\left\|f'_{A,B}\right\|_{[0,1],1}.$$

Since, by (3.9),

$$\left\|f'_{A,B}\right\|_{[0,1],1}\leq\|B-A\|\int_0^1 h'((1-t)a+tb)\,dt,$$

hence by (3.11) we get the first inequality in (3.10).

The second inequality follows by (2.5), the third inequality follows by (2.6) while the last inequality follows by (2.7). □

If we take in (3.10) $f(t)=t^r$, $r\in(0,1)$ that is operator monotone on $[0,\infty)$, then by (3.10) we derive

$$\left\|\int_0^1\alpha(t)((1-t)A+tB)^r\,dt-\int_0^1\alpha(t)\,dt\int_0^1((1-t)A+tB)^r\,dt\right\| \tag{3.12}$$
$$\leq\frac{1}{4}r\|B-A\|\times\begin{cases}\|\alpha'\|_{[0,1],1}\times\begin{cases}\frac{b^r-a^r}{r(b-a)} & \text{if } b\neq a,\\ a^{r-1} & \text{if } b=a,\end{cases}\\[2ex] \frac{1}{3}\|\alpha'\|_{[0,1],\infty}\max\left\{a^{r-1},b^{r-1}\right\},\\[1ex] \frac{1}{2}\|\alpha'\|_{[0,1],1}\max\left\{a^{r-1},b^{r-1}\right\},\\[1ex] \frac{4}{\pi^2}\|\alpha'\|_{[0,1],2}\\ \times\begin{cases}\left(\frac{b^{2r-1}-a^{2r-1}}{(2r-1)(b-a)}\right)^{1/2} & \text{if } b\neq a,\ r\neq 1/2\\[1ex] \left(\frac{\ln b-\ln a}{b-a}\right)^{1/2} & \text{if } b\neq a,\ r=1/2\\[1ex] a^{r-1} & \text{if } b=a\end{cases}\end{cases},$$

for any $\alpha : [0,1] \to \mathbb{C}$ a continuous function on $[0,1]$.

If we take in (3.10) $f(t) = \ln t$, that is operator monotone on $(0,\infty)$, then by (3.10) we derive

$$\left\| \int_0^1 \alpha(t) \ln((1-t)A + tB)\, dt - \int_0^1 \alpha(t)\, dt \int_0^1 \ln((1-t)A + tB)\, dt \right\| \tag{3.13}$$

$$\leq \frac{1}{4} \|B - A\| \times \begin{cases} \|\alpha'\|_{[0,1],1} \times \begin{cases} \frac{\ln b - \ln a}{b-a} \text{ if } b \neq a, \\ \frac{1}{a} \text{ if } b = a, \end{cases} \\ \frac{1}{3\min\{a,b\}} \|\alpha'\|_{[0,1],\infty} \\ \frac{1}{2\min\{a,b\}} \|\alpha'\|_{[0,1],1}, \\ \frac{4}{\pi^2 \sqrt{ab}} \|\alpha'\|_{[0,1],2}, \end{cases}$$

for any $\alpha : [0,1] \to \mathbb{C}$ a continuous function on $[0,1]$.

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