SOME NEW BOUNDS FOR THE ČEBYŠEV FUNCTIONAL OF DIFFERENTIABLE FUNCTIONS IN BANACH SPACES

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Abstract. Assume that $(E, \|\cdot\|)$ is a complex Banach space. In this paper we show among others that, if $Y : [a,b] \to E$ is a strongly differentiable function on the interval (a,b) and $\alpha : [a,b] \to \mathbb{C}$ is a continuous function, then

$$\begin{aligned}
&\left\|(b-a)\int_a^b \alpha(t) Y(t)\,dt - \int_a^b \alpha(t)\,dt \int_a^b Y(t)\,dt\right\| \\
&\leq \frac{1}{2}(b-a)\|\alpha'\|_{[a,b],\infty} \int_a^b (b-t)(t-a)\|Y'(t)\|\,dt \\
&\leq \frac{1}{2}(b-a)^3 \|\alpha'\|_{[a,b],\infty} \\
&\times \begin{cases} \frac{1}{4}\|Y'\|_{[a,b],1}, \\ (b-a)^{1/q}[B(q+1,q+1)]^{1/q}\|Y'\|_{[a,b],p}, \\ \frac{1}{6}(b-a)^4\|Y'\|_{[a,b],\infty}, \end{cases}
\end{aligned}$$

for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, where $\|\cdot\|_{[a,b],p}$ is L_p-norm and $\|\cdot\|_{[a,b],\infty}$ is the sup-norm. Applications for operator monotone functions in Hilbert spaces with examples for the logarithmic and power functions are also given.

1. Introduction

For two Lebesgue integrable functions f, $g : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(f,g) := (b-a)\int_a^b f(t) g(t)\,dt - \int_a^b f(t)\,dt \int_a^b g(t)\,dt.$$

In 1934, G. Grüss [19] showed that

$$|D(f,g)| \leq \frac{1}{4}(b-a)^2 (M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

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Another lesser known inequality for $D(f,g)$ was derived in 1882 by Čebyšev [11] under the assumption that f', g' exist and are continuous on $[a,b]$, and is given by

$$|D(f,g)| \leq \frac{1}{12}\|f'\|_\infty \|g'\|_\infty (b-a)^4, \tag{1.3}$$

where $\|f'\|_\infty := \sup_{t\in[a,b]} |f'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $f,\ g : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $f',\ g' \in L_\infty[a,b]$.

In 1970, A. M. Ostrowski [28] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(f,g)| \leq \frac{1}{8}(b-a)^3 (M-m)\|g'\|_\infty, \tag{1.4}$$

provided f is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $g : [a,b] \to \mathbb{R}$ is absolutely continuous and $g' \in L_\infty[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [22] (see also [24, p. 210]) obtained the following result as well:

$$|D(f,g)| \leq \frac{1}{\pi^2}\|f'\|_2 \|g'\|_2 (b-a)^3, \tag{1.5}$$

provided $f,\ g$ are absolutely continuous and $f',\ g' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [8], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(f,g)| \leq (b-a)\begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt, \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|g-\gamma\|_q \left(\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right|^p dt\right)^{\frac{1}{p}}, \\ \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(f,g)| \leq (b-a)\|g\|_\infty \int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq g \leq M$ for a.e. $x \in [a,b]$, then $\left\|g - \frac{m+M}{2}\right\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [10]

$$|D(f,g)| \leq \frac{1}{2}(b-a)(M-m)\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [9].

The following result holds [14].

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(f,g)| \leq \frac{1}{2}(b-a)\bigvee_a^b(f)\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt \tag{1.9}$$

where $\bigvee_a^b (f)$ *denotes the total variation of* f *on the interval* $[a,b]$. *The constant* $\frac{1}{2}$ *is best possible in (1.9).*

For more recent upper bounds related to the Čebyšev functional see [8], [9] and [12]-[14].

In [3] the authors obtained the following Grüss' type inequalities for functions with values in Banach spaces.

Theorem 2. *Let* F *be a Banach space over the real or complex number field* $\mathbb{K}$, $\Omega \in \mathbb{R}^n$ *a measurable set and* $\rho : \Omega \to [0,\infty)$ *a Lebesgue integrable function with* $\int_\Omega \rho(x)\,dx = 1$. *If* $\alpha : \Omega \to \mathbb{K}$ *is a Lebesgue integrable function such that there exists* $\gamma, \Gamma \in \mathbb{K}$ *with*

$$\left|\alpha(x) - \frac{\gamma+\Gamma}{2}\right| \le \frac{1}{2}|\Gamma-\gamma| \tag{1.10}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma - \alpha(x))\left(\overline{\alpha(x)} - \overline{\gamma}\right)\right] \ge 0 \tag{1.11}$$

for a.e. $x \in \Omega$, *and* $g : \Omega \to F$ *is a Bochner measurable function such that* $\rho\alpha g$ *and* ρg *are Bochner integrable on* Ω, *then,*

$$\begin{aligned} &\left\| \int_\Omega \rho(x)\alpha(x)g(x)\,dx - \int_\Omega \rho(x)\alpha(x)\,dx \int_\Omega \rho(x)g(x)\,dx \right\| \\ &\le \frac{1}{2}|\Gamma-\gamma| \int_\Omega \rho(x) \left\| g(x) - \int_\Omega \rho(y)g(y)\,dy \right\| dx. \end{aligned} \tag{1.12}$$

The constant $\frac{1}{2}$ *in (1.12) is the best possible.*

The following dual result also holds:

Theorem 3. *Let* F *and* Ω, ρ *be as above. If* $g : \Omega \to X$ *is Bochner measurable on* Ω *and there exist vector* $v \in X$ *and* $r > 0$ *such that*

$$\|g(x) - v\| \le r \text{ for a.e. } x \in \Omega$$

and $\alpha : \Omega \to \mathbb{K}$ *is a Lebesgue integrable function with* $\rho\alpha g$, ρg *Bochner integrable functions on* Ω, *then we have the sharp inequalities*

$$\begin{aligned} &\left\| \int_\Omega \rho(x)\alpha(x)g(x)\,dx - \int_\Omega \rho(x)\alpha(x)\,dx \int_\Omega \rho(x)g(x)\,dx \right\| \\ &\le r \int_\Omega \rho(x) \left| \alpha(x) - \int_\Omega \rho(y)\alpha(y)\,dy \right| dx \\ &\le r \left[\int_\Omega \rho(x)|\alpha(x)|^2\,dx - \left| \int_\Omega \rho(x)\alpha(x)\,dx \right|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{1.13}$$

Now, consider the function f defined on the open and convex subset C of the Banach space E with values in the Banach space F and $\Omega = [0,1]$. Also let $\rho(t) = 1$ and $g(t) = f((1-t)x + ty)$ for $t \in [0,1]$ and x, $y \in C$. Then we can state the following particular case of interest:

Corollary 1. *Assume that $f : C \subset E \to F$ is continuous on C and x, $y \in C$, $x \neq y$. If $p : [0,1] \to \mathbb{K}$ is a Lebesgue integrable function such that there exists γ, $\Gamma \in \mathbb{K}$ with*

$$\left| p(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \tag{1.14}$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma - p(t)) \left(\overline{p(t)} - \overline{\gamma} \right) \right] \geq 0 \tag{1.15}$$

for a.e. $t \in [0,1]$, then,

$$\begin{aligned} &\left\| \int_0^1 p(t) f((1-t)x + ty) \, dt - \int_0^1 p(t) \, dt \int_0^1 f((1-t)x + ty) \, dt \right\| \\ &\leq \frac{1}{2} |\Gamma - \gamma| \int_0^1 \left\| f((1-t)x + ty) - \int_0^1 f((1-s)x + sy) \, ds \right\| dt. \end{aligned} \tag{1.16}$$

The constant $\frac{1}{2}$ in (1.16) is the best possible.

If there exists a vector v and $r > 0$ such that

$$\| f((1-t)x + ty) - v \| \leq r \text{ for a.e. } t \in [0,1], \tag{1.17}$$

then for $q : [0,1] \to \mathbb{C}$ Lebesgue integrable, we have the sharp inequalities

$$\begin{aligned} &\left\| \int_0^1 q(t) f((1-t)x + ty) \, dt - \int_0^1 q(t) \, dt \int_0^1 f((1-t)x + ty) \, dt \right\| \\ &\leq r \int_0^1 \left| q(t) - \int_0^1 q(s) \, ds \right| dt \leq r \left[\int_0^1 |q(t)|^2 \, dt - \left| \int_\Omega q(t) \, dt \right|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{1.18}$$

We observe that, if there exists two vectors z, $w \in F$ such that

$$\left\| f((1-t)x + ty) - \frac{z + w}{2} \right\| \leq \frac{1}{2} \|w - z\| \text{ for a.e. } t \in [0,1], \tag{1.19}$$

then for $q : [0,1] \to \mathbb{C}$ Lebesgue integrable, we have the sharp inequalities

$$\begin{aligned} &\left\| \int_0^1 q(t) f((1-t)x + ty) \, dt - \int_0^1 q(t) \, dt \int_0^1 f((1-t)x + ty) \, dt \right\| \\ &\leq \frac{1}{2} \|w - z\| \int_0^1 \left| q(t) - \int_0^1 q(s) \, ds \right| dt \\ &\leq \frac{1}{2} \|w - z\| \left[\int_0^1 |q(t)|^2 \, dt - \left| \int_\Omega q(t) \, dt \right|^2 \right]^{\frac{1}{2}}. \end{aligned} \tag{1.20}$$

For some recent results on Grüss' type inequalities, see [1]-[6] and [18]-[28].

In this paper we show among others that, if $Y : [a,b] \to E$ is a strongly differentiable function on the interval (a,b) and $\alpha : [a,b] \to \mathbb{C}$ is a continuous function, then

$$\begin{aligned}
&\left\|(b-a)\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt\right\| \\
&\le \frac{1}{2}(b-a)\,\|\alpha'\|_{[a,b],\infty}\int_a^b (b-t)(t-a)\,\|Y'(t)\|\,dt \\
&\le \frac{1}{2}(b-a)^3\,\|\alpha'\|_{[a,b],\infty} \\
&\times\begin{cases}
\frac{1}{4}\|Y'\|_{[a,b],1}\,, \\[2mm]
(b-a)^{1/q}\left[B(q+1,q+1)\right]^{1/q}\|Y'\|_{[a,b],p}\,, \\[2mm]
\frac{1}{6}(b-a)^4\,\|Y'\|_{[a,b],\infty}\,,
\end{cases}
\end{aligned}$$

for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, where $\|\cdot\|_{[a,b],p}$ is L_p-norm and $\|\cdot\|_{[a,b],\infty}$ is the sup-norm. Applications for operator monotone functions in Hilbert spaces with examples for the logarithmic and power functions are also given.

2. Main Results

Let $(E,\|\cdot\|)$ be a complex Banach space. For two continuous functions $\alpha:[a,b]\to\mathbb{C}$ and $Y:[a,b]\to E$ we define the *Čebyšev functional*

$$D(\alpha,Y):=(b-a)\int_a^b \alpha(t)\,Y(t)\,dt-\int_a^b \alpha(t)\,dt\int_a^b Y(t)\,dt.$$

We have the following result of interest:

Theorem 4. *Let $\alpha:[a,b]\to\mathbb{C}$ be differentiable and $Y:[a,b]\to E$ be strongly differentiable on the interval (a,b). If $\|\alpha'\|_{[a,b],\infty}:=\sup_{t\in(a,b)}\|\alpha'(t)\|<\infty$, then*

$$\begin{aligned}
\|D(\alpha,Y)\| &\le \|\alpha'\|_{[a,b],\infty}\,D\left(\ell,\int_a^{\cdot}\|Y'(u)\|\,du\right) \\
&\le \frac{1}{8}(b-a)^3\,\|\alpha'\|_{[a,b],\infty}\,\|Y'\|_{[a,b],1}\,,
\end{aligned}\tag{2.1}$$

where $\|Y'\|_{[a,b],1}:=\int_a^b\|Y'(u)\|\,du$ and $\ell(t)=t,\ t\in[a,b]$.

Proof. Observe that

$$\begin{aligned}
&\int_a^b\int_a^b[\alpha(t)-\alpha(s)]\,[Y(t)-Y(s)]\,dtds \\
&=\int_a^b\int_a^b(\alpha(t)\,Y(t)-\alpha(s)\,Y(t)-\alpha(t)\,Y(s)+\alpha(s)\,Y(s))\,dtds \\
&=(b-a)\int_a^b\alpha(t)\,Y(t)\,dt-\int_a^b\alpha(s)\,ds\int_a^b Y(t)\,dt \\
&-\int_a^b\alpha(t)\,dt\int_a^b Y(s)\,ds+(b-a)\int_a^b\alpha(s)\,Y(s)\,ds \\
&=2(b-a)\int_a^b\alpha(t)\,Y(t)\,dt-2\int_a^b\alpha(t)\,dt\int_a^b Y(t)\,dt=2D(\alpha,Y)\,,
\end{aligned}$$

which give the Korkine's identity for functions with values in Banach spaces:

$$D(\alpha,Y)=\frac{1}{2}\int_a^b\int_a^b[\alpha(t)-\alpha(s)][Y(t)-Y(s)]\,dtds.$$

For Korkine's classical identity for real-valued functions, see [24, p. 242].

If we take the norm and use the integral's properties, we get

$$\|D(\alpha,Y)\|\le\frac{1}{2}\int_a^b\int_a^b\|[\alpha(t)-\alpha(s)][Y(t)-Y(s)]\|\,dtds \tag{2.2}$$
$$\le\frac{1}{2}\int_a^b\int_a^b|\alpha(t)-\alpha(s)|\,\|Y(t)-Y(s)\|\,dtds.$$

Observe that for $s,\,t\in[a,b]$

$$\alpha(t)-\alpha(s)=\int_s^t\alpha'(u)\,du,\ \ Y(t)-Y(s)=\int_s^tY'(u)\,du,$$

which implies that

$$\begin{aligned}|\alpha(t)-\alpha(s)|\,\|Y(t)-Y(s)\|&=\left|\int_s^t\alpha'(u)\,du\right|\left\|\int_s^tY'(u)\,du\right\|\\&\le\left|\int_s^t|\alpha'(u)|\,du\right|\left|\int_s^t\|Y'(u)\|\,du\right|\\&\le\sup_{t\in(a,b)}|\alpha'(u)|\,|t-s|\left|\int_s^t\|Y'(u)\|\,du\right|\\&=\sup_{t\in(a,b)}|\alpha'(u)|\,(t-s)\int_s^t\|Y'(u)\|\,du,\end{aligned}$$

for all $s,\,t\in[a,b]$.

By (2.2) we get

$$\|D(\alpha,Y)\|\le\sup_{t\in(a,b)}|\alpha'(u)|\,\frac{1}{2}\int_a^b\int_a^b(t-s)\left(\int_s^t\|Y'(u)\|\,du\right)dtds. \tag{2.3}$$

Since

$$(t-s)\left(\int_s^t\|Y'(u)\|\,du\right)=(t-s)\left(\int_a^t\|Y'(u)\|\,du-\int_a^s\|Y'(u)\|\,du\right),$$

hence by Korkine's identity for real valued functions $f(t)=\ell(t)$ and $g(t)=\int_a^t\|Y'(u)\|\,du$, we have

$$\begin{aligned}\frac{1}{2}\int_a^b\int_a^b(t-s)\left(\int_s^t\|Y'(u)\|\,du\right)&=(b-a)\int_a^b\ell(t)\left(\int_a^t\|Y'(u)\|\,du\right)dt\\&\quad-\int_a^b\ell(t)\,dt\int_a^b\left(\int_a^t\|Y'(u)\|\,du\right)dt\\&=D\left(\ell,\int_a^{\cdot}\|Y'(u)\|\,du\right).\end{aligned} \tag{2.4}$$

By utilising (2.3) and (2.4), we deduce the first inequality in (2.1).

Observe that

$$0\le\int_a^t\|Y'(u)\|\,du\le\int_a^b\|Y'(u)\|\,du$$

for all $t\in[a,b]$, then by (1.8) for the functions $f(t)=\ell(t)$ and $g(t)=\int_a^t\|Y'(u)\|\,du$, $t\in[a,b]$, we get

$$\begin{aligned}
&\left|D\left(\ell,\int_a^{\cdot}\|Y'(u)\|\,du\right)\right|\\
&\leq\frac{1}{2}(b-a)\int_a^b\|Y'(u)\|\,du\int_a^b\left|t-\frac{1}{b-a}\int_a^b sds\right|dt\\
&=\frac{1}{2}(b-a)\int_a^b\|Y'(u)\|\,du\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{1}{8}(b-a)^3\int_a^b\|Y'(u)\|\,du,
\end{aligned}$$

which proves the last part of (2.1). □

Remark 1. *If we apply the same inequality (1.8) for the functions $f(t)=\int_a^t\|Y'(u)\|\,du$ and $g(t)=\ell(t)$, $t\in[a,b]$, then we get*

$$\begin{aligned}
(2.5)\qquad &\left|D\left(\ell,\int_a^{\cdot}\|Y'(u)\|\,du\right)\right|\\
&\leq\frac{1}{2}(b-a)^2\int_a^b\left|\int_a^t\|Y'(u)\|\,du-\frac{1}{b-a}\int_a^b\left(\int_a^s\|Y'(u)\|\,du\right)ds\right|dt.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_a^b\left|\int_a^t\|Y'(u)\|\,du-\frac{1}{b-a}\int_a^b\left(\int_a^s\|Y'(u)\|\,du\right)ds\right|dt\\
&=\int_a^b\left|\int_a^t\|Y'(u)\|\,du-\frac{1}{b-a}\left(\left(\int_a^b\|Y'(u)\|\,du\right)b-\int_a^b\|Y'(s)\|\,sds\right)\right|dt\\
&=\int_a^b\left|\int_a^t\|Y'(u)\|\,du-\frac{1}{b-a}\left(\int_a^b(b-u)\|Y'(u)\|\,du\right)\right|dt\\
&=\frac{1}{b-a}\int_a^b\left|(b-a)\int_a^t\|Y'(u)\|\,du-\int_a^b(b-u)\|Y'(u)\|\,du\right|dt\\
&=\frac{1}{b-a}\int_a^b\left|\int_a^t(u-a)\|Y'(u)\|\,du-\int_t^b(b-u)\|Y'(u)\|\,du\right|dt.
\end{aligned}$$

Then by (2.1) and (2.5) we obtain

$$\begin{aligned}
(2.6)\qquad \|D(\alpha,Y)\|&\leq\|\alpha'\|_{[a,b],\infty}D\left(\ell,\int_a^{\cdot}\|Y'(u)\|\,du\right)\\
&\leq\frac{1}{2}(b-a)\|\alpha'\|_{[a,b],\infty}\\
&\quad\times\int_a^b\left|\int_a^t(u-a)\|Y'(u)\|\,du-\int_t^b(b-u)\|Y'(u)\|\,du\right|dt.
\end{aligned}$$

Remark 2. *Using (1.3) we have*

$$0\leq D\left(\ell,\int_a^{\cdot}\|Y'(u)\|\,du\right)\leq\frac{1}{12}\sup_{t\in(a,b)}\|Y'(u)\|(b-a)^4,$$

and by (2.1) we derive

$$\left\|D\left(\alpha ,Y\right)\right\| \leq \left\|\alpha'\right\|_{[a,b],\infty} D\left(\ell,\int_a^{\cdot}\left\|Y'\left(u\right)\right\|du\right) \tag{2.7}$$
$$\leq \frac{1}{12}\left\|\alpha'\right\|_{[a,b],\infty}\left\|Y'\right\|_{[a,b],\infty}\left(b-a\right)^4$$

provided that $\left\|\alpha'\right\|_{[a,b],\infty}, \left\|Y'\right\|_{[a,b],\infty} < \infty$.

Using (1.4) we have

$$0 \leq D\left(\ell,\int_a^{\cdot}\left\|Y'\left(u\right)\right\|du\right) \leq \frac{1}{8}\left(b-a\right)^3\int_a^b\left\|Y'\left(u\right)\right\|du,$$

and by (2.1) we obtain

$$\left\|D\left(\alpha ,Y\right)\right\| \leq \left\|\alpha'\right\|_{[a,b],\infty} D\left(\ell,\int_a^{\cdot}\left\|Y'\left(u\right)\right\|du\right) \tag{2.8}$$
$$\leq \frac{1}{8}\left(b-a\right)^3\left\|\alpha'\right\|_{[a,b],\infty}\left\|Y'\right\|_{[a,b],1},$$

provided that $\left\|\alpha'\right\|_{[a,b],\infty} < \infty$.

Corollary 2. *Let* $\alpha : [a,b] \to \mathbb{C}$ *be differentiable and* $Y : [a,b] \to E$ *be strongly differentiable on the interval* (a,b). *If*

$$\left\|Y'\right\|_{[a,b],r} := \left(\int_a^b\left\|Y'\left(u\right)\right\|^r du\right)^{1/r},\ r \geq 1,$$

then

$$\left\|D\left(\alpha ,Y\right)\right\| \leq \frac{1}{2}\left(b-a\right)\left\|\alpha'\right\|_{[a,b],\infty}\int_a^b\left(b-t\right)\left(t-a\right)\left\|Y'\left(t\right)\right\|dt \tag{2.9}$$
$$\leq \begin{cases} \frac{1}{8}\left(b-a\right)^3\left\|\alpha'\right\|_{[a,b],\infty}\left\|Y'\right\|_{[a,b],1}, \\ \\ \frac{1}{2}\left(b-a\right)^{3+1/q}\left[B\left(q+1,q+1\right)\right]^{1/q}\left\|\alpha'\right\|_{[a,b],\infty}\left\|Y'\right\|_{[a,b],p}, \\ \\ \frac{1}{12}\left(b-a\right)^4\left\|\alpha'\right\|_{[a,b],\infty}\left\|Y'\right\|_{[a,b],\infty}, \end{cases}$$

where $B\left(\cdot,\cdot\right)$ *is Beta function and* $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Observe that, integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\int_a^b (b-t)(t-a)\left\|Y'(t)\right\| dt \\
&= \frac{1}{2}\int_a^b (b-t)(t-a)\, d\left(\int_a^t \left\|Y'(u)\right\| du\right) \\
&= \frac{1}{2}\left[(b-t)(t-a)\int_a^t \left\|Y'(u)\right\| du\Big|_a^b + \int_a^b (2t-a-b)\left(\int_a^t \left\|Y'(u)\right\| du\right) dt\right] \\
&= \int_a^b \left(t-\frac{a+b}{2}\right)\left(\int_a^t \left\|Y'(u)\right\| du\right) dt \\
&= \int_a^b t\left(\int_a^t \left\|Y'(u)\right\| du\right) dt - \frac{a+b}{2}\int_a^b \left(\int_a^t \left\|Y'(u)\right\| du\right) dt \\
&= \frac{1}{b-a} D\left(\ell, \int_a^{\cdot} \left\|Y'(u)\right\| du\right),
\end{aligned}$$

namely

$$D\left(\ell, \int_a^{\cdot} \left\|Y'(u)\right\| du\right) = \frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\left\|Y'(t)\right\| dt. \tag{2.10}$$

By utilising the first inequality (2.1) we deduce the first inequality in (2.9).

By Hölder's integral inequality we have for p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\begin{aligned}
&\int_a^b (b-t)(t-a)\left\|Y'(t)\right\| dt \\
&\leq \begin{cases}
\sup_{t\in[a,b]}\left[(b-t)(t-a)\right]\int_a^b \left\|Y'(t)\right\| dt, \\[2mm]
\left(\int_a^b \left[(b-t)(t-a)\right]^q dt\right)^{1/q}\left(\int_a^b \left\|Y'(t)\right\|^p dt\right)^{1/p}, \\[2mm]
\int_a^b (b-t)(t-a)\, dt \sup_{t\in[a,b]}\left\|Y'(t)\right\|,
\end{cases} \\
&= \begin{cases}
\frac{1}{4}(b-a)^2\int_a^b \left\|Y'(t)\right\| dt, \\[2mm]
(b-a)^{2+1/q}\left[B(q+1,q+1)\right]^{1/q}\left(\int_a^b \left\|Y'(t)\right\|^p dt\right)^{1/p}, \\[2mm]
\frac{1}{6}(b-a)^3\sup_{t\in[a,b]}\left\|Y'(t)\right\|,
\end{cases}
\end{aligned}$$

which proves the last part of (2.9). □

Theorem 5. *Let $\alpha : [a,b] \to \mathbb{C}$ be differentiable and $Y : [a,b] \to E$ be strongly differentiable on the interval (a,b). If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\|D(\alpha, Y)\| \le \left[D\left(\ell, \int_a^{\cdot} |\alpha'(u)|^p du \right) \right]^{1/p} \left[D\left(\ell, \int_a^{\cdot} \|Y'(u)\|^q du \right) \right]^{1/q} \tag{2.11}$$
$$= \frac{1}{2}(b-a) \left[\int_a^b (b-t)(t-a) |\alpha'(t)|^p dt \right]^{1/p} \times \left[\int_a^b (b-t)(t-a) \|Y'(t)\|^q dt \right]^{1/q}$$
$$\le \frac{1}{8}(b-a)^3 \|\alpha'\|_{[a,b],p} \|Y'\|_{[a,b],q}.$$

In particular, we have for $p = q = 2$ that

$$\|D(\alpha, Y)\| \le \left[D\left(\ell, \int_a^{\cdot} |\alpha'(u)|^2 du \right) \right]^{1/2} \left[D\left(\ell, \int_a^{\cdot} \|Y'(u)\|^2 du \right) \right]^{1/2} \tag{2.12}$$
$$= \frac{1}{2}(b-a) \left[\int_a^b (b-t)(t-a) |\alpha'(t)|^2 dt \right]^{1/2} \times \left[\int_a^b (b-t)(t-a) \|Y'(t)\|^2 dt \right]^{1/2}$$
$$\le \frac{1}{8}(b-a)^3 \|\alpha'\|_{[a,b],2} \|Y'\|_{[a,b],2}.$$

Proof. Using Hölder's inequality for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$|\alpha(t) - \alpha(s)| \|Y(t) - Y(s)\|$$
$$= \left\| \int_s^t \alpha'(u) du \right\| \left\| \int_s^t Y'(u) du \right\|$$
$$\le \left| \int_s^t |\alpha'(u)| du \right| \left| \int_s^t \|Y'(u)\| du \right|$$
$$\le |t-s|^{1/q} \left| \int_s^t |\alpha'(u)|^p du \right|^{1/p} |t-s|^{1/p} \left| \int_s^t \|Y'(u)\|^q du \right|^{1/q}$$
$$= |t-s| \left| \int_s^t |\alpha'(u)|^p du \right|^{1/p} \left| \int_s^t \|Y'(u)\|^q du \right|^{1/q}.$$

By the weighted Hölder's inequality for double integral, we also have

$$\begin{aligned}(2.13)\qquad &\int_a^b\int_a^b |\alpha(t)-\alpha(s)|\,\|Y(t)-Y(s)\|\,dtds \\ &\leq \int_a^b\int_a^b |t-s|\left|\int_s^t |\alpha'(u)|^p\,du\right|^{1/p}\left|\int_s^t \|Y'(u)\|^q\,du\right|^{1/q} dtds \\ &\leq \left(\int_a^b\int_a^b |t-s|\left(\left|\int_s^t |\alpha'(u)|^p\,du\right|^{1/p}\right)^p dtds\right)^{1/p} \\ &\times \left(\int_a^b\int_a^b |t-s|\left(\left|\int_s^t \|Y'(u)\|^q\,du\right|^{1/q}\right)^q dtds\right)^{1/q} \\ &= \left(\int_a^b\int_a^b |t-s|\left|\int_s^t |\alpha'(u)|^p\,du\right| dtds\right)^{1/p} \\ &\times \left(\int_a^b\int_a^b |t-s|\left|\int_s^t \|Y'(u)\|^q\,du\right| dtds\right)^{1/q}.\end{aligned}$$

Observe that

$$\begin{aligned}&\int_a^b\int_a^b |t-s|\left|\int_s^t |\alpha'(u)|^p\,du\right| dtds \\ &= \int_a^b\int_a^b (t-s)\left(\int_s^t |\alpha'(u)|^p\,du\right) dtds \\ &= \int_a^b\int_a^b (t-s)\left(\int_a^t |\alpha'(u)|^p\,du - \int_a^s |\alpha'(u)|^p\,du\right) dtds \\ &= 2D\left(\ell, \int_a^{\cdot} |\alpha'(u)|^p\,du\right)\end{aligned}$$

and

$$\int_a^b\int_a^b |t-s|\left|\int_s^t \|Y'(u)\|^q\,du\right| dtds = 2D\left(\ell, \int_a^{\cdot} \|Y'(u)\|^q\,du\right).$$

Therefore, by (2.2)

$$\begin{aligned}\|D(\alpha,Y)\| &\leq \frac{1}{2}\int_a^b\int_a^b |\alpha(t)-\alpha(s)|\,\|Y(t)-Y(s)\|\,dtds \\ &\leq \frac{1}{2}\left[2D\left(\ell, \int_a^{\cdot} |\alpha'(u)|^p\,du\right)\right]^{1/p}\left[2D\left(\ell, \int_a^{\cdot} \|Y'(u)\|^q\,du\right)\right]^{1/q} \\ &= \left[D\left(\ell, \int_a^{\cdot} |\alpha'(u)|^p\,du\right)\right]^{1/p}\left[D\left(\ell, \int_a^{\cdot} \|Y'(u)\|^q\,du\right)\right]^{1/q}.\end{aligned}$$

From (2.10) we have

$$D\left(\ell, \int_a^{\cdot} |\alpha'(u)|^p\,du\right) = \frac{1}{2}(b-a)\int_a^b (b-t)(t-a)|\alpha'(t)|^p\,dt$$

and

$$D\left(\ell, \int_a^{\cdot} \|Y'(u)\|^q\,du\right) = \frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\|Y'(t)\|^q\,dt.$$

Therefore

$$\left[D\left(\ell,\int_a^{\cdot}\left|\alpha'(u)\right|^p du\right)\right]^{1/p}\left[D\left(\ell,\int_a^{\cdot}\left\|Y'(u)\right\|^q du\right)\right]^{1/q}$$
$$=\left[\frac{1}{2}(b-a)\int_a^b(b-t)(t-a)\left|\alpha'(t)\right|^p dt\right]^{1/p}$$
$$\times\left[\frac{1}{2}(b-a)\int_a^b(b-t)(t-a)\left\|Y'(t)\right\|^q dt\right]^{1/q}$$
$$=\frac{1}{2}(b-a)\left[\int_a^b(b-t)(t-a)\left|\alpha'(t)\right|^p dt\right]^{1/p}$$
$$\times\left[\int_a^b(b-t)(t-a)\left\|Y'(t)\right\|^q dt\right]^{1/q}$$

and the first part of the theorem is proved.

Now, observe that

$$\int_a^b(b-t)(t-a)\left|\alpha'(t)\right|^p dt\le\frac{1}{4}(b-a)^2\int_a^b\left|\alpha'(t)\right|^p dt$$

and

$$\int_a^b(b-t)(t-a)\left\|Y'(t)\right\|^q dt\le\frac{1}{4}(b-a)^2\int_a^b\left\|Y'(t)\right\|^q dt,$$

which gives the last part of (2.11). □

Remark 3. *Assume that* $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *and* $\gamma,\ \delta>1$ *with* $\frac{1}{\gamma}+\frac{1}{\delta}=1$. *Then by Hölder's inequality we get*

$$\int_a^b(b-t)(t-a)\left|\alpha'(t)\right|^p dt$$
$$\le(b-a)^{2+1/\beta}\left[B(\beta+1,\beta+1)\right]^{1/\beta}\left(\int_a^b\left|\alpha'(t)\right|^{\alpha p}dt\right)^{1/\alpha}$$

and

$$\int_a^b(b-t)(t-a)\left\|Y'(t)\right\|^q dt$$
$$\le(b-a)^{2+1/\delta}\left[B(\delta+1,\delta+1)\right]^{1/\delta}\left(\int_a^b\left\|Y'(t)\right\|^{\gamma q}dt\right)^{1/\gamma}.$$

Then

$$\left[\int_a^b(b-t)(t-a)\left|\alpha'(t)\right|^p dt\right]^{1/p}$$
$$\le(b-a)^{(2\beta+1)/(\beta p)}\left[B(\beta+1,\beta+1)\right]^{1/(\beta p)}\left(\int_a^b\left|\alpha'(t)\right|^{\alpha p}dt\right)^{1/(\alpha p)}$$

and

$$\left[\int_a^b (b-t)(t-a)\|Y'(t)\|^q dt\right]^{1/q}$$
$$\le (b-a)^{(2\delta+1)/(\delta q)} [B(\delta+1,\delta+1)]^{1/(\delta q)} \left(\int_a^b \|Y'(t)\|^{\gamma q} dt\right)^{1/(\gamma q)}.$$

Therefore

$$\frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)|\alpha'(t)|^p dt\right]^{1/p}$$
$$\times\left[\int_a^b (b-t)(t-a)\|Y'(t)\|^q dt\right]^{1/q}$$
$$\le \frac{1}{2}(b-a)(b-a)^{(2\beta+1)/(\beta p)} [B(\beta+1,\beta+1)]^{1/(\beta p)} \left(\int_a^b |\alpha'(t)|^{\alpha p} dt\right)^{1/(\alpha p)}$$
$$\times (b-a)^{(2\delta+1)/(\delta q)} [B(\delta+1,\delta+1)]^{1/(\delta q)} \left(\int_a^b \|Y'(t)\|^{\gamma q} dt\right)^{1/(\gamma q)}$$
$$= \frac{1}{2}[B(\beta+1,\beta+1)]^{1/(\beta p)} [B(\delta+1,\delta+1)]^{1/(\delta q)}$$
$$\times (b-a)^{1+(2\beta+1)/(\beta p)+(2\delta+1)/(\delta q)}$$
$$\times \left(\int_a^b |\alpha'(t)|^{\alpha p} dt\right)^{1/(\alpha p)} \left(\int_a^b \|Y'(t)\|^{\gamma q} dt\right)^{1/(\gamma q)}$$

and by (2.11) we get

$$\|D(\alpha,Y)\| \le \frac{1}{2}[B(\beta+1,\beta+1)]^{1/(\beta p)} [B(\delta+1,\delta+1)]^{1/(\delta q)} \times (b-a)^{1+(2\beta+1)/(\beta p)+(2\delta+1)/(\delta q)} \|\alpha'\|_{[a,b],\alpha p} \|Y'\|_{[a,b],\gamma q}, \tag{2.14}$$

where $p, q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, $\alpha, \beta > 1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *and* $\gamma, \delta > 1$ *with* $\frac{1}{\gamma}+\frac{1}{\delta}=1$.

3. Inequalities for Operator Monotone Functions

A real valued continuous function h on $[0,\infty)$ is said to be operator monotone if $h(A) \ge h(B)$ holds for any $A \ge B \ge 0$.

We have the following representation of operator monotone functions, see for instance [4, p. 144-145]:

Theorem 6. *A function* $h:[0,\infty)\to\mathbb{R}$ *is operator monotone in* $[0,\infty)$ *if and only if it has the representation*

$$h(t) = h(0) + bt + \int_0^\infty \frac{t\lambda}{t+\lambda} d\mu(\lambda), \tag{3.1}$$

where $b \ge 0$ *and a positive measure* μ *on* $[0,\infty)$ *such that*

$$\int_0^\infty \frac{\lambda}{1+\lambda} d\mu(\lambda) < \infty. \tag{3.2}$$

We have the following representation result:

Lemma 1. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U\geq 0$, then for all selfadjoint operators V we have*

$$Dh(U)(V)=bV+\int_0^\infty \lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda). \tag{3.3}$$

Proof. From (3.1) we get

$$h(t)=h(0)+bt+\int_0^\infty\left(\lambda-\frac{\lambda^2}{t+\lambda}\right)d\mu(\lambda).$$

Assume that $U\geq 0$, then for all selfadjoint operator V we have, by the representation of h and for t in a small open interval around 0, that

$$\begin{aligned}
&h(U+tV)-h(U)\\
&=btV+\int_0^\infty\left(\lambda-\lambda^2(U+tV+\lambda)^{-1}\right)d\mu(\lambda)-\int_0^\infty\left(\lambda-\lambda^2(U+\lambda)^{-1}\right)d\mu(\lambda)\\
&=btV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}-(\lambda+U+tV)^{-1}\right]d\mu(\lambda)\\
&=btV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}(\lambda+U+tV-\lambda-U)(\lambda+U+tV)^{-1}\right]d\mu(\lambda)\\
&=btV+t\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U+tV)^{-1}\right]d\mu(\lambda).
\end{aligned}$$

Dividing by $t\neq 0$, we get

$$\frac{h(U+tV)-h(U)}{t}=bV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U+tV)^{-1}\right]d\mu(\lambda)$$

and by taking the limit over $t\to 0$, we get

$$Dh(U)(V)=bV+\int_0^\infty\lambda^2\left[(\lambda+U)^{-1}V(\lambda+U)^{-1}\right]d\mu(\lambda)$$

for all selfadjoint operator V we have (3.3). □

Theorem 7. *Let $h:[0,\infty)\to\mathbb{R}$ be operator monotone in $[0,\infty)$. Assume that $U\geq u>0$, then for all selfadjoint operators V we have*

$$\|Dh(U)(V)\|\leq h'(u)\|V\|. \tag{3.4}$$

Proof. From (3.3) we get

$$\begin{aligned}
\|Dh(U)(V)-bV\|&\leq\int_0^\infty\lambda^2\left\|(\lambda+U)^{-1}V(\lambda+U)^{-1}\right\|d\mu(\lambda)\\
&\leq\|V\|\int_0^\infty\lambda^2\left\|(\lambda+U)^{-1}\right\|^2d\mu(\lambda).
\end{aligned}\tag{3.5}$$

Observe that $\lambda+U\geq\lambda+u>0$ for $\lambda\in[0,\infty)$. Then $0<(\lambda+U)^{-1}\leq(\lambda+u)^{-1}$, which implies that $\left\|(\lambda+U)^{-1}\right\|\leq(\lambda+u)^{-1}$, namely $\left\|(\lambda+U)^{-1}\right\|^2\leq(\lambda+u)^{-2}$ for $\lambda\in[0,\infty)$.

Therefore by (3.5) we get

$$\|Dh(U)(V)-bV\|\leq\|V\|\int_0^\infty\lambda^2(\lambda+u)^{-2}d\mu(\lambda). \tag{3.6}$$

If we take the derivative over t in (3.1) then we have

$$h'(t) = b + \int_0^\infty \frac{\lambda(t+\lambda) - \lambda t}{(t+\lambda)^2} d\mu(\lambda) = b + \int_0^\infty \frac{\lambda^2}{(t+\lambda)^2} d\mu(\lambda) \tag{3.7}$$

for $t > 0$.

From (3.7) we get

$$\int_0^\infty \lambda^2 (\lambda + u)^{-2} d\mu(\lambda) = h'(u) - b,$$

and by (3.6) we derive

$$\|Dh(U)(V) - bV\| \le \|V\| h'(u) - b\|V\|.$$

Finally, by the triangle inequality and by the fact that $b \ge 0$, we obtain that

$$\|Dh(U)(V)\| - b\|V\| \le \|Dh(U)(V) - bV\|,$$

which proves the desired result (3.4). □

For a continuous function h on $(0,\infty)$ and A, $B > 0$ we consider the auxiliary function $h_{A,B} : [0,1] \to \mathbb{R}$ defined by

$$h_{A,B}(t) := h((1-t)A + tB), \ t \in [0,1].$$

We have the following representations of the derivatives:

Lemma 2. *Assume that the operator function generated by h is Fréchet differentiable in each $A \ge 0$, then for $B \ge 0$ we have that $h_{A,B}$ is differentiable on $[0,1]$ and*

$$h'_{A,B}(t) = D(h)((1-t)A + tB)(B - A) \tag{3.8}$$

for $t \in [0,1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t \in (0,1)$. Let h be in a small interval around 0 such that $t + h \in (0,1)$. Then for $h \neq 0$,

$$\begin{aligned} &\frac{h_{A,B}(t+h) - h(t)}{h} \\ &= \frac{h((1-(t+h))A + (t+h)B) - h((1-t)A + tB)}{h} \\ &= \frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h} \end{aligned}$$

and by taking the limit over $h \to 0$, we get

$$\begin{aligned} h'_{A,B}(t) &= \lim_{h\to 0} \frac{h_{A,B}(t+h) - h(t)}{h} \\ &= \lim_{h\to 0} \left[\frac{h((1-t)A + tB + h(B-A)) - h((1-t)A + tB)}{h}\right] \\ &= D(h)((1-t)A + tB)(B-A), \end{aligned}$$

which proves (3.8). □

Corollary 3. *Let $h : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$. Then for all $A \geq a > 0$, $B \geq b > 0$ we have*

$$\begin{aligned} \left\| h'_{A,B}(t) \right\| &= \left\| D(h)((1-t)A + tB)(B-A) \right\| \\ &\leq h'((1-t)a + tb) \left\| B - A \right\| \end{aligned} \tag{3.9}$$

for all $t \in [0,1]$.

The proof follows by Theorem 7 and Lemma 2.

One can observe that the inequality (3.9) remains valid for operator monotone functions on $(0,\infty)$. This follows by considering the function $h_\varepsilon(t) := h(t+\varepsilon)$ for $\varepsilon > 0$, which is operator monotone on $[0,\infty)$ and then by letting $\varepsilon \to 0+$ and using the continuity of h and h'.

Theorem 8. *Let $f : [0,\infty) \to \mathbb{R}$ be operator monotone in $[0,\infty)$ and $\alpha : [0,1] \to \mathbb{C}$ a continuous function on $[0,1]$. Then for all $A \geq a > 0$, $B \geq b > 0$ we have*

$$\begin{aligned} &\left\| \int_0^1 \alpha(t) f((1-t)A + tB)\,dt - \int_0^1 \alpha(t)\,dt \int_0^1 f((1-t)A + tB)\,dt \right\| \\ &\leq \frac{1}{2} \left\| B - A \right\| \left\| \alpha' \right\|_{[0,1],\infty} \int_0^1 (1-t)\,t h'((1-t)a + tb)\,dt \\ &\leq \frac{1}{2} \left\| B - A \right\| \left\| \alpha' \right\|_{[0,1],\infty} \\ &\times \begin{cases} \frac{1}{4} \int_0^1 h'((1-t)a + tb)\,dt, \\ \\ \left[B(q+1,q+1) \right]^{1/q} \left(\int_0^1 \left[h'((1-t)a + tb) \right]^p dt \right)^{1/p}, \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \frac{1}{6} \sup_{t \in [0,1]} h'((1-t)a + tb). \end{cases} \end{aligned} \tag{3.10}$$

Proof. We use the inequality (2.9) for α and $f_{A,B}$ on $[0,1]$ to get

$$\begin{aligned} &\left\| \int_0^1 \alpha(t) f((1-t)A + tB)\,dt - \int_0^1 \alpha(t)\,dt \int_0^1 f((1-t)A + tB)\,dt \right\| \\ &\leq \frac{1}{2} \left\| \alpha' \right\|_{[0,1],\infty} \int_0^1 (1-t)\,t \left\| f'_{A,B}(t) \right\| dt \\ &\leq \begin{cases} \frac{1}{8} \left\| \alpha' \right\|_{[0,1],\infty} \left\| f'_{A,B} \right\|_{[a,b],1}, \\ \\ \frac{1}{2} \left[B(q+1,q+1) \right]^{1/q} \left\| \alpha' \right\|_{[0,1],\infty} \left\| f'_{A,B} \right\|_{[a,b],p}, \\ \\ \frac{1}{12} \left\| \alpha' \right\|_{[0,1],\infty} \left\| f'_{A,B} \right\|_{[a,b],\infty}. \end{cases} \end{aligned} \tag{3.11}$$

Since, by (3.9),

$$\left\| f'_{A,B} \right\|_{[0,1],1} \leq \left\| B - A \right\| \int_0^1 h'((1-t)a + tb)\,dt,$$

hence by (3.11) we get the desired inequality in (3.10). $\square$

Theorem 9. *With the assumptions of Theorem 8 we have*

$$\begin{aligned}
(3.12)\quad &\left\|\int_0^1 \alpha(t) f((1-t)A+tB)\,dt - \int_0^1 \alpha(t)\,dt \int_0^1 f((1-t)A+tB)\,dt\right\| \\
&\le \frac{1}{2}\left[\int_0^1 (1-t)t\,|\alpha'(t)|^p\,dt\right]^{1/p}\left[\int_0^1 (1-t)t\,[h'((1-t)a+tb)]^q\,dt\right]^{1/q} \\
&\le \frac{1}{8}\|\alpha'\|_{[0,1],p}\left[\int_0^1 [h'((1-t)a+tb)]^q\,dt\right]^{1/q}
\end{aligned}$$

for all $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

In particular, we have

$$\begin{aligned}
(3.13)\quad &\left\|\int_0^1 \alpha(t) f((1-t)A+tB)\,dt - \int_0^1 \alpha(t)\,dt \int_0^1 f((1-t)A+tB)\,dt\right\| \\
&\le \frac{1}{2}\left[\int_0^1 (1-t)t\,|\alpha'(t)|^2\,dt\right]^{1/2}\left[\int_0^1 (1-t)t\,[h'((1-t)a+tb)]^2\,dt\right]^{1/2} \\
&\le \frac{1}{8}\|\alpha'\|_{[0,1],2}\left[\int_0^1 [h'((1-t)a+tb)]^2\,dt\right]^{1/2}.
\end{aligned}$$

The proof follows in a similar way from (2.11) and (3.9).

If we take in (3.10) $f(t)=t^r$, $r\in(0,1)$ that is operator monotone on $[0,\infty)$, then by (3.10) we derive

$$\begin{aligned}
(3.14)\quad &\left\|\int_0^1 \alpha(t)((1-t)A+tB)^r\,dt - \int_0^1 \alpha(t)\,dt \int_0^1 ((1-t)A+tB)^r\,dt\right\| \\
&\le \frac{1}{2}r\|B-A\|\,\|\alpha'\|_{[0,1],\infty}\int_0^1 (1-t)t((1-t)a+tb)^{r-1}\,dt \\
&\le \frac{1}{2}r\|B-A\|\,\|\alpha'\|_{[0,1],\infty} \\
&\times\begin{cases}
\frac{1}{4}\times\begin{cases}\frac{b^r-a^r}{r(b-a)} \text{ if } b\neq a,\\[1ex] a^{r-1} \text{ if } b=a,\end{cases}, \\[3ex]
[B(q+1,q+1)]^{1/q}\times\begin{cases}\left(\frac{b^{p(r-1)+1}-a^{p(r-1)+1}}{(p(r-1)+1)(b-a)}\right)^{1/p} \text{ if } b\neq a,\\ p(r-1)+1\neq 0,\\[1ex] a^{r-1} \text{ if } b=a\end{cases}, \\
p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1, \\[2ex]
\frac{1}{6\min\{a,b\}},
\end{cases}
\end{aligned}$$

for all $A\ge a>0$, $B\ge b>0$.

If we take in (3.10) $f(t) = \ln t$, that is operator monotone on $(0, \infty)$, then by (3.10) we derive

$$\left\| \int_0^1 \alpha(t) \ln((1-t)A + tB)\, dt - \int_0^1 \alpha(t)\, dt \int_0^1 \ln((1-t)A + tB)\, dt \right\| \tag{3.15}$$
$$\le \frac{1}{2} \|B - A\| \|\alpha'\|_{[0,1],\infty} \int_0^1 t(1-t)((1-t)a + tb)^{-1}\, dt$$
$$\le \frac{1}{2} \|B - A\| \|\alpha'\|_{[0,1],\infty}$$
$$\times \begin{cases} \frac{1}{4} \times \begin{cases} \frac{\ln b - \ln a}{b-a} \text{ if } b \neq a, \\ \frac{1}{a} \text{ if } b = a, \end{cases}, \\ [B(q+1, q+1)]^{1/q} \times \begin{cases} \left(\frac{b^{1-p} - a^{1-p}}{(1-p)(b-a)}\right)^{1/p} \text{ if } b \neq a, \\ \frac{1}{a} \text{ if } b = a, \end{cases}, \\ p,\ q > 1,\ \frac{1}{p} + \frac{1}{q} = 1, \\ \frac{1}{6 \min\{a,b\}}, \end{cases}$$

for all $A \ge a > 0$, $B \ge b > 0$.

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