SEVERAL UPPER BOUNDS FOR THE ČEBYŠEV FUNCTIONAL OF MONOTONIC AND ABSOLUTELY CONTINUOUS FUNCTIONS WITH APPLICATIONS

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Abstract. For two Lebesgue integrable functions $h,\ k:[a,b]\to\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(h,k):=(b-a)\int_a^b h(t)\,k(t)\,dt-\int_a^b h(t)\,dt\int_a^b k(t)\,dt.$$

In this paper we show among others that, if $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and absolutely continuous while $g:[a,b]\to\mathbb{C}$ is absolutely continuous with $\int_a^b |g'(u)|\,du>0$, then

$$\begin{aligned}|D(f,g)| &\le D\left(f,\int_a^{\cdot}|g'(u)|\,du\right)\\ &\le \frac{1}{2}(b-a)\frac{\|f'\|_{[a,b],\infty}}{\|g'\|_{[a,b],1}}\int_a^b (b-t)\,|g'(t)|\,dt\int_a^b (t-a)\,|g'(t)|\,dt\\ &\le \frac{1}{8}(b-a)^3\,\|f'\|_{[a,b],\infty}\,\|g'\|_{[a,b],1}\,,\end{aligned}$$

provided $\|f'\|_{[a,b],\infty}<\infty$. Applications for trapezoid inequality related to convex functions with examples for norms and semi-inner products are also provided.

1. Introduction

For two Lebesgue integrable functions $h,\ k:[a,b]\to\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(h,k):=(b-a)\int_a^b h(t)\,k(t)\,dt-\int_a^b h(t)\,dt\int_a^b k(t)\,dt.$$

In 1934, G. Grüss [23] showed that

$$|D(h,k)|\le\frac{1}{4}(b-a)^2(M-m)(N-n)\,, \tag{1.1}$$

provided $m,\ M,\ n,\ N$ are real numbers with the property that

$$-\infty<m\le h\le M<\infty,\quad -\infty<n\le k\le N<\infty\quad\text{a.e. on}\quad [a,b]\,. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D10, 46B05.

Key words and phrases. Čebyšev functional, Grüss inequality, Ostrowski inequality, Trapezoid inequality, Norm inequalities, Semi-inner products.

RGMIA Res. Rep. Coll. 24 (2021), Art. 27, 13 pp. Received 11/03/21

Another lesser known inequality for $D(h,k)$ was derived in 1882 by Čebyšev [10] under the assumption that h', k' exist and are continuous on $[a,b]$, and is given by

$$|D(h,k)| \le \frac{1}{12} \|h'\|_\infty \|k'\|_\infty (b-a)^4, \tag{1.3}$$

where $\|h'\|_\infty := \sup_{t\in[a,b]} |h'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $h,\ k : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $h',\ k' \in L_\infty[a,b]$.

In 1970, A. M. Ostrowski [30] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(h,k)| \le \frac{1}{8}(b-a)^3 (M-m) \|k'\|_\infty, \tag{1.4}$$

provided h is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $k : [a,b] \to \mathbb{R}$ is absolutely continuous and $k' \in L_\infty[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [26] (see also [27, p. 210]) obtained the following result as well:

$$|D(h,k)| \le \frac{1}{\pi^2} \|h'\|_2 \|k'\|_2 (b-a)^3, \tag{1.5}$$

provided $h,\ k$ are absolutely continuous and $h',\ k' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [7], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(h,k)| \le (b-a) \times \begin{cases} \inf_{\gamma\in\mathbb{R}} \|h-\gamma\|_\infty \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt, \\ \\ \inf_{\gamma\in\mathbb{R}} \|h-\gamma\|_q \left(\int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right|^p dt \right)^{\frac{1}{p}}, \\ \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(h,k)| \le (b-a) \|h\|_\infty \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \le h \le M$ for a.e. $x \in [a,b]$, then $\left\| h - \frac{m+M}{2} \right\|_\infty \le \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [9]

$$|D(h,k)| \le \frac{1}{2}(b-a)(M-m) \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [8].

The following result holds [16].

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(f,g)| \le \frac{1}{2}(b-a) \bigvee_a^b (f) \int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| dt$$

where $\bigvee_a^b(f)$ *denotes the total variation of* f *on the interval* $[a,b]$*. The constant* $\frac{1}{2}$ *is best possible in (1.9).*

For more recent upper bounds related to the Čebyšev functional see [7], [8] and [14]-[16].

In [13] we obtained the following refinement of Ostrowski's inequality (1.4)

$$\text{(1.9)} \quad |D(h,k)| \le \frac{1}{2}(b-a)^3 \frac{\left(\frac{1}{b-a}\int_a^b h(t)\,dt - m\right)\left(M - \frac{1}{b-a}\int_a^b h(t)\,dt\right)}{M-m}\|k'\|_\infty \le \frac{1}{8}(b-a)^3(M-m)\|k'\|_\infty,$$

provided $m \le h \le M$ a.e. on $[a,b]$ and k is absolutely continuous on $[a,b]$.

In this paper we show among others that, if $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and absolutely continuous while $g:[a,b]\to\mathbb{C}$ is absolutely continuous with $\int_a^b |g'(u)|\,du > 0$, then

$$|D(f,g)| \le D\left(f, \int_a^\cdot |g'(u)|\,du\right) \le \frac{1}{2}(b-a)\frac{\|f'\|_{[a,b],\infty}}{\|g'\|_{[a,b],1}}\int_a^b (b-t)|g'(t)|\,dt\int_a^b (t-a)|g'(t)|\,dt \le \frac{1}{8}(b-a)^3\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],1},$$

provided $\|f'\|_{[a,b],\infty} < \infty$. Applications for trapezoid inequality related to convex functions with examples for norms and semi-inner products are also provided.

2. Grüss Type Inequalities

We define the Lebesgue r-norms as

$$\|h\|_{[a,b],r} := \int_a^b |h(u)|^r\,du,\ r\ge 1$$

and

$$\|h\|_{[a,b],\infty} := \operatorname*{esssup}_{u\in[a,b]} |h(u)|.$$

We have the following result:

Theorem 2. *Assume that* $f:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing and* $g:[a,b]\to\mathbb{C}$ *is absolutely continuous. Then*

$$\text{(2.1)} \qquad |D(f,g)| \le D\left(f, \int_a^\cdot |g'(u)|\,du\right)$$

$$\leq \frac{1}{2}(b-a)\|g'\|_{[a,b],1}\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds \right| dt$$
$$\leq \frac{1}{2}(b-a)^2 \|g'\|_{[a,b],1}$$
$$\times \left[\frac{1}{b-a}\int_a^b f^2(t)\,dt - \left(\frac{1}{b-a}\int_a^b f(s)\,ds \right)^2 \right]^{1/2}$$
$$\leq \frac{1}{4}(b-a)^2 [f(b) - f(a)] \|g'\|_{[a,b],1}$$

and

$$|D(f,g)| \leq D\left(f, \int_a^{\cdot} |g'(u)|\,du \right) \tag{2.2}$$
$$\leq \frac{1}{2}[f(b) - f(a)]$$
$$\times \int_a^b \left| \int_a^t (u-a)|g'(u)|\,du - \int_t^b (b-u)|g'(u)|\,du \right| dt.$$

Proof. Observe that for $f,\ g : [a,b] \to \mathbb{C}$ we have Korkine's identity

$$D(f,g) = \frac{1}{2}\int_a^b \int_a^b [f(t) - f(s)][g(t) - g(s)]\,dtds.$$

For Korkine's classical identity for real-valued functions, see [27, p. 242].

If we take the modulus and use the integral's properties, we get

$$|D(f,g)| \leq \frac{1}{2}\int_a^b \int_a^b |[f(t) - f(s)][g(t) - g(s)]|\,dtds \tag{2.3}$$
$$= \frac{1}{2}\int_a^b \int_a^b |f(t) - f(s)||g(t) - g(s)|\,dtds.$$

Since $g : [a,b] \to \mathbb{C}$ is absolutely continuous, then

$$|g(t) - g(s)| = \left| \int_s^t g'(u)\,du \right| \leq \left| \int_s^t |g'(u)|\,du \right|$$

for all $t,\ s \in [a,b]$.

Therefore

$$\frac{1}{2}\int_a^b \int_a^b |f(t) - f(s)||g(t) - g(s)|\,dtds \tag{2.4}$$
$$\leq \frac{1}{2}\int_a^b \int_a^b |f(t) - f(s)| \left| \int_s^t |g'(u)|\,du \right| dtds$$
$$= \frac{1}{2}\int_a^b \int_a^b \left| (f(t) - f(s)) \int_s^t |g'(u)|\,du \right| dtds.$$

Since f is monotone nondecreasing, then

$$\left| (f(t) - f(s)) \int_s^t |g'(u)|\,du \right| = (f(t) - f(s)) \int_s^t |g'(u)|\,du \geq 0$$

for all t, $s \in [a,b]$, which implies that

$$\begin{aligned}
&\frac{1}{2}\int_a^b\int_a^b\left|(f(t)-f(s))\int_s^t|g'(u)|\,du\right|dtds\\
&=\frac{1}{2}\int_a^b\int_a^b(f(t)-f(s))\int_s^t|g'(u)|\,dudtds\\
&=\frac{1}{2}\int_a^b\int_a^b(f(t)-f(s))\left(\int_a^t|g'(u)|\,du-\int_a^s|g'(u)|\,du\right)dtds\\
&=D\left(f,\int_a^{\cdot}|g'(u)|\,du\right).
\end{aligned}$$

By utilising (2.3) and (2.4) we derive

$$|D(f,g)|\leq D\left(f,\int_a^{\cdot}|g'(u)|\,du\right),$$

and the first inequality is proved.

Now, if we write the inequality (1.8) for $k=f$ and $h=\int_a^{\cdot}|g'(u)|\,du$, which satisfy the bounds

$$0\leq\int_a^t|g'(u)|\,du\leq\int_a^b|g'(u)|\,du \text{ for all } t\in[a,b],$$

then we get

$$D\left(f,\int_a^{\cdot}|g'(u)|\,du\right)\leq\frac{1}{2}(b-a)\int_a^b|g'(u)|\,du\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt.$$

By Schwarz and Grüss' inequalities we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt.\\
&\leq\left(\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2dt\right)^{1/2}\\
&=\left[\frac{1}{b-a}\int_a^b f^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)^2\right]^{1/2}\leq\frac{1}{2}[f(b)-f(a)],
\end{aligned}$$

which proves the second part of (2.1).

If we write the inequality (1.8) for $k=\int_a^{\cdot}|g'(u)|\,du$ and $h=f$, then we get

$$\begin{aligned}
&\left|D\left(\int_a^{\cdot}|g'(u)|\,du,f\right)\right|\\
&\leq\frac{1}{2}(b-a)[f(b)-f(a)]\\
&\times\int_a^b\left|\int_a^t|g'(u)|\,du-\frac{1}{b-a}\int_a^b\left(\int_a^s|g'(u)|\,du\right)ds\right|dt.
\end{aligned}\tag{2.5}$$

Observe that

$$\begin{aligned}
&\int_a^b \left| \int_a^t |g'(u)|\,du - \frac{1}{b-a}\int_a^b \left(\int_a^s |g'(u)|\,du\right) ds \right| dt \\
&= \int_a^b \left| \int_a^t |g'(u)|\,du - \frac{1}{b-a}\left(\left(\int_a^b |g'(u)|\,du\right) b - \int_a^b |g'(s)|\, s ds\right) \right| dt \\
&= \int_a^b \left| \int_a^t |g'(u)|\,du - \frac{1}{b-a}\left(\int_a^b (b-u)\,|g'(u)|\,du\right) \right| dt \\
&= \frac{1}{b-a}\int_a^b \left| (b-a)\int_a^t |g'(u)|\,du - \int_a^b (b-u)\,|g'(u)|\,du \right| dt \\
&= \frac{1}{b-a}\int_a^b \left| \int_a^t (u-a)\,|g'(u)|\,du - \int_t^b (b-u)\,|g'(u)|\,du \right| dt
\end{aligned}$$

and by (2.5) we get (2.2). □

Theorem 3. *Assume that $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and absolutely continuous while $g:[a,b]\to\mathbb{C}$ is absolutely continuous with $\int_a^b |g'(u)|\,du > 0$. Then*

$$\begin{aligned}
(2.6)\qquad |D(f,g)| &\le D\left(f, \int_a^{\cdot} |g'(u)|\,du\right) \\
&\le \frac{1}{2}(b-a)\frac{\|f'\|_{[a,b],\infty}}{\|g'\|_{[a,b],1}} \int_a^b (b-t)\,|g'(t)|\,dt \int_a^b (t-a)\,|g'(t)|\,dt \\
&\le \frac{1}{8}(b-a)^3 \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],1},
\end{aligned}$$

provided $\|f'\|_{[a,b],\infty} < \infty$.

Proof. If we use the inequality (1.9) for $h(t) = \int_a^t |g'(u)|\,du$ and $k = f$, then

$$\begin{aligned}
(2.7)\qquad &\left| D\left(\int_a^{\cdot} |g'(u)|\,du, f\right) \right| \\
&\le \frac{1}{2}(b-a)^3 \|f'\|_\infty \\
&\times \frac{\left(\frac{1}{b-a}\int_a^b \left(\int_a^t |g'(u)|\,du\right) dt\right)\left(\int_a^b |g'(u)|\,du - \frac{1}{b-a}\int_a^b \left(\int_a^t |g'(u)|\,du\right) dt\right)}{\int_a^b |g'(u)|\,du} \\
&\le \frac{1}{8}(b-a)^3 \|f'\|_\infty \int_a^b |g'(u)|\,du.
\end{aligned}$$

Observe that

$$\begin{aligned}
&\int_a^b \left(\int_a^t |g'(u)|\,du\right) dt \\
&= \left(\int_a^t |g'(u)|\,du\right) t \Big|_a^b - \int_a^b t\,|g'(t)|\,dt \\
&= \left(\int_a^b |g'(u)|\,du\right) b - \int_a^b t\,|g'(t)|\,dt = \int_a^b (b-t)\,|g'(t)|\,dt
\end{aligned}$$

and

$$\begin{aligned}
&(b-a)\int_a^b |g'(u)|\,du-\int_a^b\left(\int_a^t |g'(u)|\,du\right)dt\\
&=(b-a)\int_a^b |g'(u)|\,du-\int_a^b (b-t)\,|g'(t)|\,dt=\int_a^b (t-a)\,|g'(t)|\,dt,
\end{aligned}$$

which gives

$$\begin{aligned}
&\frac{\left(\frac{1}{b-a}\int_a^b\left(\int_a^t |g'(u)|\,du\right)dt\right)\left(\int_a^b |g'(u)|\,du-\frac{1}{b-a}\int_a^b\left(\int_a^t |g'(u)|\,du\right)dt\right)}{\int_a^b |g'(u)|\,du}\\
&=\frac{\frac{1}{b-a}\int_a^b (b-t)\,|g'(t)|\,dt\frac{1}{b-a}\int_a^b (t-a)\,|g'(t)|\,dt}{\int_a^b |g'(u)|\,du}\\
&=\frac{\int_a^b (b-t)\,|g'(t)|\,dt\int_a^b (t-a)\,|g'(t)|\,dt}{(b-a)^2\int_a^b |g'(u)|\,du}
\end{aligned}$$

and by (2.7) we derive (2.6). □

Remark 1. *If we use Hölder's inequality for* $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$, *then we have*

$$\begin{aligned}
\int_a^b (b-t)\,|g'(t)|\,dt &\le \left(\int_a^b (b-t)^p\,dt\right)^{1/p}\left(\int_a^b |g'(t)|^q\,dt\right)^{1/q}\\
&=\left[\frac{(b-a)^{p+1}}{p+1}\right]^{1/p}\left(\int_a^b |g'(t)|^q\,dt\right)^{1/q}\\
&=\frac{(b-a)^{1+1/p}}{(p+1)^{1/p}}\,\|g'\|_{[a,b],q}
\end{aligned}$$

and

$$\int_a^b (t-a)\,|g'(t)|\,dt\le\frac{(b-a)^{1+1/q}}{(q+1)^{1/q}}\,\|g'\|_{[a,b],p}\,,$$

which implies that

$$\int_a^b (b-t)\,|g'(t)|\,dt\int_a^b (t-a)\,|g'(t)|\,dt\le\frac{(b-a)^3}{(p+1)^{1/p}(q+1)^{1/q}}\,\|g'\|_{[a,b],q}\,\|g'\|_{[a,b],p}$$

and from (2.6) we derive

$$\begin{aligned}
|D(f,g)| &\le D\left(f,\int_a^{\cdot}|g'(u)|\,du\right)\\
&\le \frac{1}{2}\frac{(b-a)^4}{(p+1)^{1/p}(q+1)^{1/q}}\frac{\|f'\|_{[a,b],\infty}}{\|g'\|_{[a,b],1}}\,\|g'\|_{[a,b],q}\,\|g'\|_{[a,b],p}
\end{aligned}\tag{2.8}$$

for $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1.$

For $p=q=2$ *we get*

$$|D(f,g)|\le D\left(f,\int_a^{\cdot}|g'(u)|\,du\right)\le\frac{1}{6}(b-a)^4\frac{\|f'\|_{[a,b],\infty}}{\|g'\|_{[a,b],1}}\,\|g'\|_{[a,b],2}^2\,.\tag{2.9}$$

3. Applications for Trapezoid Inequality

We use the following trapezoid identity in terms of the first derivative

$$\frac{h(a)+h(b)}{2}(b-a)-\int_a^b h(t)\,dt=\int_a^b\left(t-\frac{a+b}{2}\right)h'(t)\,dt.$$

This shows that

$$\frac{h(a)+h(b)}{2}(b-a)-\int_a^b h(t)\,dt=\frac{1}{b-a}D\left(\cdot-\frac{a+b}{2},h'\right).$$

If h is convex, then h' is monotone increasing almost everywhere. By applying inequality (2.1) for $f(t)=h'(t)$ and $g(t)=t-\frac{a+b}{2}$, then we get

$$\begin{aligned}0&\le\frac{h(a)+h(b)}{2}(b-a)-\int_a^b h(t)\,dt\\&\le\frac{1}{2}(b-a)\int_a^b\left|h'(t)-\frac{h(b)-h(a)}{b-a}\right|dt\\&\le\frac{1}{2}(b-a)^2\left[\frac{1}{b-a}\int_a^b[h'(t)]^2\,dt-\left(\frac{h(b)-h(a)}{b-a}\right)^2\right]^{1/2}.\end{aligned}\tag{3.1}$$

Observe that

$$\begin{aligned}&\int_a^b\left|\int_a^t(u-a)\left|g'(u)\right|du-\int_t^b(b-u)\left|g'(u)\right|du\right|dt\\&=\int_a^b\left|\int_a^t(u-a)\,du-\int_t^b(b-u)\,du\right|dt=\frac{1}{2}\int_a^b\left|(t-a)^2-(b-t)^2\right|dt\\&=(b-a)\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{1}{4}(b-a)^3.\end{aligned}$$

Then by (2.2) for we get the known inequality

$$0\le\frac{h(a)+h(b)}{2}(b-a)-\int_a^b h(t)\,dt\le\frac{1}{8}\left[h'_-(b)-h'_+(a)\right](b-a)^2.\tag{3.2}$$

If we take $f(t)=t-\frac{a+b}{2}$ and $g(t)=h'(t)$ in (2.6), then we get

$$\begin{aligned}&D\left(\cdot-\frac{a+b}{2},h'\right)\\&\le\frac{1}{2}(b-a)\frac{1}{\int_a^b h''(t)\,dt}\int_a^b(b-t)h''(t)\,dt\int_a^b(t-a)h''(t)\,dt\\&\le\frac{1}{8}(b-a)^3\int_a^b h''(t)\,dt.\end{aligned}\tag{3.3}$$

Since

$$\int_a^b(b-t)h''(t)\,dt=h(b)-h(a)-(b-a)h'_+(a),$$

$$\int_a^b(t-a)h''(t)\,dt=(b-a)h'_-(b)-h(b)+h(a)$$

and

$$\int_a^b h''(t)\,dt = h'_-(b) - h'_+(a),$$

then by (3.3) we get

$$\begin{aligned}
&\frac{1}{(b-a)}D\left(\cdot - \frac{a+b}{2}, h'\right)\\
&\leq \frac{\left[h(b) - h(a) - (b-a)h'_+(a)\right]\left[(b-a)h'_-(b) - h(b) + h(a)\right]}{h'_-(b) - h'_+(a)}\\
&\leq \frac{1}{8}(b-a)^2\left[h'_-(b) - h'_+(a)\right].
\end{aligned}$$

Therefore, if h is convex and $h'_-(b) - h'_+(a) > 0$, then we have the following inequality of interest:

$$\begin{aligned}
0 &\leq \frac{h(a)+h(b)}{2}(b-a) - \int_a^b h(t)\,dt\\
&\leq \frac{\left[h(b) - h(a) - (b-a)h'_+(a)\right]\left[(b-a)h'_-(b) - h(b) + h(a)\right]}{h'_-(b) - h'_+(a)}\\
&\leq \frac{1}{8}(b-a)^2\left[h'_-(b) - h'_+(a)\right].
\end{aligned} \tag{3.4}$$

4. Applications for Norms and Semi-Inner Products

Let X be a real linear space, $a, b \in X$, $a \neq b$ and let $[a,b] := \{(1-\lambda)a + \lambda b,\ \lambda \in [0,1]\}$ be the *segment* generated by a and b. We consider the function $f : [a,b] \to \mathbb{R}$ and the attached function $g(a,b) : [0,1] \to \mathbb{R}$, $g(a,b)(t) := f[(1-t)a + tb]$, $t \in [0,1]$.

It is well known that f is convex on $[a,b]$ iff $g(a,b)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_{\pm}(a,b)(s) = \nabla_{\pm}f[(1-s)a + sb](b-a)$, $s \in [0,1)$,
(ii) $g'_+(a,b)(0) = \nabla_+ f(a)(b-a)$,
(iii) $g'_-(a,b)(1) = \nabla_- f(b)(b-a)$,

where $\nabla_{\pm}f(x)(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\begin{aligned}
\nabla_+ f(x)(y) &:= \lim_{h\to 0+}\frac{f(x+hy) - f(x)}{h},\\
\nabla_- f(x)(y) &:= \lim_{k\to 0-}\frac{f(x+ky) - f(x)}{k}, \quad x,\ y \in X.
\end{aligned}$$

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[a,b] \subset X$:

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f[(1-t)a + tb]\,dt \leq \frac{f(a)+f(b)}{2}, \tag{HH}$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $g(a,b) : [0,1] \to \mathbb{R}$

$$g(a,b)\left(\frac{1}{2}\right) \leq \int_0^1 g(a,b)(t)\,dt \leq \frac{g(a,b)(0) + g(a,b)(1)}{2}.$$

For other related results see the monograph on line [6].

Now, assume that $(X, \|\cdot\|)$ is a normed linear space. The function $f_0(s) = \frac{1}{2}\|x\|^2$, $x \in X$ is convex and thus the following limits exist

(iv) $\langle x, y\rangle_s := \nabla_+ f_0(y)(x) = \lim_{t\to 0+} \frac{\|y+tx\|^2 - \|y\|^2}{2t}$;
(v) $\langle x, y\rangle_i := \nabla_- f_0(y)(x) = \lim_{s\to 0-} \frac{\|y+sx\|^2 - \|y\|^2}{2s}$;

for any x, $y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [21]), assuming that p, $q \in \{s, i\}$ and $p \neq q$:

(a) $\langle x, x\rangle_p = \|x\|^2$ for all $x \in X$;
(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta \langle x, y\rangle_p$ if α, $\beta \geq 0$ and x, $y \in X$;
(aaa) $\left|\langle x, y\rangle_p\right| \leq \|x\| \|y\|$ for all x, $y \in X$;
(av) $\langle \alpha x + y, x\rangle_p = \alpha \langle x, x\rangle_p + \langle y, x\rangle_p$ if x, $y \in X$ and $\alpha \in \mathbb{R}$;
(v) $\langle -x, y\rangle_p = -\langle x, y\rangle_q$ for all x, $y \in X$;
(va) $\langle x + y, z\rangle_p \leq \|x\| \|z\| + \langle y, z\rangle_p$ for all x, y, $z \in X$;
(vaa) The mapping $\langle \cdot, \cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);
(vaaa) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X \setminus \{0\}$ if and only if $\langle y, x_0\rangle_s = \langle y, x_0\rangle_i$ for all $y \in X$; in general $\langle y, x\rangle_i \leq \langle y, x\rangle_s$ for all x, $y \in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot\rangle$, then $\langle y, x\rangle_i = \langle y, x\rangle = \langle y, x\rangle_s$ for all x, $y \in X$.

Applying inequality (HH) for the convex function $f_0(x) = \frac{1}{2}\|x\|^2$, one may deduce the inequality

$$\left\|\frac{x+y}{2}\right\|^2 \leq \int_0^1 \|(1-t)x + ty\|^2 dt \leq \frac{\|x\|^2 + \|y\|^2}{2} \tag{4.1}$$

for any $x, y \in X$. The same (HH) inequality applied for $f_1(x) = \|x\|$, will give the following refinement of the triangle inequality:

$$\left\|\frac{x+y}{2}\right\| \leq \int_0^1 \|(1-t)x + ty\| dt \leq \frac{\|x\| + \|y\|}{2}, \quad x,\ y \in X. \tag{4.2}$$

Assume that $f : C \subset X \to \mathbb{R}$ is convex on the convex subset C, then for all x, $y \in C$ we have

$$\begin{aligned} 0 &\leq \frac{f(x) + f(y)}{2} - \int_0^1 f((1-t)x + ty)\, dt \\ &\leq \frac{[f(y) - f(x) - \nabla_+ f(x)(y-x)][\nabla_- f(y)(y-x) - f(y) + f(x)]}{\nabla_- f(y)(y-x) - \nabla_+ f(x)(y-x)} \\ &\leq \frac{1}{8}[\nabla_- f(y)(y-x) - \nabla_+ f(x)(y-x)], \end{aligned} \tag{4.3}$$

provided that $\nabla_- f(y)(y-x) - \nabla_+ f(x)(y-x) > 0$.

If we write the inequality (4.3) for $f_0(x) = \frac{1}{2}\|x\|^2$, then we get

$$0 \le \frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \|(1-t)x + ty\|^2 dt \tag{4.4}$$
$$\le \frac{\left[\|y\|^2 - \|x\|^2 - 2\langle y-x, x\rangle_s\right]\left[2\langle y-x, y\rangle_i - \|y\|^2 + \|x\|^2\right]}{2\left[\langle y-x, y\rangle_i - \langle y-x, x\rangle_s\right]}$$
$$\le \frac{1}{4}\left[\langle y-x, y\rangle_i - \langle y-x, x\rangle_s\right],$$

for all $x, y \in X$ with $\langle y-x, y\rangle_i - \langle y-x, x\rangle_s > 0$.

If we write the inequality (4.3) for $f(x) = \|x\|$, then we get

$$0 \le \frac{\|x\| + \|y\|}{2} - \int_0^1 \|(1-t)x + ty\| dt \tag{4.5}$$
$$\le \frac{\left[\|y\| - \|x\| - \langle y-x, x/\|x\|\rangle_s\right]\left[\langle y-x, y/\|y\|\rangle_i - \|y\| + \|x\|\right]}{\langle y-x, y/\|y\|\rangle_i - \langle y-x, x/\|x\|\rangle_s}$$
$$\le \frac{1}{8}\left[\langle y-x, y/\|y\|\rangle_i - \langle y-x, x/\|x\|\rangle_s\right],$$

for all $x, y \in X \setminus \{0\}$ with $\langle y-x, y/\|y\|\rangle_i - \langle y-x, x/\|x\|\rangle_s > 0$.

For $p \ge 1$ the function $f_p(x) = \|x\|^p$ is convex on X. Therefore

$$\nabla_{+(-)} f_p(y)(x) = p\|y\|^{p-2}\langle x, y\rangle_{s(i)} \tag{4.6}$$

which exists for all $x, y \in X$ whenever $p \ge 2$. If $1 \le p < 2$ the equality (4.6) for all $x \in X$ and nonzero $y \in X$.

If we use the inequality (4.3) for the function f_p, then we get

$$0 \le \frac{\|x\|^p + \|y\|^p}{2} - \int_0^1 \|(1-t)x + ty\|^p dt \tag{4.7}$$
$$\le \frac{\left[\|y\|^p - \|x\|^p - p\|x\|^{p-2}\langle y-x, x\rangle_s\right]\left[p\|y\|^{p-2}\langle y-x, y\rangle_i - \|y\|^p + \|x\|^p\right]}{p\left(\|y\|^{p-2}\langle y-x, y\rangle_i - \|x\|^{p-2}\langle y-x, x\rangle_s\right)}$$
$$\le \frac{1}{8}p\left(\|y\|^{p-2}\langle y-x, y\rangle_i - \|x\|^{p-2}\langle y-x, x\rangle_s\right),$$

which holds for all $x, y \in X$ whenever $p \ge 2$ and for $y \ne 0$ when $1 \le p < 2$.

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