BOUNDS FOR THE ČEBYŠEV FUNCTIONAL OF TWO ABSOLUTELY CONTINUOUS FUNCTIONS WITH APPLICATIONS TO FINITE FOURIER TRANSFORM

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Abstract. For two Lebesgue integrable functions $h,\ k:[a,b]\to\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(h,k):=(b-a)\int_a^b h(t)\,k(t)\,dt-\int_a^b h(t)\,dt\int_a^b k(t)\,dt.$$

In this paper we show among others that, if $f,\ g:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$, then

$$\begin{aligned}|D(f,g)|&\le D\left(\int_a^{\cdot}\left|f'(u)\right|du,\int_a^{\cdot}\left|g'(u)\right|du\right)\\&\le\frac{1}{4}(b-a)^2\left\|f'\right\|_{[a,b],1}\left\|g'\right\|_{[a,b],1},\end{aligned}$$

where $\|h'\|_{[a,b],1}:=\int_a^b|h'(u)|\,du$. Applications for finite Fourier transform are also provided.

1. Introduction

For two Lebesgue integrable functions $h,\ k:[a,b]\to\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(h,k):=(b-a)\int_a^b h(t)\,k(t)\,dt-\int_a^b h(t)\,dt\int_a^b k(t)\,dt.$$

In 1934, G. Grüss [24] showed that

$$|D(h,k)|\le\frac{1}{4}(b-a)^2(M-m)(N-n),\tag{1.1}$$

provided $m,\ M,\ n,\ N$ are real numbers with the property that

$$-\infty<m\le h\le M<\infty,\quad-\infty<n\le k\le N<\infty\quad\text{a.e. on}\quad[a,b].\tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $D(h,k)$ was derived in 1882 by Čebyšev [11] under the assumption that h',k' exist and are continuous on $[a,b]$, and is given by

$$|D(h,k)|\le\frac{1}{12}\|h'\|_\infty\|k'\|_\infty(b-a)^4,\tag{1.3}$$

where $\|h'\|_\infty:=\sup_{t\in[a,b]}|h'(t)|<\infty$.

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The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $h,\ k : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $h',\ k' \in L_\infty [a,b]$.

In 1970, A. M. Ostrowski [32] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(h,k)| \leq \frac{1}{8}(b-a)^3 (M-m)\|k'\|_\infty , \tag{1.4}$$

provided h is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $k : [a,b] \to \mathbb{R}$ is absolutely continuous and $k' \in L_\infty [a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [28] (see also [29, p. 210]) obtained the following result as well:

$$|D(h,k)| \leq \frac{1}{\pi^2}\|h'\|_2 \|k'\|_2 (b-a)^3 , \tag{1.5}$$

provided $h,\ k$ are absolutely continuous and $h',\ k' \in L_2 [a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [8], P. Cerone and S. S. Dragomir proved the following inequalities:

$$\begin{aligned} &|D(h,k)| \\ &\leq (b-a) \times \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|h-\gamma\|_\infty \int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right| dt, \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|h-\gamma\|_q \left(\int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right|^p dt\right)^{\frac{1}{p}}, \\ \text{where } p>1,\ 1/p+1/q=1. \end{cases} \end{aligned} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(h,k)| \leq (b-a)\|h\|_\infty \int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq h \leq M$ for a.e. $x \in [a,b]$, then $\left\|h - \frac{m+M}{2}\right\|_\infty \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [10]

$$|D(h,k)| \leq \frac{1}{2}(b-a)(M-m)\int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [9].

The following result holds [17].

Theorem 1. *Let $h : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $k : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(h,k)| \leq \frac{1}{2}(b-a)\bigvee_a^b (h) \int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right| dt$$

where $\bigvee_a^b (f)$ denotes the total variation of f on the interval $[a,b]$. The constant $\frac{1}{2}$ is best possible in (1.9).

For more recent upper bounds related to the Čebyšev functional see [8], [9] and [15]-[17].

In [14] we obtained the following refinement of Ostrowski's inequality (1.4)

$$|D(h,k)| \leq \frac{1}{2}(b-a)^3 \frac{\left(\frac{1}{b-a}\int_a^b h(t)\,dt - m\right)\left(M - \frac{1}{b-a}\int_a^b h(t)\,dt\right)}{M-m}\|k'\|_\infty \leq \frac{1}{8}(b-a)^3(M-m)\|k'\|_\infty, \tag{1.9}$$

provided $m \leq h \leq M$ a.e. on $[a,b]$ and k is absolutely continuous on $[a,b]$.

In this paper we show among others that, if $f,\ g : [a,b] \to \mathbb{C}$ are absolutely continuous on $[a,b]$, then

$$|D(f,g)| \leq D\left(\int_a^{\cdot} |f'(u)|\,du, \int_a^{\cdot} |g'(u)|\,du\right) \leq \frac{1}{4}(b-a)^2 \|f'\|_{[a,b],1}\|g'\|_{[a,b],1},$$

where $\|h'\|_{[a,b],1} := \int_a^b |h'(u)|\,du$. Applications for finite Fourier transform are also provided.

2. Main Results

For two continuous functions $f,\ g : [a,b] \to \mathbb{C}$, we define the *Čebyšev functional*

$$D(f,g) := (b-a)\int_a^b f(t)\,g(t)\,dt - \int_a^b f(t)\,dt \int_a^b g(t)\,dt.$$

We have the following result of interest:

Theorem 2. *Let $f,\ g : [a,b] \to \mathbb{C}$ be absolutely continuous on $[a,b]$. Then*

$$|D(f,g)| \leq D\left(\int_a^{\cdot} |f'(u)|\,du, \int_a^{\cdot} |g'(u)|\,du\right) \leq \frac{1}{4}(b-a)^2 \|f'\|_{[a,b],1}\|g'\|_{[a,b],1}, \tag{2.1}$$

where $\|h'\|_{[a,b],1} := \int_a^b |h'(u)|\,du$.

Proof. Observe that

$$\begin{aligned}
&\int_a^b \int_a^b [f(t)-f(s)][g(t)-g(s)]\,dtds \\
&= \int_a^b \int_a^b (f(t)g(t) - f(s)g(t) - f(t)g(s) + f(s)g(s))\,dtds \\
&= (b-a)\int_a^b f(t)g(t)\,dt - \int_a^b f(s)\,ds \int_a^b g(t)\,dt \\
&- \int_a^b f(t)\,dt \int_a^b g(s)\,ds + (b-a)\int_a^b f(s)g(s)\,ds \\
&= 2(b-a)\int_a^b f(t)g(t)\,dt - 2\int_a^b f(t)\,dt\int_a^b g(t)\,dt = 2D(f,g),
\end{aligned}$$

which give the Korkine's identity for complex valued functions

$$D(f,g) = \frac{1}{2}\int_a^b \int_a^b [f(t) - f(s)][g(t) - g(s)]\,dtds.$$

For Korkine's classical identity for real-valued functions, see [29, p. 242].

If we take the modulus and use the integral's properties, we get

$$\begin{aligned} |D(f,g)| &\le \frac{1}{2}\int_a^b \int_a^b |[f(t) - f(s)][g(t) - g(s)]|\,dtds \\ &\le \frac{1}{2}\int_a^b \int_a^b |f(t) - f(s)|\,|g(t) - g(s)|\,dtds. \end{aligned} \tag{2.2}$$

Observe that for $s,\ t \in [a,b]$

$$f(t) - f(s) = \int_s^t f'(u)\,du,\ \ g(t) - g(s) = \int_s^t g'(u)\,du,$$

which implies that

$$\begin{aligned} |f(t) - f(s)|\,|g(t) - g(s)| &= \left|\int_s^t f'(u)\,du\right|\left|\int_s^t g'(u)\,du\right| \\ &\le \left|\int_s^t |f'(u)|\,du\right|\left|\int_s^t |g'(u)|\,du\right| \\ &= \left|\int_s^t |f'(u)|\,du \int_s^t |g'(u)|\,du\right| \\ &= \int_s^t |f'(u)|\,du \int_s^t |g'(u)|\,du, \end{aligned}$$

for all $s,\ t \in [a,b]$.

By (2.2) we get

$$|D(f,g)| \le \frac{1}{2}\int_a^b \int_a^b \left(\int_s^t |f'(u)|\,du\right)\left(\int_s^t |g'(u)|\,du\right) dtds. \tag{2.3}$$

Since

$$\begin{aligned} &\int_s^t |f'(u)|\,du \int_s^t |g'(u)|\,du \\ &= \left(\int_a^t |f'(u)|\,du - \int_a^s |f'(u)|\,du\right)\left(\int_a^t |g'(u)|\,du - \int_a^s |g'(u)|\,du\right), \end{aligned}$$

hence by Korkine's identity for real valued functions $f(t) = \int_a^t |f'(u)|\,du$ and $g(t) = \int_a^t |g'(u)|\,du$, we have

$$\begin{aligned}(2.4)\qquad &\frac{1}{2}\int_a^b\int_a^b\left(\int_a^t |f'(u)|\,du-\int_a^s |f'(u)|\,du\right)\\ &\times\left(\int_a^t |g'(u)|\,du-\int_a^s |g'(u)|\,du\right)\\ &=(b-a)\int_a^b\left(\int_a^t |f'(u)|\,du\right)\left(\int_a^t |g'(u)|\,du\right)dt\\ &-\int_a^b\left(\int_a^t |f'(u)|\,du\right)dt\int_a^b\left(\int_a^t |g'(u)|\,du\right)dt\\ &=D\left(\int_a^{\cdot} |f'(u)|\,du,\int_a^{\cdot} |g'(u)|\,du\right).\end{aligned}$$

By utilising (2.3) and (2.4), we deduce the first inequality in (2.1).

Observe that

$$0\le\int_a^t |f'(u)|\,du\le\int_a^b |f'(u)|\,du$$

and

$$0\le\int_a^t |g'(u)|\,du\le\int_a^b |g'(u)|\,du$$

for all $t\in[a,b]$, then by Grüss's inequality for the functions $f(t)=\int_a^t |f'(u)|\,du$ and $g(t)=\int_a^t |g'(u)|\,du$, $t\in[a,b]$, we get the last part of (2.1). □

We have the Čebyšev's inequality:

Corollary 1. *Let $f,\ g:[a,b]\to\mathbb{C}$ be absolutely continuous on $[a,b]$ with*

$$\|f'\|_{[a,b],\infty}:=\sup_{u\in(a,b)}|f'(u)|,\quad \|g'\|_{[a,b],\infty}<\infty,$$

then

$$\begin{aligned}(2.5)\qquad |D(f,g)| &\le D\left(\int_a^{\cdot} |f'(u)|\,du,\int_a^{\cdot} |g'(u)|\right)\\ &\le\frac{1}{12}(b-a)^4\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\infty}.\end{aligned}$$

Proof. If we use Čebyšev's inequality (1.3) for $f(t)=\int_a^t |f'(u)|\,du$ and $g(t)=\int_a^t |g'(u)|$, $t\in[a,b]$, then we get

$$\begin{aligned}D\left(\int_a^{\cdot} |f'(u)|\,du,\int_a^{\cdot} |g'(u)|\right) &\le\frac{1}{12}(b-a)^4\|f'\|_\infty\|g'\|_\infty\\ &=\frac{1}{12}(b-a)^4\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\infty},\end{aligned}$$

which by the first inequality in (2.1) gives the desired result (2.5). □

By the use of Ostrowski's inequality (1.4) we derive:

Corollary 2. *Let $f,g:[a,b]\to\mathbb{C}$ be absolutely continuous on $[a,b]$ with$\|g'\|_{[a,b],\infty}<\infty$, then*

$$(2.6)\quad |D(f,g)|\le D\left(\int_a^{\cdot} |f'(u)|\,du,\int_a^{\cdot} |g'(u)|\right)\le\frac{1}{8}(b-a)^3\|f'\|_{[a,b],1}\|g'\|_{[a,b],\infty}.$$

For a differentiable complex valued function z on (a,b), we define

$$\|z'\|_{[a,b],2} := \left(\int_a^b |z'(u)|^2 du\right)^{1/2}.$$

By the use of Lupaş inequality for $f(t) = \int_a^t |f'(u)| du$ and $g(t) = \int_a^t |g'(u)| du$, $t \in [a,b]$, we get:

Corollary 3. *Let $f, g : [a,b] \to \mathbb{C}$ be absolutely continuous on $[a,b]$ with $\|f'\|_{[a,b],2}$, $\|g'\|_{[a,b],2} < \infty$, then*

$$|D(f,g)| \le \left(\int_a^{\cdot} |f'(u)| du, \int_a^{\cdot} |g'(u)|\right) \le \frac{1}{\pi^2}(b-a)^3 \|f'\|_{[a,b],2} \|g'\|_{[a,b],2}. \tag{2.7}$$

Observe that for $f(t) = \int_a^t |f'(u)| du$, we get integrating by parts that

$$\begin{aligned}
&\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\, ds \right| dt \\
&= \int_a^b \left| \int_a^t |f'(u)| du - \frac{1}{b-a}\int_a^b \left(\int_a^s |f'(u)| du\right) ds \right| dt \\
&= \int_a^b \left| \int_a^t |f'(u)| du - \frac{1}{b-a}\left(\left(\int_a^b |f'(u)| du\right) b - \int_a^b |f'(s)| s ds\right) \right| dt
\end{aligned}$$

$$\begin{aligned}
&= \int_a^b \left| \int_a^t |f'(u)| du - \frac{1}{b-a}\left(\int_a^b (b-u)|f'(u)| du\right) \right| dt \\
&= \frac{1}{b-a}\int_a^b \left| (b-a)\int_a^t |f'(u)| du - \int_a^b (b-u)|f'(u)| du \right| dt \\
&= \frac{1}{b-a}\int_a^b \left| \int_a^t (u-a)|f'(u)| du - \int_t^b (b-u)|f'(u)| du \right| dt.
\end{aligned}$$

By utilising (1.8) for $f(t) = \int_a^t |f'(u)| du$ and $g(t) = \int_a^t |g'(u)| du$, $t \in [a,b]$, we get:

Corollary 4. *Let $f, g : [a,b] \to \mathbb{C}$ be absolutely continuous on $[a,b]$, then*

$$\begin{aligned}
|D(f,g)| &\le \frac{1}{2}\|g'\|_{[a,b],1} \\
&\quad \times \int_a^b \left| \int_a^t (u-a)|f'(u)| du - \int_t^b (b-u)|f'(u)| du \right| dt.
\end{aligned} \tag{2.8}$$

Remark 1. *We observe that*

$$\int_a^b \left| \int_a^t (u-a)\,|f'(u)|\,du - \int_t^b (b-u)\,|f'(u)|\,du \right| dt$$
$$\leq \int_a^b \left[\left| \int_a^t (u-a)\,|f'(u)|\,du \right| + \left| \int_t^b (b-u)\,|f'(u)|\,du \right| \right] dt$$
$$\leq \int_a^b \left[\int_a^t (u-a)\,|f'(u)|\,du + \int_t^b (b-u)\,|f'(u)|\,du \right] dt$$

$$= \left[\int_a^t (u-a)\,|f'(u)|\,du + \int_t^b (b-u)\,|f'(u)|\,du \right] t \Big|_a^b$$
$$- \int_a^b t\left((t-a)\,|f'(t)| - (b-t)\,|f'(t)|\right) dt$$

$$= b\int_a^b (u-a)\,|f'(u)|\,du - a\int_a^b (b-u)\,|f'(u)|\,du$$
$$- \int_a^b t\left((t-a)\,|f'(t)| - (b-t)\,|f'(t)|\right) dt$$
$$= 2\int_a^b (b-t)(t-a)\,|f'(t)|\,dt$$

and by (2.8) we get

$$\|D(f,g)\| \leq \|g'\|_{[a,b],1} \int_a^b (b-t)(t-a)\,|f'(t)|\,dt. \tag{2.9}$$

Theorem 3. *Let $f,\ g : [a,b] \to \mathbb{C}$ be absolutely continuous on $[a,b]$, then*

$$|D(f,g)| \leq \begin{cases} \inf_{\beta\in\mathbb{C}} \|f-\beta\|_{[a,b],\infty} \int_a^b \left|(b-a)\,g(t) - \int_a^b g(s)\,ds\right| dt, \\ \\ \inf_{\beta\in\mathbb{C}} \|f-\beta\|_{[a,b],q} \left(\int_a^b \left|(b-a)\,g(t) - \int_a^b g(s)\,ds\right|^p dt\right)^{1/p}, \\ \\ \inf_{\beta\in\mathbb{C}} \|f-\beta\|_{[a,b],1} \sup_{t\in[a,b]} \left|(b-a)\,g(t) - \int_a^b g(s)\,ds\right| \end{cases} \tag{2.10}$$

for all $p,\ q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. For all $\beta \in \mathbb{C}$ we have

$$\int_a^b [f(t) - \beta]\left[(b-a)\,g(t) - \int_a^b g(s)\,ds\right] dt$$
$$= \int_a^b f(t)\left[(b-a)\,g(t) - \int_a^b g(s)\,ds\right] dt$$

$$-\beta\int_a^b\left[(b-a)\,g(t)-\int_a^b g(s)\,ds\right]dt$$
$$=(b-a)\int_a^b f(t)\,g(t)\,dt-\int_a^b f(t)\,dt\int_a^b g(s)\,ds$$
$$-\beta\left[(b-a)\int_a^b g(t)\,dt-(b-a)\int_a^b g(s)\,ds\right]=D(f,g).$$

Taking the norm in this equality, we get by Hölder's inequality that

$$\|D(f,g)\|\le\int_a^b\left|[f(t)-\beta]\left[(b-a)\,g(t)-\int_a^b g(s)\,ds\right]\right|dt$$

$$\le\int_a^b|f(t)-\beta|\left|(b-a)\,g(t)-\int_a^b g(s)\,ds\right|dt$$
$$\le\begin{cases}\sup_{t\in[a,b]}|f(t)-\beta|\int_a^b\left|(b-a)\,g(t)-\int_a^b g(s)\,ds\right|dt,\\[2ex]\left(\int_a^b|f(t)-\beta|^q\,dt\right)^{1/q}\left(\int_a^b\left|(b-a)\,g(t)-\int_a^b g(s)\,ds\right|^p dt\right)^{1/p},\\[2ex]\int_a^b|f(t)-\beta|\sup_{t\in[a,b]}\left|(b-a)\,g(t)-\int_a^b g(s)\,ds\right|\end{cases}$$

for all $\beta\in\mathbb{C}$.

By taking the infimum over $\beta\in\mathbb{C}$, we obtain the desired result (2.10). □

Corollary 5. *With the assumptions of Theorem 3 and if there exists $\beta\in\mathbb{C}$ and $M>0$ such that*

$$|f(t)-\beta|\le M\text{ for all }t\in[a,b],$$

then

$$|D(f,g)|\le M\int_a^b\left|(b-a)\,g(t)-\int_a^b g(s)\,ds\right|dt. \tag{2.11}$$

Remark 2. *If there exists γ, $\Gamma\in\mathbb{C}$ with*

$$\left|f(t)-\frac{\gamma+\Gamma}{2}\right|\le\frac{1}{2}|\Gamma-\gamma|\text{ for all }t\in[a,b]$$

or, equivalently,

$$\operatorname{Re}\left[(\Gamma-f(t))\left(\overline{f(t)}-\overline{\gamma}\right)\right]\ge0,\text{ for all }t\in[a,b],$$

then by (2.11) we get

$$|D(f,g)|\le\frac{1}{2}|\Gamma-\gamma|\int_a^b\left|(b-a)\,g(t)-\int_a^b g(s)\,ds\right|dt. \tag{2.12}$$

The proof is obvious from the first branch of (2.10).

Corollary 6. *Let f, $g : [a,b] \to \mathbb{C}$ be absolutely continuous on $[a,b]$, then for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$
\begin{aligned}
|D(f,g)| &\leq \sup_{t\in[a,b]} \left| (b-a)\, g(t) - \int_a^b g(s)\, ds \right| \\
&\quad \times \left[\int_a^{\frac{a+b}{2}} (t-a)\, |f'(t)|\, dt + \int_{\frac{a+b}{2}}^b (b-t)\, |f'(t)|\, dt \right] \\
&\leq \sup_{t\in[a,b]} \left| (b-a)\, g(t) - \int_a^b g(s)\, ds \right| \\
&\quad \times \begin{cases} \frac{1}{8}(b-a)^2 \left[\|f'\|_{[a,\frac{a+b}{2}],\infty} + \|f'\|_{[\frac{a+b}{2},b],\infty} \right], \\[2mm] \frac{1}{(q+1)^{1/q} 2^{1+1/q}} \left[\|f'\|_{[a,\frac{a+b}{2}],p} + \|f'\|_{[\frac{a+b}{2},b],p} \right], \\[2mm] \frac{1}{2}(b-a)\, \|f'\|_{[a,b],1}\, . \end{cases}
\end{aligned} \tag{2.13}
$$

Proof. We have

$$
\begin{aligned}
&\int_a^b \left| f(t) - f\left(\frac{a+b}{2}\right) \right| dt \\
&= \int_a^b \left| \int_{\frac{a+b}{2}}^t f'(s)\, ds \right| dt \leq \int_a^b \left| \int_{\frac{a+b}{2}}^t |f'(s)|\, ds \right| dt \\
&= \int_a^{\frac{a+b}{2}} \left(\int_t^{\frac{a+b}{2}} |f'(s)|\, ds \right) dt + \int_{\frac{a+b}{2}}^b \left(\int_{\frac{a+b}{2}}^t |f'(s)|\, ds \right) dt \\
&\quad = \left(\int_t^{\frac{a+b}{2}} |f'(s)|\, ds \right) t \Big|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} t\, |f'(t)|\, dt \\
&\quad + \left(\int_{\frac{a+b}{2}}^t |f'(s)|\, ds \right) t \Big|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b t\, |f'(t)|\, dt \\
&\quad = \int_a^{\frac{a+b}{2}} t\, |f'(t)|\, dt - a \int_a^{\frac{a+b}{2}} |f'(s)|\, ds \\
&\quad + b \int_{\frac{a+b}{2}}^b |f'(s)|\, ds - \int_{\frac{a+b}{2}}^b t\, |f'(t)|\, dt \\
&\quad = \int_a^{\frac{a+b}{2}} (t-a)\, |f'(t)|\, dt + \int_{\frac{a+b}{2}}^b (b-t)\, |f'(t)|\, dt,
\end{aligned}
$$

which, by the third branch of (2.10), gives the first part of (2.13).

The last part follows by Hölder's inequality. □

3. Applications for Uni-Dimensional Fourier Transform

The *Fourier Transform* has applications in a wide variety of fields in science and engineering [6, p. xi].

Let $g:[a,b]\to\mathbb{C}$ be a Lebesgue integrable mapping defined on the finite interval $[a,b]$ and $\mathcal{F}(g)$ its finite Fourier transform, i.e.,

$$\mathcal{F}(g)(t):=\int_a^b g(s)\,e^{-2\pi its}ds. \tag{3.1}$$

Define E, the exponential mean of two complex numbers by

$$E(z,w):=\begin{cases}\dfrac{e^z-e^w}{z-w}, & \text{if } z\neq w\\ \exp(w) & \text{if } z=w\end{cases},\quad z,w\in\mathbb{C}. \tag{3.2}$$

In [25], Hanna, Dragomir and Roumeliotis obtained the following result:

Theorem 4. *Let $g:[a,b]\to\mathbb{K}$ be a complex-valued integrable function and there exists the constants $\varphi,\ \phi\in\mathbb{C}$ with the property that, either*

$$\left|g(s)-\frac{\phi+\varphi}{2}\right|\leq\frac{1}{2}\left|\phi-\varphi\right| \text{ for } \mu\text{-a.e. } s\in[a,b] \tag{3.3}$$

or, equivalently

$$\operatorname{Re}\left[\left(\phi-g(s)\right)\left(\overline{g(s)}-\overline{\varphi}\right)\right]\geq 0 \text{ for a.e. } s\in[a,b] \tag{3.4}$$

holds. Then we have the inequality

$$\left|\mathcal{F}(g)(t)-E(-2\pi ita,-2\pi itb)\int_a^b g(s)\,ds\right| \tag{3.5}$$
$$\leq\frac{1}{2}\left|\phi-\varphi\right|(b-a)\left[1-\frac{\sin^2\left[\pi t(b-a)\right]}{\pi^2t^2(b-a)^2}\right]^{\frac{1}{2}},$$

for each $t\in[a,b]$ $(t\neq 0)$, where $E(\cdot,\cdot)$ is the exponential mean defined above.

Let $f(s)=e^{-2\pi its}$, $s,\ t\in[a,b]$ $(t\neq 0)$. Then

$$\int_a^b e^{-2\pi its}ds=(b-a)\,E(-2\pi ita,-2\pi itb),$$
$$\left|e^{2\pi its}\right|^2=1,$$
$$f'(s)=-2\pi ite^{-2\pi its},\quad \left|f'(s)\right|=2\pi\left|t\right|$$

and

$$\left\|f'\right\|_{[a,b],1}=\int_a^b\left|f'(s)\right|ds=2\pi\left|t\right|(b-a)$$

for $s,\ t\in[a,b]$.

Applying (2.1), we get for any absolutely continuous function $g:[a,b]\to\mathbb{C}$ that

$$\left|\mathcal{F}(g)(t)-E(-2\pi ita,-2\pi itb)\int_a^b g(s)\,ds\right|\leq\frac{1}{2}\pi\left|t\right|(b-a)^2\left\|g'\right\|_{[a,b],1}, \tag{3.6}$$

for all $t\in[a,b]$ $(t\neq 0)$.

Further, if we use the inequality (2.5), then we get for any absolutely continuous function $g:[a,b]\to\mathbb{C}$ with $\left\|g'\right\|_{[a,b],\infty}<\infty$ that

$$\left|\mathcal{F}(g)(t)-E(-2\pi ita,-2\pi itb)\int_a^b g(s)\,ds\right|\leq\frac{1}{6}\pi\left|t\right|(b-a)^3\left\|g'\right\|_{[a,b],\infty} \tag{3.7}$$

for all $t \in [a,b]$ $(t \neq 0)$.

Also, if we apply the inequality (1.9) we get for $m \leq g \leq M$ that

$$\left|\mathcal{F}(g)(t) - E(-2\pi ita, -2\pi itb)\int_a^b g(s)\,ds\right| \leq \pi|t|(b-a)^2 \frac{\left(\frac{1}{b-a}\int_a^b g(t)\,dt - m\right)\left(M - \frac{1}{b-a}\int_a^b g(t)\,dt\right)}{M-m} \leq \frac{1}{4}\pi|t|(b-a)^2(M-m), \tag{3.8}$$

for all $t \in [a,b]$ $(t \neq 0)$.

Finally, by utilising (2.13) we get

$$\left|\mathcal{F}(g)(t) - E(-2\pi ita, -2\pi itb)\int_a^b g(s)\,ds\right| \leq \frac{1}{2}\pi|t|(b-a)^2 \sup_{t\in[a,b]}\left|(b-a)g(t) - \int_a^b g(s)\,ds\right| \tag{3.9}$$

for all $t \in [a,b]$ $(t \neq 0)$.

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