p-NORMS BOUNDS FOR THE ČEBYŠEV FUNCTIONAL OF MONOTONIC AND ABSOLUTELY CONTINUOUS FUNCTIONS WITH APPLICATIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For two Lebesgue integrable functions $h,\ k:[a,b]\to\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(h,k):=(b-a)\int_a^b h(t)\,k(t)\,dt-\int_a^b h(t)\,dt\int_a^b k(t)\,dt.$$

In this paper we show among others that, if $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and absolutely continuous while $g:[a,b]\to\mathbb{C}$ is absolutely continuous with $\|g'\|_{[a,b],\infty}:=\operatorname{esssup}_{t\in[a,b]}|g'(t)|<\infty$, then

$$\begin{aligned}|D(f,g)|&\le(b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q}\\&\quad\times\left[D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p}\\&\le(b-a)\,\|g'\|_{[a,b],\infty}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right),\end{aligned}$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Applications for trapezoid inequality related to convex functions with examples for norms and semi-inner products are also provided.

1. Introduction

For two Lebesgue integrable functions $h,\ k:[a,b]\to\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(h,k):=(b-a)\int_a^b h(t)\,k(t)\,dt-\int_a^b h(t)\,dt\int_a^b k(t)\,dt.$$

In 1934, G. Grüss [23] showed that

$$|D(h,k)|\le\frac{1}{4}(b-a)^2(M-m)(N-n),\tag{1.1}$$

provided $m,\ M,\ n,\ N$ are real numbers with the property that

$$-\infty<m\le h\le M<\infty,\quad -\infty<n\le k\le N<\infty\quad\text{a.e. on}\quad[a,b].\tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

2000 *Mathematics Subject Classification.* Primary 26D15; Secondary 26D10, 46B05.

Key words and phrases. Čebyšev functional, Grüss inequality, Ostrowski inequality, Trapezoid inequality, Norm inequalities, Semi-inner products.

RGMIA Res. Rep. Coll. 24 (2021), Art. 29, 20 pp. Received 11/03/21

Another lesser known inequality for $D(h,k)$ was derived in 1882 by Čebyšev [10] under the assumption that h', k' exist and are continuous on $[a,b]$, and is given by

$$|D(h,k)| \leq \frac{1}{12} \|h'\|_{\infty} \|k'\|_{\infty} (b-a)^4, \tag{1.3}$$

where $\|h'\|_{\infty} := \sup_{t\in[a,b]} |h'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if h, $k : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and h', $k' \in L_{\infty}[a,b]$.

In 1970, A. M. Ostrowski [30] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(h,k)| \leq \frac{1}{8} (b-a)^3 (M-m) \|k'\|_{\infty}, \tag{1.4}$$

provided h is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $k : [a,b] \to \mathbb{R}$ is absolutely continuous and $k' \in L_{\infty}[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [26] (see also [27, p. 210]) obtained the following result as well:

$$|D(h,k)| \leq \frac{1}{\pi^2} \|h'\|_2 \|k'\|_2 (b-a)^3, \tag{1.5}$$

provided h, k are absolutely continuous and h', $k' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [7], P. Cerone and S. S. Dragomir proved the following inequalities:

$$\begin{aligned} &|D(h,k)| \\ &\leq (b-a) \times \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|h-\gamma\|_{\infty} \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt, \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|h-\gamma\|_{q} \left(\int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right|^p dt \right)^{\frac{1}{p}}, \\ \text{where } p>1,\ 1/p+1/q=1. \end{cases} \end{aligned} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(h,k)| \leq (b-a) \|h\|_{\infty} \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq h \leq M$ for a.e. $x \in [a,b]$, then $\left\| h - \frac{m+M}{2} \right\|_{\infty} \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [9]

$$|D(h,k)| \leq \frac{1}{2}(b-a)(M-m) \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [8].

The following result holds [16].

Theorem 1. *Let $h : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $k : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(h,k)| \leq \frac{1}{2}(b-a) \bigvee_a^b (h) \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt$$

where $\bigvee_a^b (f)$ *denotes the total variation of* f *on the interval* $[a,b]$. *The constant* $\frac{1}{2}$ *is best possible in (1.9).*

For more recent upper bounds related to the Čebyšev functional see [7], [8] and [14]-[16].

In [13] we obtained the following refinement of Ostrowski's inequality (1.4)

$$\begin{aligned}(1.9)\quad |D(h,k)| &\le \frac{1}{2}(b-a)^3 \frac{\left(\frac{1}{b-a}\int_a^b h(t)\,dt - m\right)\left(M - \frac{1}{b-a}\int_a^b h(t)\,dt\right)}{M-m}\|k'\|_\infty \\ &\le \frac{1}{8}(b-a)^3 (M-m)\|k'\|_\infty,\end{aligned}$$

provided $m \le h \le M$ a.e. on $[a,b]$ and k is absolutely continuous on $[a,b]$.

In this paper we show among others that, if $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and absolutely continuous while $g:[a,b]\to\mathbb{C}$ is absolutely continuous with $\|g'\|_{[a,b],\infty} := \operatorname{esssup}_{t\in[a,b]}|g'(t)| < \infty$, then

$$\begin{aligned}|D(f,g)| &\le (b-a)^{1/q}\left(\int_a^b tf(t)\,dt - \frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q} \\ &\quad\times\left[D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p} \\ &\le (b-a)\|g'\|_{[a,b],\infty}\left(\int_a^b tf(t)\,dt - \frac{a+b}{2}\int_a^b f(t)\,dt\right),\end{aligned}$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Applications for trapezoid inequality related to convex functions with examples for norms and semi-inner products are also provided.

2. Main Results

We define the Lebesgue r-norms as

$$\|h\|_{[a,b],r} := \int_a^b |h(u)|^r\,du,\ \ r\ge 1$$

and

$$\|h\|_{[a,b],\infty} := \operatorname*{esssup}_{u\in[a,b]}|h(u)|.$$

Theorem 2. *Assume that* $f:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing and* $g:[a,b]\to\mathbb{C}$ *is absolutely continuous with* $\|g'\|_{[a,b],\infty} := \operatorname{esssup}_{t\in[a,b]}|g'(t)| < \infty$.

Then

$$|D(f,g)| \leq (b-a)\|g'\|_{[a,b],\infty}\left(\int_a^b tf(t)\,dt - \frac{a+b}{2}\int_a^b f(t)\,dt\right) \tag{2.1}$$
$$\leq \frac{1}{2}(b-a)^2\|g'\|_{[a,b],\infty}\int_a^b \left|f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt$$
$$\leq \frac{1}{2}(b-a)^3\|g'\|_{[a,b],\infty}$$
$$\times\left[\frac{1}{b-a}\int_a^b f^2(t) - \left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)^2\right]^{1/2}$$

and

$$|D(f,g)| \leq (b-a)\|g'\|_{[a,b],\infty}\left(\int_a^b tf(t)\,dt - \frac{a+b}{2}\int_a^b f(t)\,dt\right) \tag{2.2}$$
$$\leq \frac{1}{8}(b-a)^3[f(b)-f(a)]\|g'\|_{[a,b],\infty}.$$

In fact, we have the better result:

$$|D(f,g)| \leq (b-a)\|g'\|_{[a,b],\infty}\left(\int_a^b tf(t)\,dt - \frac{a+b}{2}\int_a^b f(t)\,dt\right) \tag{2.3}$$
$$\leq \frac{1}{2}(b-a)\|g'\|_{[a,b],\infty}\frac{\int_a^b [f(t)-f(a)]\,dt\int_a^b [f(b)-f(t)]\,dt}{f(b)-f(a)}$$
$$\leq \frac{1}{8}(b-a)^3[f(b)-f(a)]\|g'\|_{[a,b],\infty}.$$

Proof. Observe that for $f,\ g:[a,b]\to\mathbb{C}$ we have Korkine's identity

$$D(f,g) = \frac{1}{2}\int_a^b\int_a^b [f(t)-f(s)][g(t)-g(s)]\,dtds.$$

For Korkine's classical identity for real-valued functions, see [27, p. 242].

If we take the modulus and use the integral's properties, we get

$$|D(f,g)| \leq \frac{1}{2}\int_a^b\int_a^b |[f(t)-f(s)][g(t)-g(s)]|\,dtds \tag{2.4}$$
$$= \frac{1}{2}\int_a^b\int_a^b |f(t)-f(s)||g(t)-g(s)|\,dtds.$$

Since $g:[a,b]\to\mathbb{C}$ is absolutely continuous, then

$$|g(t)-g(s)| = \left|\int_s^t g'(u)\,du\right| \leq \left|\int_s^t |g'(u)|\,du\right| \leq |t-s|\,\|g'\|_{[a,b],\infty}$$

for all $t,\ s\in[a,b]$.

Therefore

$$
\begin{aligned}
(2.5)\qquad & \frac{1}{2}\int_a^b\int_a^b |f(t)-f(s)|\,|g(t)-g(s)|\,dtds \\
&\leq \frac{1}{2}\|g'\|_{[a,b],\infty}\int_a^b\int_a^b |f(t)-f(s)|\,|t-s|\,dtds \\
&= \frac{1}{2}\|g'\|_{[a,b],\infty}\int_a^b\int_a^b |(f(t)-f(s))(t-s)|\,dtds \\
&= \frac{1}{2}\|g'\|_{[a,b],\infty}\int_a^b\int_a^b (f(t)-f(s))(t-s)\,dtds = \|g'\|_{[a,b],\infty} D(f,\ell),
\end{aligned}
$$

where $\ell(t)=t$, $t\in[a,b]$.

Since

$$
\begin{aligned}
D(f,\ell) &= (b-a)\int_a^b tf(t)\,dt - \frac{b^2-a^2}{2}\int_a^b f(t)\,dt \\
&= (b-a)\left[\int_a^b tf(t)\,dt - \frac{a+b}{2}\int_a^b f(t)\,dt\right],
\end{aligned}
$$

then by (2.4) and (2.5) we derive the first inequality in (2.1).

Now, if we use (1.8) for $k=f$ and $h=\ell$, then we get

$$
\begin{aligned}
D(\ell,f) &\leq \frac{1}{2}(b-a)^2\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \\
&= \frac{1}{2}(b-a)^3\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt \\
&\leq \frac{1}{2}(b-a)^3\left[\frac{1}{b-a}\int_a^b f^2(t)-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)^2\right]^{1/2},
\end{aligned}
$$

which proves the last part of (2.1).

If we use (1.8) for $k=\ell$ and $h=f$, then we get

$$
\begin{aligned}
(2.6)\qquad D(f,\ell) &\leq \frac{1}{2}(b-a)(f(b)-f(a))\int_a^b\left|t-\frac{a+b}{2}\right|dt \\
&= \frac{1}{8}(b-a)^3[f(b)-f(a)],
\end{aligned}
$$

which proves the last part of (2.2).

Also, from (1.9) for $h=f$ and $k=\ell$, we get

$$
\begin{aligned}
D(f,\ell) &\leq \frac{1}{2}(b-a)^3\frac{\left(\frac{1}{b-a}\int_a^b f(t)\,dt - f(a)\right)\left(f(b)-\frac{1}{b-a}\int_a^b h(t)\,dt\right)}{f(b)-f(a)} \\
&\leq \frac{1}{8}(b-a)^3[f(b)-f(a)],
\end{aligned}
$$

namely

$$D(f,\ell) \leq \frac{1}{2}(b-a)\frac{\left(\int_a^b f(t)\,dt-(b-a)f(a)\right)\left((b-a)f(b)-\int_a^b h(t)\,dt\right)}{f(b)-f(a)}$$
$$\leq \frac{1}{8}(b-a)^3[f(b)-f(a)],$$

which proves (2.3). □

Corollary 1. *With the assumptions of Theorem 2, we have*

$$|D(f,g)| \leq (b-a)\|g'\|_{[a,b],\infty}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right) \tag{2.7}$$
$$\leq \frac{1}{2}(b-a)^2\|g'\|_{[a,b],\infty}\begin{cases}\|f\|_{[a,b],1}, \\ \frac{(b-a)^{1/q}}{(q+1)^{1/q}}\|f\|_{[a,b],p}, \ p,\ q>1, \\ \frac{1}{p}+\frac{1}{q}=1, \\ \frac{1}{2}(b-a)\max\{|f(a)|,|f(b)|\}.\end{cases}$$

Proof. Using Hölder's inequality, we have

$$\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt=\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt$$
$$\leq\begin{cases}\sup_{t\in[a,b]}\left|t-\frac{a+b}{2}\right|\int_a^b|f(t)|\,dt \\ \left(\int_a^b\left|t-\frac{a+b}{2}\right|^q dt\right)^{1/q}\left(\int_a^b|f(t)|^p\,dt\right)^{1/p}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \sup_{t\in[a,b]}|f(t)|\int_a^b\left|t-\frac{a+b}{2}\right|dt\end{cases}$$
$$=\begin{cases}\frac{1}{2}(b-a)\|f\|_{[a,b],1} \\ \frac{(b-a)^{1+1/q}}{2(q+1)^{1/q}}\|f\|_{[a,b],p}, \\ p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1 \\ \frac{1}{4}(b-a)^2\max\{|f(a)|,|f(b)|\}\end{cases}$$

and the inequality (2.5) is proved. □

Theorem 3. *Assume that $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and $g:[a,b]\to\mathbb{C}$ is absolutely continuous with $\|g'\|_{[a,b],p} := \left(\int_a^b|g'(t)|^p\right)^{1/p}<\infty$. Then*

$$|D(f,g)| \leq (b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q} \tag{2.8}$$
$$\times\left[D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p}$$

for all $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

In particular, we have

$$|D(f,g)| \le (b-a)^{1/2}\left(\int_a^b tf(t)\,dt - \frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/2} \times \left[D\left(f,\int_a^{\cdot} |g'(u)|^2\,du\right)\right]^{1/2}. \tag{2.9}$$

Proof. Using Hölder's inequality, we also have for all $s,\ t\in[a,b]$ that

$$|g(t)-g(s)| = \left|\int_s^t g'(u)\,du\right| \le \left|\int_s^t |g'(u)|\,du\right| \le |t-s|^{1/q}\left|\int_s^t |g'(u)|^p\,du\right|^{1/p}$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Therefore

$$\begin{aligned} &\frac{1}{2}\int_a^b\int_a^b |f(t)-f(s)|\,|g(t)-g(s)|\,dtds \\ &\le \frac{1}{2}\int_a^b\int_a^b |f(t)-f(s)|\,|t-s|^{1/q}\left|\int_s^t |g'(u)|^p\,du\right|^{1/p} dtds. \end{aligned} \tag{2.10}$$

On making use of weighted Hölder's inequality for double integral, we have for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\begin{aligned} &\frac{1}{2}\int_a^b\int_a^b |f(t)-f(s)|\,|t-s|^{1/q}\left|\int_s^t |g'(u)|^p\,du\right|^{1/p} dtds \\ &\le \left[\frac{1}{2}\int_a^b\int_a^b |f(t)-f(s)|\left(|t-s|^{1/q}\right)^q dtds\right]^{1/q} \\ &\quad\times \left[\frac{1}{2}\int_a^b\int_a^b |f(t)-f(s)|\left(\left|\int_s^t |g'(u)|^p\,du\right|^{1/p}\right)^p dtds\right]^{1/p} \\ &= \left[\frac{1}{2}\int_a^b\int_a^b |f(t)-f(s)|\,|t-s|\,dtds\right]^{1/q} \\ &\quad\times \left[\frac{1}{2}\int_a^b\int_a^b |f(t)-f(s)|\left|\int_s^t |g'(u)|^p\,du\right| dtds\right]^{1/p} \\ &= \left[\frac{1}{2}\int_a^b\int_a^b [f(t)-f(s)]\,(t-s)\,dtds\right]^{1/q} \\ &\quad\times \left[\frac{1}{2}\int_a^b\int_a^b [f(t)-f(s)]\left(\int_a^t |g'(u)|^p\,du - \int_a^s |g'(u)|^p\,du\right) dtds\right]^{1/p} \\ &= [D(f,\ell)]^{1/q}\left[D\left(f,\int_a^{\cdot} |g'(u)|^p\,du\right)\right]^{1/p}. \end{aligned} \tag{2.11}$$

By making use (2.4), (2.10) and (2.11), we derive the desired inequality in (2.8). □

Corollary 2. *With the assumptions of Theorem 3 we have the following improvement of the first inequality in (2.1)*

$$\begin{aligned} (2.12)\qquad & |D(f,g)| \\ & \leq (b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q} \\ & \times\left[D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p} \\ & \leq (b-a)\|g'\|_{[a,b],\infty}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right), \end{aligned}$$

where $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

In particular,

$$\begin{aligned} (2.13)\qquad & |D(f,g)| \\ & \leq (b-a)^{1/2}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/2} \\ & \times\left[D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/2} \\ & \leq (b-a)\|g'\|_{[a,b],\infty}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right). \end{aligned}$$

Proof. If we apply the first inequality in (2.1), we derive

$$\begin{aligned} & D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right) \\ & \leq (b-a)\left\|\left(\int_a^{\cdot}|g'(u)|^p\,du\right)'\right\|_{[a,b],\infty}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right) \end{aligned}$$

namely

$$D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\leq (b-a)\|g'\|_{[a,b],\infty}^p\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right).$$

By taking the power $1/p$ we get

$$\begin{aligned} & \left[D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p} \\ & \leq (b-a)^{1/p}\|g'\|_{[a,b],\infty}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/p}. \end{aligned}$$

Therefore

$$\begin{aligned}
&(b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q}\left[D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p}\\
&\leq (b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q}\\
&\times (b-a)^{1/p}\|g'\|_{[a,b],\infty}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/p}\\
&=(b-a)\|g'\|_{[a,b],\infty}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right),
\end{aligned}$$

which proves the last part of (2.12). □

Corollary 3. *With the assumptions of Theorem 3 we have*

$$\begin{aligned}
(2.14)\qquad |D(f,g)| &\leq (b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q}\\
&\times\left[D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p}\\
&\leq \frac{1}{2^{1/p}}(b-a)\|g'\|_{[a,b],p}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q}\\
&\times\left(\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt\right)^{1/p}\\
&\leq \frac{1}{2^{1/p}}(b-a)^{1+1/p}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q}\\
&\times\left[\frac{1}{b-a}\int_a^b f^2(t)-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)^2\right]^{\frac{1}{2p}}\|g'\|_{[a,b],p}\\
&\leq \frac{1}{4^{1/p}}(b-a)^{1+1/p}[f(b)-f(a)]^{1/p}\|g'\|_{[a,b],p}\\
&\times\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q}.
\end{aligned}$$

Proof. If we use (1.8) for $h=\int_a^{\cdot}|g'(u)|^p du$ and $k=f$ and observe that $0\le h\le \int_a^b |g'(u)|^p du$, then we get

$$\begin{aligned}
& D\left(f,\int_a^{\cdot}|g'(u)|^p du\right)\\
&\le \frac{1}{2}(b-a)\int_a^b |g'(u)|^p du \int_a^b \left| f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| dt\\
&\le \frac{1}{2}(b-a)^2\int_a^b |g'(u)|^p du \left[\frac{1}{b-a}\int_a^b f^2(t)-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)^2\right]^{1/2}\\
&\le \frac{1}{4}(b-a)^2\left[f(b)-f(a)\right]\int_a^b |g'(u)|^p du,
\end{aligned}$$

which implies that

$$\begin{aligned}
& \left[D\left(f,\int_a^{\cdot}|g'(u)|^p du\right)\right]^{1/p}\\
&\le \frac{1}{2^{1/p}}(b-a)^{1/p}\|g'\|_{[a,b],p}\left(\int_a^b \left| f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| dt\right)^{1/p}\\
&\le \frac{1}{2^{1/p}}(b-a)^{2/p}\|g'\|_{[a,b],p}\left[\frac{1}{b-a}\int_a^b f^2(t)-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)^2\right]^{\frac{1}{2p}}\\
&\le \frac{1}{4^{1/p}}(b-a)^{2/p}\left[f(b)-f(a)\right]^{1/p}\|g'\|_{[a,b],p}.
\end{aligned}$$

Then we get

$$\begin{aligned}
& (b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q}\\
&\times\left[D\left(f,\int_a^{\cdot}|g'(u)|^p du\right)\right]^{1/p}
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{2^{1/p}}(b-a)^{1/p}\|g'\|_{[a,b],p}(b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q}\\
&\left(\int_a^b \left| f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right| dt\right)^{1/p}
\end{aligned}$$

$$\leq \frac{1}{2^{1/p}}(b-a)^{2/p}\left\|g'\right\|_{[a,b],p}(b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q}$$
$$\left[\frac{1}{b-a}\int_a^b f^2(t)-\left(\frac{1}{b-a}\int_a^b f(s)\,ds\right)^2\right]^{\frac{1}{2p}}$$
$$\leq \frac{1}{4^{1/p}}(b-a)^{2/p}\left[f(b)-f(a)\right]^{1/p}\left\|g'\right\|_{[a,b],p}$$
$$\times(b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q},$$

which gives by (2.8) the desired result (2.14). □

Remark 1. *Since by (2.6) we have*

$$0\leq\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\leq\frac{1}{8}(b-a)^2\left[f(b)-f(a)\right], \tag{2.15}$$

then

$$\frac{1}{4^{1/p}}(b-a)^{1+1/p}\left[f(b)-f(a)\right]^{1/p}\left\|g'\right\|_{[a,b],p}$$
$$\times\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q}$$
$$\leq\frac{1}{4^{1/p}}(b-a)^{1+1/p}\left[f(b)-f(a)\right]^{1/p}\left\|g'\right\|_{[a,b],p}$$
$$\times\left(\frac{1}{8}(b-a)^2\left[f(b)-f(a)\right]\right)^{1/q}$$
$$=\frac{1}{4\cdot 2^{1/q}}(b-a)^{2+1/q}\left[f(b)-f(a)\right]\left\|g'\right\|_{[a,b],p}$$

and by (2.14) we get

$$|D(f,g)|\leq(b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q} \tag{2.16}$$
$$\times\left[D\left(f,\int_a^{\cdot}\left|g'(u)\right|^p du\right)\right]^{1/p}$$
$$\leq\frac{1}{4\cdot 2^{1/q}}(b-a)^{2+1/q}\left[f(b)-f(a)\right]\left\|g'\right\|_{[a,b],p},$$

where $p,\, q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

In particular, we have

$$|D(f,g)|\leq(b-a)^{1/2}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/2} \tag{2.17}$$
$$\times\left[D\left(f,\int_a^{\cdot}\left|g'(u)\right|^p du\right)\right]^{1/2}$$
$$\leq\frac{1}{4\cdot 2^{1/2}}(b-a)^{2+1/2}\left[f(b)-f(a)\right]\left\|g'\right\|_{[a,b],2}.$$

Corollary 4. *With the assumptions of Theorem 3 we have*

$$
\begin{aligned}
|D(f,g)| &\leq (b-a)^{1/q}\left(\int_a^b tf(t)\,dt - \frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q} \\
&\quad\times\left[D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p} \\
&\leq \frac{1}{8^{1/p}}(b-a)^{1+2/p}\|f'\|_\infty^{1/p}\|g'\|_{[a,b],p} \\
&\quad\times\left(\int_a^b tf(t)\,dt - \frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q} \\
&\leq \frac{1}{8}(b-a)^3\|f'\|_\infty^{1/p}[f(b)-f(a)]^{1/q}\|g'\|_{[a,b],p} \\
&\leq \frac{1}{8}(b-a)^{3+1/q}\|f'\|_\infty\|g'\|_{[a,b],p},
\end{aligned}
\tag{2.18}
$$

where $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. If we use Ostrowski's inequality (1.4) for $h=\int_a^{\cdot}|g'(u)|^p\,du$ and $k=f$ and observe that $0\leq h\leq\int_a^b|g'(u)|^p\,du$, then we get

$$D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\leq\frac{1}{8}(b-a)^3\|f'\|_\infty\int_a^b|g'(u)|^p\,du,$$

which implies that

$$\left[D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p}\leq\frac{1}{8^{1/p}}(b-a)^{3/p}\|f'\|_\infty^{1/p}\|g'\|_{[a,b],p}.$$

Then we get

$$
\begin{aligned}
&(b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q} \\
&\times\left[D\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p} \\
&\leq(b-a)^{1/q}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q} \\
&\times\frac{1}{8^{1/p}}(b-a)^{3/p}\|f'\|_\infty^{1/p}\|g'\|_{[a,b],p} \\
&=\frac{1}{8^{1/p}}(b-a)^{1+2/p}\|f'\|_\infty^{1/p}\|g'\|_{[a,b],p}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)^{1/q},
\end{aligned}
$$

which proves the second inequality in (2.18).

The third inequality follows by (2.15), while the last part follows by

$$0\leq f(b)-f(a)\leq(b-a)\|f'\|_\infty.$$

□

3. Applications for Perturbed Trapezoid Rule

Using integration by parts twice we have the trapezoid equality in terms of the second derivative:

$$\frac{h(a)+h(b)}{2}(b-a)-\int_a^b h(t)\,dt=\frac{1}{2}\int_a^b (b-t)(t-a)\,h''(t)\,dt, \tag{3.1}$$

provided that h' is absolutely continuous on $[a,b]$.

We also have

$$\begin{aligned}
&D((b-\cdot)(\cdot-a),h'') \\
&=(b-a)\int_a^b (b-t)(t-a)\,h''(t)\,dt-\int_a^b (b-t)(t-a)\,dt\int_a^b h''(t)\,dt \\
&=(b-a)\int_a^b (b-t)(t-a)\,h''(t)\,dt-\frac{1}{6}(b-a)^3\left[h'(b)-h'(a)\right],
\end{aligned}$$

namely

$$\begin{aligned}
&\frac{1}{2}\int_a^b (b-t)(t-a)\,h''(t)\,dt \\
&=\frac{1}{2(b-a)}D((b-\cdot)(\cdot-a),h'')+\frac{1}{12}(b-a)^2\left[h'(b)-h'(a)\right].
\end{aligned} \tag{3.2}$$

By (3.1) and (3.2) we get

$$\begin{aligned}
&\frac{h(a)+h(b)}{2}(b-a)-\int_a^b h(t)\,dt \\
&=\frac{1}{2(b-a)}D((b-\cdot)(\cdot-a),h'')+\frac{1}{12}(b-a)^2\left[h'(b)-h'(a)\right],
\end{aligned}$$

namely

$$\begin{aligned}
&\frac{h(a)+h(b)}{2}(b-a)-\frac{1}{12}(b-a)^2\left[h'(b)-h'(a)\right]-\int_a^b h(t)\,dt \\
&=\frac{1}{2(b-a)}D((b-\cdot)(\cdot-a),h'').
\end{aligned} \tag{3.3}$$

Let $f=h''$ and $g(t)=-t^2+(a+b)t-ab$, then

$$g'(t)=a+b-2t=2\left(\frac{a+b}{2}-t\right),$$

which implies that

$$\|g'\|_{[a,b],\infty}\le b-a.$$

By using the inequality (2.1), we get

$$\begin{aligned}
&\left|D\left((b-\cdot)(\cdot-a),h''\right)\right| \\
&\leq (b-a)^2\left(\int_a^b th''(t)\,dt-\frac{a+b}{2}\int_a^b h''(t)\,dt\right) \\
&\leq \frac{1}{2}(b-a)^3\int_a^b\left|h''(t)-\frac{1}{b-a}\int_a^b h''(s)\,ds\right|dt \\
&\leq \frac{1}{2}(b-a)^3\left[\frac{1}{b-a}\int_a^b [h''(t)]^2\,dt-\left(\frac{1}{b-a}\int_a^b h''(s)\,ds\right)^2\right]^{1/2},
\end{aligned}$$

namely

$$\begin{aligned}
&\left|D\left((b-\cdot)(\cdot-a),h''\right)\right| \\
&\leq (b-a)^2\left(\frac{h'(a)+h'(b)}{2}(b-a)-[h(b)-h(a)]\right) \\
&\leq \frac{1}{2}(b-a)^3\int_a^b\left|h''(t)-\frac{h'(b)-h'(a)}{b-a}\right|dt \\
&\leq \frac{1}{2}(b-a)^3\left[\frac{1}{b-a}\int_a^b [h''(t)]^2\,dt-\left(\frac{h'(b)-h'(a)}{b-a}\right)^2\right]^{1/2}.
\end{aligned}$$

By dividing with $2(b-a)$, we get

$$\begin{aligned}
&\frac{1}{2(b-a)}\left|D\left((b-\cdot)(\cdot-a),h''\right)\right| \\
&\leq \frac{1}{2}(b-a)\left(\frac{h'(a)+h'(b)}{2}(b-a)-[h(b)-h(a)]\right) \\
&\leq \frac{1}{4}(b-a)^2\int_a^b\left|h''(t)-\frac{h'(b)-h'(a)}{b-a}\right|dt \\
&\leq \frac{1}{4}(b-a)^2\left[\frac{1}{b-a}\int_a^b [h''(t)]^2\,dt-\left(\frac{h'(b)-h'(a)}{b-a}\right)^2\right]^{1/2}.
\end{aligned}$$

By (3.3) then get

$$\begin{aligned}
&\left|\frac{h(a)+h(b)}{2}(b-a)-\frac{1}{12}(b-a)^2[h'(b)-h'(a)]-\int_a^b h(t)\,dt\right| \\
&\leq \frac{1}{2}(b-a)\left(\frac{h'(a)+h'(b)}{2}(b-a)-[h(b)-h(a)]\right) \\
&\leq \frac{1}{4}(b-a)^2\int_a^b\left|h''(t)-\frac{h'(b)-h'(a)}{b-a}\right|dt \\
&\leq \frac{1}{4}(b-a)^2\left[\frac{1}{b-a}\int_a^b [h''(t)]^2\,dt-\left(\frac{h'(b)-h'(a)}{b-a}\right)^2\right]^{1/2}
\end{aligned} \tag{3.4}$$

for any twice differentiable function h for which the second derivative h'' is monotonic nondecreasing on $[a,b]$.

By using inequality (2.3) $f = h''$ and $g(t) = -t^2 + (a+b)t - ab$ we have

$$
\begin{aligned}
&|D((b-\cdot)(\cdot-a), h'')| \\
&\leq (b-a)^2 \left(\frac{h'(a)+h'(b)}{2}(b-a) - [h(b)-h(a)] \right) \\
&\leq \frac{1}{2}(b-a)^2 \frac{\int_a^b [h''(t)-h''(a)]\,dt \int_a^b [h''(b)-h''(t)]\,dt}{h''(b)-h''(a)} \\
&\leq \frac{1}{8}(b-a)^4 [h''(b)-h''(a)],
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\left| \frac{h(a)+h(b)}{2}(b-a) - \frac{1}{12}(b-a)^2[h'(b)-h'(a)] - \int_a^b h(t)\,dt \right| \qquad (3.5)\\
&\leq \frac{1}{2}(b-a)^2 \left(\frac{h'(a)+h'(b)}{2}(b-a) - [h(b)-h(a)] \right) \\
&\leq \frac{1}{4}(b-a) \frac{\int_a^b [h''(t)-h''(a)]\,dt \int_a^b [h''(b)-h''(t)]\,dt}{h''(b)-h''(a)} \\
&\leq \frac{1}{16}(b-a)^3 [h''(b)-h''(a)],
\end{aligned}
$$

for any twice differentiable function h for which the second derivative h'' is monotonic nondecreasing on $[a,b]$ and $h''(b) - h''(a) > 0$.

4. Applications for Norms and Semi-Inner Products

Let X be a real linear space, $a, b \in X$, $a \neq b$ and let $[a,b] := \{(1-\lambda)a + \lambda b,\ \lambda \in [0,1]\}$ be the *segment* generated by a and b. We consider the function $f : [a,b] \to \mathbb{R}$ and the attached function $g(a,b) : [0,1] \to \mathbb{R}$, $g(a,b)(t) := f[(1-t)a + tb]$, $t \in [0,1]$.

It is well known that f is convex on $[a,b]$ iff $g(a,b)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_{\pm}(a,b)(s) = \triangledown_{\pm} f[(1-s)a + sb](b-a)$, $s \in [0,1)$,
(ii) $g'_{+}(a,b)(0) = \triangledown_{+} f(a)(b-a)$,
(iii) $g'_{-}(a,b)(1) = \triangledown_{-} f(b)(b-a)$,

where $\triangledown_{\pm} f(x)(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$
\begin{aligned}
\triangledown_{+} f(x)(y) &:= \lim_{h \to 0+} \frac{f(x+hy) - f(x)}{h}, \\
\triangledown_{-} f(x)(y) &:= \lim_{k \to 0-} \frac{f(x+ky) - f(x)}{k}, \quad x,\ y \in X.
\end{aligned}
$$

The following inequality is the well-known Hermite-Hadamard integral inequality for convex functions defined on a segment $[a,b] \subset X$:

$$
f\left(\frac{a+b}{2}\right) \leq \int_0^1 f[(1-t)a + tb]\,dt \leq \frac{f(a)+f(b)}{2}, \qquad \text{(HH)}
$$

which easily follows by the classical Hermite-Hadamard inequality for the convex function $g(a,b) : [0,1] \to \mathbb{R}$

$$
g(a,b)\left(\frac{1}{2}\right) \leq \int_0^1 g(a,b)(t)\,dt \leq \frac{g(a,b)(0) + g(a,b)(1)}{2}.
$$

For other related results see the monograph on line [6].

Now, assume that $(X, \|\cdot\|)$ is a normed linear space. The function $f_0(s) = \frac{1}{2}\|x\|^2$, $x \in X$ is convex and thus the following limits exist

(iv) $\langle x, y\rangle_s := \nabla_+ f_0(y)(x) = \lim_{t\to 0+} \frac{\|y+tx\|^2 - \|y\|^2}{2t}$;

(v) $\langle x, y\rangle_i := \nabla_- f_0(y)(x) = \lim_{s\to 0-} \frac{\|y+sx\|^2 - \|y\|^2}{2s}$;

for any x, $y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [21]), assuming that p, $q \in \{s, i\}$ and $p \neq q$:

(a) $\langle x, x\rangle_p = \|x\|^2$ for all $x \in X$;
(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta \langle x, y\rangle_p$ if α, $\beta \geq 0$ and x, $y \in X$;
(aaa) $\left|\langle x, y\rangle_p\right| \leq \|x\| \|y\|$ for all x, $y \in X$;
(av) $\langle \alpha x + y, x\rangle_p = \alpha \langle x, x\rangle_p + \langle y, x\rangle_p$ if x, $y \in X$ and $\alpha \in \mathbb{R}$;
(v) $\langle -x, y\rangle_p = -\langle x, y\rangle_q$ for all x, $y \in X$;
(va) $\langle x + y, z\rangle_p \leq \|x\| \|z\| + \langle y, z\rangle_p$ for all x, y, $z \in X$;
(vaa) The mapping $\langle \cdot, \cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);
(vaaa) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X \setminus \{0\}$ if and only if $\langle y, x_0\rangle_s = \langle y, x_0\rangle_i$ for all $y \in X$; in general $\langle y, x\rangle_i \leq \langle y, x\rangle_s$ for all x, $y \in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot\rangle$, then $\langle y, x\rangle_i = \langle y, x\rangle = \langle y, x\rangle_s$ for all x, $y \in X$.

Applying inequality (HH) for the convex function $f_0(x) = \frac{1}{2}\|x\|^2$, one may deduce the inequality

$$\left\|\frac{x+y}{2}\right\|^2 \leq \int_0^1 \|(1-t)x + ty\|^2 \, dt \leq \frac{\|x\|^2 + \|y\|^2}{2} \tag{4.1}$$

for any $x, y \in X$. The same (HH) inequality applied for $f_1(x) = \|x\|$, will give the following refinement of the triangle inequality:

$$\left\|\frac{x+y}{2}\right\| \leq \int_0^1 \|(1-t)x + ty\| \, dt \leq \frac{\|x\| + \|y\|}{2}, \quad x, \ y \in X. \tag{4.2}$$

Assume that $f : C \subset X \to \mathbb{R}$ is convex on the convex subset C in the linear space X. Consider the function

$$h_{x,y}(t) = \int_0^t f[(1-s)x + sy] \, ds, \ t \in [0, 1]$$

defined for distinct x, $y \in C$. Then

$$h'_{x,y}(t) = f[(1-t)x + ty], \ t \in [0, 1].$$

We have

$$\begin{aligned}\int_0^1 h_{x,y}(t)\,dt &= \int_0^1\left(\int_0^t f[(1-s)x+sy]\,ds\right)dt\\ &= t\int_0^t f[(1-s)x+sy]\,ds\Big|_0^1 - \int_0^1 tf[(1-t)x+ty]\,dt\\ &= \int_0^1 f[(1-s)x+sy]\,ds - \int_0^1 tf[(1-t)x+ty]\,dt.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{h_{x,y}(0)+h_{x.y}(1)}{2} - \frac{1}{12}\left[h'_{x,y}(1)-h'_{x,y}(0)\right] - \int_0^1 h_{x,y}(t)\,dt\\ &= \frac{1}{2}\int_0^1 f[(1-s)x+sy]\,ds - \frac{1}{12}[f(y)-f(x)]\\ &- \int_0^1 f[(1-s)x+sy]\,ds + \int_0^1 tf[(1-t)x+ty]\,dt\\ &= \int_0^1 tf[(1-t)x+ty]\,dt - \frac{1}{2}\int_0^1 f[(1-s)x+sy]\,ds\\ &- \frac{1}{12}[f(y)-f(x)].\end{aligned}$$

We then get from (3.5) that

$$\begin{aligned}&\left|\int_0^1 tf[(1-t)x+ty]\,dt - \frac{1}{2}\int_0^1 f[(1-s)x+sy]\,ds\right.\\ &\left.-\frac{1}{12}[f(y)-f(x)]\right|\\ &\le \frac{1}{2}\left(\frac{f(y)+f(x)}{2} - \int_0^1 f[(1-s)x+sy]\,ds\right)\\ &\le \frac{1}{4}\frac{[f(y)-f(x)-\nabla_+ f(x)(y-x)][\nabla_- f(y)(y-x)-f(y)+f(x)]}{\nabla_- f(y)(y-x)-\nabla_+ f(x)(y-x)}\\ &\le \frac{1}{16}[\nabla_- f(y)(y-x)-\nabla_+ f(x)(y-x)],\end{aligned}\tag{4.3}$$

provided that $\nabla_- f(y)(y-x)-\nabla_+ f(x)(y-x)>0$.

For $p\ge 1$ the function $f_p(x)=\|x\|^p$ is convex on X. Therefore

$$\nabla_{+(-)} f_p(y)(x) = p\|y\|^{p-2}\langle x,y\rangle_{s(i)},\tag{4.4}$$

which exists for all x, $y\in X$ whenever $p\ge 2$. If $1\le p<2$ the equality (4.4) for all $x\in X$ and nonzero $y\in X$.

If we use the inequality (4.3) for the function f_p, then we get

$$0 \leq \left| \int_0^1 t \left\| (1-t)x + ty \right\|^p dt - \frac{1}{2} \int_0^1 \left\| (1-s)x + sy \right\|^p ds - \frac{1}{12} \left[\|y\|^p - \|x\|^p \right] \right| \tag{4.5}$$
$$\leq \frac{1}{2} \left(\frac{\|x\|^p + \|y\|^p}{2} - \int_0^1 \left\| (1-t)x + ty \right\|^p dt \right)$$
$$\leq \frac{\left[\|y\|^p - \|x\|^p - p\|x\|^{p-2} \langle y-x, x \rangle_s \right] \left[p\|y\|^{p-2} \langle y-x, y \rangle_i - \|y\|^p + \|x\|^p \right]}{2p \left(\|y\|^{p-2} \langle y-x, y \rangle_i - \|x\|^{p-2} \langle y-x, x \rangle_s \right)}$$
$$\leq \frac{1}{16} p \left(\|y\|^{p-2} \langle y-x, y \rangle_i - \|x\|^{p-2} \langle y-x, x \rangle_s \right),$$

which holds for all $x, y \in X$ whenever $p \geq 2$ and for $y \neq 0$ when $1 \leq p < 2$.

In particular, for $p = 2$ we get

$$0 \leq \left| \int_0^1 t \left\| (1-t)x + ty \right\|^2 dt - \frac{1}{2} \int_0^1 \left\| (1-s)x + sy \right\|^2 ds - \frac{1}{12} \left[\|y\|^2 - \|x\|^2 \right] \right| \tag{4.6}$$
$$\leq \frac{1}{2} \left(\frac{\|x\|^2 + \|y\|^2}{2} - \int_0^1 \left\| (1-t)x + ty \right\|^2 dt \right)$$
$$\leq \frac{\left[\|y\|^2 - \|x\|^2 - 2\langle y-x, x \rangle_s \right] \left[2\langle y-x, y \rangle_i - \|y\|^2 + \|x\|^2 \right]}{4 \left(\langle y-x, y \rangle_i - \langle y-x, x \rangle_s \right)}$$
$$\leq \frac{1}{8} \left(\langle y-x, y \rangle_i - \langle y-x, x \rangle_s \right),$$

for all $x, y \in X$.

For $p = 1$ we obtain

$$0 \leq \left| \int_0^1 t \left\| (1-t)x + ty \right\| dt - \frac{1}{2} \int_0^1 \left\| (1-s)x + sy \right\| ds - \frac{1}{12} \left[\|y\| - \|x\| \right] \right| \tag{4.7}$$
$$\leq \frac{1}{2} \left(\frac{\|x\| + \|y\|}{2} - \int_0^1 \left\| (1-t)x + ty \right\| dt \right)$$
$$\leq \frac{\left[\|y\| - \|x\| - \langle y-x, x \rangle_s \|x\|^{-1} \right] \left[\langle y-x, y \rangle_i \|y\|^{-1} - \|y\| + \|x\| \right]}{2 \left(\langle y-x, y \rangle_i \|y\|^{-1} - \langle y-x, x \rangle_s \|x\|^{-1} \right)}$$
$$\leq \frac{1}{16} \left(\langle y-x, y \rangle_i \|y\|^{-1} - \langle y-x, x \rangle_s \|x\|^{-1} \right),$$

for all $x, y \in X$ with $y \neq 0$.

References

[1] G. A. Anastassiou, Complex multivariate Fink type identity applied to complex multivariate Ostrowski and Grüss inequalities. *Indian J. Math.* **61** (2019), no. 2, 199–237.

[2] G. A. Anastassiou, Selfadjoint operator Chebyshev-Grüss type inequalities. *Appl. Math.* (Warsaw) **46** (2019), no. 1, 99–114.

[3] N. S. Barnett, P. Cerone, S. S. Dragomir and C. Buşe, Some Grüss type inequalities for vector-valued functions in Banach spaces and applications. *Tamsui Oxf. J. Math. Sci.* **23** (2007), no. 1, 91–103. Preprint *RGMIA Res. Rep. Coll.* **8** (2005), No. 2, Art. 12 [Online https://rgmia.org/papers/v8n2/GTIVVFBSApp.pdf].

[4] H. Budak and M. Z. Sarikaya, On weighted Grüss type inequalities for double integrals. *Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.* **66** (2017), no. 2, 53–61.

[5] H. Budak and M. Z. Sarikaya, An inequality of Ostrowski-Grüss type for double integrals. *Stud. Univ. Babeş-Bolyai Math.* **62** (2017), no. 2, 163–173.

[6] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, 1990.

[7] P. Cerone and S. S. Dragomir, New bounds for the Čebyšev functional, *Appl. Math. Lett.,* **18** (2005), 603-611.

[8] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.* **38**(2007), No. 1, 37-49. Preprint available at *RGMIA Res. Rep. Coll.*, **5**(2) (2002), Art. 14. [ONLINE http://rgmia.vu.edu.au/v5n2.html].

[9] X.-L. Cheng and J. Sun, Note on the perturbed trapezoid inequality, *J. Inequal. Pure Appl. Math.,* **3**(2) (2002), Art. 29. [ONLINEhttp://jipam.vu.edu.au/article.php?sid=181

[10] P. L. Chebyshev, Sur les expressions approximatives des intègrals dèfinis par les outres prises entre les même limites, *Proc. Math. Soc. Charkov,* **2** (1882), 93-98.

[11] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31, 8 pp. [Online http://www.emis.de/journals/JIPAM/article183.html?sid=183].

[12] S. S. Dragomir, Improvements of Ostrowski and generalised trapezoid inequality in terms of the upper and lower bounds of the first derivative. *Tamkang J. Math.* **34** (2003), No. 3, 213–222.

[13] S. S. Dragomir, A refinement of Ostrowski's inequality for the Čebyšev functional and applications. *Analysis* (Munich) **23** (2003), no. 4, 287–297.

[14] S. S. Dragomir, Inequalities of Grüss type for the Stieltjes integral and applications, *Kragujevac J. Math.*, **26** (2004), 89-112.

[15] S. S. Dragomir, Inequalities for Stieltjes integrals with convex integrators and applications, *Appl. Math. Lett.,* **20** (2007), 123-130.

[16] S. S. Dragomir, New Grüss' type inequalities for functions of bounded variation and applications. *Appl. Math. Lett.* **25** (2012), no. 10, 1475–1479.

[17] S. S. Dragomir, Inequalities of Lipschitz type for power series in Banach algebras. *Ann. Math. Sil.* No. **29** (2015), 61–83.

[18] S. S. Dragomir, General Lebesgue integral inequalities of Jensen and Ostrowski type for differentiable functions whose derivatives in absolute value are convex and applications. *Ann. Univ. Mariae Curie-Skłodowska Sect.* A **69** (2015), no. 2, 17–45.

[19] S. S. Dragomir, Integral inequalities for Lipschitzian mappings between two Banach spaces and applications. *Kodai Math. J.* **39** (2016), no. 1, 227–251.

[20] S. S. Dragomir, On some Grüss' type inequalities for the complex integral. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **113** (2019), no. 4, 3531–3543.

[21] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. (ONLINE: http://rgmia.vu.edu.au/monographs)

[22] J. V. da C. Sousa, D. S. Oliveira and E. de Oliveira, Capelas Grüss-type inequalities by means of generalized fractional integrals. *Bull. Braz. Math. Soc.* (N.S.) **50** (2019), no. 4, 1029–1047.

[23] G. Grüss, Über das maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b h(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b h(x)\,dx \cdot \int_a^b g(x)\,dx$, *Math. Z.,* **39** (1934), 215-226.

[24] S. Joshi, E. Mittal, R. M. Pandey and S. D. Purohit, Some Grüss type inequalities involving generalized fractional integral operator. *Bull. Transilv. Univ. Braşov Ser. III* **12** (**61**) (2019), no. 1, 41–52.

[25] S. Kermausuor and E. R. Nwaeze, New Ostrowski and Ostrowski-Grüss type inequalities for double integrals on time scales involving a combination of Δ-integral means. *Tamkang J. Math.* **49** (2018), no. 4, 277–289.

[26] A. Lupaş, The best constant in an integral inequality, *Mathematica* (Cluj), **15** (**38**) (1973), No. 2, 219-222.

[27] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht, 1993.

[28] E. R. Nwaeze, Generalized weighted trapezoid and Grüss type inequalities on time scales. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 4, 13 pp.

[29] E. R. Nwaeze, N. Kaplan, F. G. Tuna, and A. Tuna, Some new inequalities of the Ostrowski-Grüss, Čebyšev, and trapezoid types on time scales. *J. Nonlinear Sci. Appl.* **12** (2019), no. 4, 192–205.

[30] A. M. Ostrowski, On an integral inequality, *Aequationes Math.,* **4** (1970), 358-373.

[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: `sever.dragomir@vu.edu.au`

URL: `http://rgmia.org/dragomir`

[2]DST-NRF Centre of Excellence, in the Mathematical and Statistical Sciences, School of Computer Science & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa