NEW BOUNDS FOR THE ČEBYŠEV FUNCTIONAL OF TWO ABSOLUTELY CONTINUOUS FUNCTIONS WITH APPLICATIONS TO FINITE FOURIER TRANSFORM

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Abstract. For two Lebesgue integrable functions h, $k : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(h,k) := (b-a)\int_a^b h(t)\,k(t)\,dt - \int_a^b h(t)\,dt \int_a^b k(t)\,dt.$$

In this paper we show among others that, if f, $g : [a,b] \to \mathbb{C}$ are absolutely continuous on $[a,b]$, then for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} |D(f,g)| &\le \left[D\left(\ell, \int_a^{\cdot} |f'(u)|^p\,du\right)\right]^{1/p} \left[D\left(\ell, \int_a^{\cdot} |g'(u)|^q\,du\right)\right]^{1/q} \\ &= \frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\,|f'(t)|^p\,dt\right]^{1/p} \\ &\times \left[\int_a^b (b-t)(t-a)\,|g'(t)|^q\,dt\right]^{1/q} \\ &\le \frac{1}{8}(b-a)^3 \|f'\|_{[a,b],p} \|g'\|_{[a,b],q}, \end{aligned}$$

where $\|\cdot\|_{[a,b],p}$ is the usual Lebesgue p-norm. Applications for finite Fourier transform are also provided.

1. Introduction

For two Lebesgue integrable functions h, $k : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(h,k) := (b-a)\int_a^b h(t)\,k(t)\,dt - \int_a^b h(t)\,dt \int_a^b k(t)\,dt.$$

In 1934, G. Grüss [24] showed that

$$|D(h,k)| \le \frac{1}{4}(b-a)^2 (M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \le h \le M < \infty, \quad -\infty < n \le k \le N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

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Another lesser known inequality for $D(h,k)$ was derived in 1882 by Čebyšev [11] under the assumption that h', k' exist and are continuous on $[a,b]$, and is given by

$$|D(h,k)| \leq \frac{1}{12} \|h'\|_{\infty} \|k'\|_{\infty} (b-a)^{4}, \tag{1.3}$$

where $\|h'\|_{\infty} := \sup_{t\in[a,b]} |h'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if h, $k : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and h', $k' \in L_{\infty}[a,b]$.

In 1970, A. M. Ostrowski [32] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(h,k)| \leq \frac{1}{8}(b-a)^{3}(M-m)\|k'\|_{\infty}, \tag{1.4}$$

provided h is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $k : [a,b] \to \mathbb{R}$ is absolutely continuous and $k' \in L_{\infty}[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [28] (see also [29, p. 210]) obtained the following result as well:

$$|D(h,k)| \leq \frac{1}{\pi^{2}} \|h'\|_{2} \|k'\|_{2} (b-a)^{3}, \tag{1.5}$$

provided h, k are absolutely continuous and h', $k' \in L_{2}[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [8], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(h,k)| \leq (b-a) \times \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|h-\gamma\|_{\infty} \int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right| dt, \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|h-\gamma\|_{q} \left(\int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right|^{p} dt\right)^{\frac{1}{p}}, \\ \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(h,k)| \leq (b-a)\|h\|_{\infty} \int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \leq h \leq M$ for a.e. $x \in [a,b]$, then $\left\|h - \frac{m+M}{2}\right\|_{\infty} \leq \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [10]

$$|D(h,k)| \leq \frac{1}{2}(b-a)(M-m)\int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [9].

The following result holds [17].

Theorem 1. *Let $h : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $k : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(h,g)| \leq \frac{1}{2}(b-a)\bigvee_a^b(h)\int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right| dt,$$

where $\bigvee_a^b (f)$ *denotes the total variation of* f *on the interval* $[a,b]$. *The constant* $\frac{1}{2}$ *is best possible in (1.9).*

For more recent upper bounds related to the Čebyšev functional see [8], [9] and [15]-[17].

In [14] we obtained the following refinement of Ostrowski's inequality (1.4)

$$
\begin{aligned}
(1.9)\quad |D(h,k)| &\le \frac{1}{2}(b-a)^3 \frac{\left(\frac{1}{b-a}\int_a^b h(t)\,dt - m\right)\left(M - \frac{1}{b-a}\int_a^b h(t)\,dt\right)}{M-m}\|k'\|_\infty \\
&\le \frac{1}{8}(b-a)^3 (M-m)\|k'\|_\infty ,
\end{aligned}
$$

provided $m \le h \le M$ a.e. on $[a,b]$ and k is absolutely continuous on $[a,b]$.

In this paper we show among others that, if $f,\ g : [a,b] \to \mathbb{C}$ are absolutely continuous on $[a,b]$, then for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$
\begin{aligned}
|D(f,g)| &\le \left[D\left(\ell, \int_a^{\cdot} |f'(u)|^p\,du\right)\right]^{1/p}\left[D\left(\ell, \int_a^{\cdot} |g'(u)|^q\,du\right)\right]^{1/q} \\
&= \frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)|f'(t)|^p\,dt\right]^{1/p} \\
&\quad\times \left[\int_a^b (b-t)(t-a)|g'(t)|^q\,dt\right]^{1/q} \\
&\le \frac{1}{8}(b-a)^3 \|f'\|_{[a,b],p}\|g'\|_{[a,b],q},
\end{aligned}
$$

where $\|\cdot\|_{[a,b],p}$ is the usual Lebesgue p-norm. Applications for finite Fourier transform are also provided.

2. Main Results

For two continuous functions $f,\ g : [a,b] \to \mathbb{C}$ we define the *Čebyšev functional*

$$D(f,g) := (b-a)\int_a^b f(t)\,g(t)\,dt - \int_a^b f(t)\,dt \int_a^b g(t)\,dt.$$

We have the following result of interest:

Theorem 2. *Let* $f,\ g : [a,b] \to \mathbb{C}$ *be absolutely continuous on* $[a,b]$. *If* $\|f'\|_{[a,b],\infty} := \sup_{t\in(a,b)} \|f'(t)\| < \infty$, *then*

$$
\begin{aligned}
(2.1)\quad |D(f,g)| &\le \|f'\|_{[a,b],\infty} D\left(\ell, \int_a^{\cdot} |g'(u)|\,du\right) \\
&\le \frac{1}{8}(b-a)^3 \|f'\|_{[a,b],\infty}\|g'\|_{[a,b],1},
\end{aligned}
$$

where $\|g'\|_{[a,b],1} := \int_a^b |g'(u)|\,du$ *and* $\ell(t) = t,\ t \in [a,b]$.

Proof. Observe that

$$\begin{aligned}
&\int_a^b \int_a^b [f(t)-f(s)][g(t)-g(s)]\,dtds \\
&= \int_a^b \int_a^b (f(t)g(t)-f(s)g(t)-f(t)g(s)+f(s)g(s))\,dtds \\
&= (b-a)\int_a^b f(t)g(t)\,dt - \int_a^b f(s)\,ds \int_a^b g(t)\,dt \\
&- \int_a^b f(t)\,dt \int_a^b g(s)\,ds + (b-a)\int_a^b f(s)g(s)\,ds \\
&= 2(b-a)\int_a^b f(t)g(t)\,dt - 2\int_a^b f(t)\,dt \int_a^b g(t)\,dt = 2D(f,g),
\end{aligned}$$

which give the Korkine's identity for functions with complex values

$$D(f,g) = \frac{1}{2}\int_a^b \int_a^b [f(t)-f(s)][g(t)-g(s)]\,dtds.$$

For Korkine's classical identity for real-valued functions, see [29, p. 242].

If we take the norm and use the integral's properties, we get

$$\begin{aligned}
|D(f,g)| &\le \frac{1}{2}\int_a^b \int_a^b |[f(t)-f(s)][g(t)-g(s)]|\,dtds \\
&= \frac{1}{2}\int_a^b \int_a^b |f(t)-f(s)|\,|g(t)-g(s)|\,dtds.
\end{aligned} \tag{2.2}$$

Observe that for $s,\ t\in[a,b]$

$$f(t)-f(s) = \int_s^t f'(u)\,du,\ \ g(t)-g(s) = \int_s^t g'(u)\,du,$$

which implies that

$$\begin{aligned}
|f(t)-f(s)|\,|g(t)-g(s)| &= \left|\int_s^t f'(u)\,du\right| \left|\int_s^t g'(u)\,du\right| \\
&\le \left|\int_s^t |f'(u)|\,du\right| \left|\int_s^t |g'(u)|\,du\right| \\
&\le \sup_{t\in(a,b)} |f'(u)|\,|t-s| \left|\int_s^t |g'(u)|\,du\right| \\
&= \sup_{t\in(a,b)} |f'(u)|\,(t-s)\int_s^t |g'(u)|\,du,
\end{aligned}$$

for all $s,\ t\in[a,b]$.

By (2.2) we get

$$|D(f,g)| \le \sup_{t\in(a,b)} |f'(u)|\,\frac{1}{2}\int_a^b \int_a^b (t-s)\left(\int_s^t |g'(u)|\,du\right) dtds. \tag{2.3}$$

Since

$$(t-s)\left(\int_s^t |g'(u)|\,du\right) = (t-s)\left(\int_a^t |g'(u)|\,du - \int_a^s |g'(u)|\,du\right),$$

hence by Korkine's identity for real valued functions $f(t) = \ell(t)$ and $g(t) = \int_a^t |g'(u)|\, du$, we have

$$\frac{1}{2}\int_a^b\int_a^b (t-s)\left(\int_s^t |g'(u)|\, du\right) = (b-a)\int_a^b \ell(t)\left(\int_a^t |g'(u)|\, du\right) dt - \int_a^b \ell(t)\, dt \int_a^b \left(\int_a^t |g'(u)|\, du\right) dt = D\left(\ell, \int_a^{\cdot} |g'(u)|\, du\right). \tag{2.4}$$

By utilising (2.3) and (2.4), we deduce the first inequality in (2.1).

Observe that

$$0 \le \int_a^t |g'(u)|\, du \le \int_a^b |g'(u)|\, du$$

for all $t \in [a,b]$, then by (1.8) for the functions $f(t) = \ell(t)$ and $g(t) = \int_a^t |g'(u)|\, du$, $t \in [a,b]$, we get

$$\left|D\left(\ell, \int_a^{\cdot} |g'(u)|\, du\right)\right| \le \frac{1}{2}(b-a)\int_a^b |g'(u)|\, du \int_a^b \left|t - \frac{1}{b-a}\int_a^b s ds\right| dt = \frac{1}{2}(b-a)\int_a^b |g'(u)|\, du \int_a^b \left|t - \frac{a+b}{2}\right| dt = \frac{1}{8}(b-a)^3\int_a^b |g'(u)|\, du,$$

which proves the last part of (2.1). □

Remark 1. *If we apply the same inequality (1.8) for the functions $f(t) = \int_a^t |g'(u)|\, du$ and $g(t) = \ell(t)$, $t \in [a,b]$, then we get*

$$\left|D\left(\ell, \int_a^{\cdot} |g'(u)|\, du\right)\right| \le \frac{1}{2}(b-a)^2\int_a^b \left|\int_a^t |g'(u)|\, du - \frac{1}{b-a}\int_a^b \left(\int_a^s |g'(u)|\, du\right) ds\right| dt. \tag{2.5}$$

Observe that

$$\int_a^b \left|\int_a^t |g'(u)|\, du - \frac{1}{b-a}\int_a^b \left(\int_a^s |g'(u)|\, du\right) ds\right| dt$$
$$= \int_a^b \left|\int_a^t |g'(u)|\, du - \frac{1}{b-a}\left(\left(\int_a^b |g'(u)|\, du\right) b - \int_a^b |g'(u)|\, s ds\right)\right| dt$$
$$= \int_a^b \left|\int_a^t |g'(u)|\, du - \frac{1}{b-a}\left(\int_a^b (b-u)\, |g'(u)|\, du\right)\right| dt$$
$$= \frac{1}{b-a}\int_a^b \left|(b-a)\int_a^t |g'(u)|\, du - \int_a^b (b-u)\, |g'(u)|\, du\right| dt$$
$$= \frac{1}{b-a}\int_a^b \left|\int_a^t (u-a)\, |g'(u)|\, du - \int_t^b (b-u)\, |g'(u)|\, du\right| dt.$$

Then by (2.1) and (2.5) we obtain

$$\begin{aligned}|D(f,g)| &\leq \|f'\|_{[a,b],\infty} D\left(\ell, \int_a^{\cdot} |g'(u)|\,du\right) \\ &\leq \frac{1}{2}(b-a)\|f'\|_{[a,b],\infty} \\ &\quad\times \int_a^b \left| \int_a^t (u-a)|g'(u)|\,du - \int_t^b (b-u)|g'(u)|\,du \right| dt.\end{aligned} \tag{2.6}$$

Remark 2. *Using (1.3) we have*

$$0 \leq D\left(\ell, \int_a^{\cdot} |g'(u)|\,du\right) \leq \frac{1}{12} \sup_{t\in(a,b)} |g'(u)|(b-a)^4,$$

and by (2.1) we derive

$$\begin{aligned}|D(f,g)| &\leq \|f'\|_{[a,b],\infty} D\left(\ell, \int_a^{\cdot} |g'(u)|\,du\right) \\ &\leq \frac{1}{12}\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\infty}(b-a)^4\end{aligned} \tag{2.7}$$

provided that $\|f'\|_{[a,b],\infty}, \|g'\|_{[a,b],\infty} < \infty$.

Using (1.4) we have

$$0 \leq D\left(\ell, \int_a^{\cdot} |g'(u)|\,du\right) \leq \frac{1}{8}(b-a)^3 \int_a^b |g'(u)|\,du,$$

and by (2.1) we obtain

$$\begin{aligned}|D(f,g)| &\leq \|f'\|_{[a,b],\infty} D\left(\ell, \int_a^{\cdot} |g'(u)|\,du\right) \\ &\leq \frac{1}{8}(b-a)^3\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],1},\end{aligned} \tag{2.8}$$

provided that $\|f'\|_{[a,b],\infty} < \infty$.

Corollary 1. *Let* $f, g : [a,b] \to \mathbb{C}$ *be absolutely continuous on* $[a,b]$. *If*

$$\|g'\|_{[a,b],r} := \left(\int_a^b |g'(u)|^r\,du\right)^{1/r}, \ r \geq 1,$$

then

$$\begin{aligned}&|D(f,g)| \\ &\leq \frac{1}{2}(b-a)\|f'\|_{[a,b],\infty}\int_a^b (b-t)(t-a)|g'(t)|\,dt \\ &\leq \frac{1}{2}(b-a)\|f'\|_{[a,b],\infty} \times \begin{cases} \frac{1}{4}(b-a)^2\|g'\|_{[a,b],1}, \\ \\ (b-a)^{2+1/q}[B(q+1,q+1)]^{1/q}\|g'\|_{[a,b],p}, \\ \\ \frac{1}{6}(b-a)^3\|g'\|_{[a,b],\infty}, \end{cases}\end{aligned} \tag{2.9}$$

where $B(\cdot,\cdot)$ *is Beta function and* $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Observe that, integrating by parts, we have

$$
\begin{aligned}
&\frac{1}{2}\int_a^b (b-t)(t-a)\left|g'(t)\right| dt \\
&= \frac{1}{2}\int_a^b (b-t)(t-a)\, d\left(\int_a^t \left|g'(u)\right| du\right) \\
&= \frac{1}{2}\left[(b-t)(t-a)\int_a^t \left|g'(u)\right| du\Big|_a^b + \int_a^b (2t-a-b)\left(\int_a^t \left|g'(u)\right| du\right) dt\right] \\
&= \int_a^b \left(t-\frac{a+b}{2}\right)\left(\int_a^t \left|g'(u)\right| du\right) dt \\
&= \int_a^b t\left(\int_a^t \left|g'(u)\right| du\right) dt - \frac{a+b}{2}\int_a^b \left(\int_a^t \left|g'(u)\right| du\right) dt \\
&= \frac{1}{b-a} D\left(\ell, \int_a^{\cdot} \left|g'(u)\right| du\right),
\end{aligned}
$$

namely

$$
D\left(\ell, \int_a^{\cdot} \left|g'(u)\right| du\right) = \frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\left|g'(t)\right| dt. \tag{2.10}
$$

By utilising the first inequality (2.1) we deduce the first inequality in (2.9).

By Hölder's integral inequality we have for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$
\begin{aligned}
&\int_a^b (b-t)(t-a)\left|g'(t)\right| dt \\
&\le \begin{cases}
\sup_{t\in[a,b]}\left[(b-t)(t-a)\right]\int_a^b \left|g'(t)\right| dt, \\[2ex]
\left(\int_a^b \left[(b-t)(t-a)\right]^q dt\right)^{1/q}\left(\int_a^b \left|g'(t)\right|^p dt\right)^{1/p}, \\[2ex]
\int_a^b (b-t)(t-a)\, dt \sup_{t\in[a,b]}\left|g'(t)\right|,
\end{cases} \\
&= \begin{cases}
\frac{1}{4}(b-a)^2\int_a^b \left|g'(t)\right| dt, \\[2ex]
(b-a)^{2+1/q}\left[B(q+1,q+1)\right]^{1/q}\left(\int_a^b \left|g'(t)\right|^p dt\right)^{1/p}, \\[2ex]
\frac{1}{6}(b-a)^3\sup_{t\in[a,b]}\left|g'(t)\right|,
\end{cases}
\end{aligned}
$$

which proves the last part of (2.9). □

Theorem 3. *Let $f,\ g : [a,b] \to \mathbb{C}$ be absolutely continuous on $[a,b]$. If $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$
\begin{aligned}
(2.11) \qquad |D(f,g)| &\le \left[D\left(\ell, \int_a^{\cdot} |f'(u)|^p\, du \right) \right]^{1/p} \left[D\left(\ell, \int_a^{\cdot} |g'(u)|^q\, du \right) \right]^{1/q} \\
&= \frac{1}{2}(b-a) \left[\int_a^b (b-t)(t-a) |f'(t)|^p\, dt \right]^{1/p} \\
&\quad \times \left[\int_a^b (b-t)(t-a) |g'(t)|^q\, dt \right]^{1/q} \\
&\le \frac{1}{8}(b-a)^3 \|f'\|_{[a,b],p} \|g'\|_{[a,b],q}.
\end{aligned}
$$

In particular, we have for $p = q = 2$ that

$$
\begin{aligned}
(2.12) \qquad |D(f,g)| &\le \left[D\left(\ell, \int_a^{\cdot} |f'(u)|^2\, du \right) \right]^{1/2} \left[D\left(\ell, \int_a^{\cdot} |g'(u)|^2\, du \right) \right]^{1/2} \\
&= \frac{1}{2}(b-a) \left[\int_a^b (b-t)(t-a) |f'(t)|^2\, dt \right]^{1/2} \\
&\quad \times \left[\int_a^b (b-t)(t-a) |g'(u)|^2\, dt \right]^{1/2} \\
&\le \frac{1}{8}(b-a)^3 \|f'\|_{[a,b],2} \|g'\|_{[a,b],2}.
\end{aligned}
$$

Proof. Using Hölder's inequality for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
\begin{aligned}
&|f(t) - f(s)|\, |g(t) - g(s)| \\
&= \left| \int_s^t f'(u)\, du \right| \left| \int_s^t g'(u)\, du \right| \\
&\le \left| \int_s^t |f'(u)|\, du \right| \left| \int_s^t |g'(u)|\, du \right| \\
&\le |t-s|^{1/q} \left| \int_s^t |f'(u)|^p\, du \right|^{1/p} |t-s|^{1/p} \left| \int_s^t |g'(u)|^q\, du \right|^{1/q} \\
&= |t-s| \left| \int_s^t |f'(u)|^p\, du \right|^{1/p} \left| \int_s^t |g'(u)|^q\, du \right|^{1/q}.
\end{aligned}
$$

By the weighted Hölder's inequality for double integral, we also have

$$\begin{aligned}
&\int_a^b\int_a^b |f(t)-f(s)|\,|g(t)-g(s)|\,dtds \\
&\le \int_a^b\int_a^b |t-s|\left|\int_s^t |f'(u)|^p du\right|^{1/p}\left|\int_s^t |g'(u)|^q du\right|^{1/q} dtds \\
&\le \left(\int_a^b\int_a^b |t-s|\left(\left|\int_s^t |f'(u)|^p du\right|^{1/p}\right)^p dtds\right)^{1/p} \\
&\times \left(\int_a^b\int_a^b |t-s|\left(\left|\int_s^t |g'(u)|^q du\right|^{1/q}\right)^q dtds\right)^{1/q} \\
&= \left(\int_a^b\int_a^b |t-s|\left|\int_s^t |f'(u)|^p du\right| dtds\right)^{1/p} \\
&\times \left(\int_a^b\int_a^b |t-s|\left|\int_s^t |g'(u)|^q du\right| dtds\right)^{1/q}.
\end{aligned} \tag{2.13}$$

Observe that

$$\begin{aligned}
&\int_a^b\int_a^b |t-s|\left|\int_s^t |f'(u)|^p du\right| dtds \\
&= \int_a^b\int_a^b (t-s)\left(\int_s^t |f'(u)|^p du\right) dtds \\
&= \int_a^b\int_a^b (t-s)\left(\int_a^t |f'(u)|^p du - \int_a^s |f'(u)|^p du\right) dtds \\
&= 2D\left(\ell, \int_a^{\cdot} |f'(u)|^p du\right)
\end{aligned}$$

and

$$\int_a^b\int_a^b |t-s|\left|\int_s^t |g'(u)|^q du\right| dtds = 2D\left(\ell, \int_a^{\cdot} |g'(u)|^q du\right).$$

Therefore, by (2.2)

$$\begin{aligned}
|D(f,g)| &\le \frac{1}{2}\int_a^b\int_a^b |f(t)-f(s)|\,|g(t)-g(s)|\,dtds \\
&\le \frac{1}{2}\left[2D\left(\ell, \int_a^{\cdot} |f'(u)|^p du\right)\right]^{1/p}\left[2D\left(\ell, \int_a^{\cdot} |g'(u)|^q du\right)\right]^{1/q} \\
&= \left[D\left(\ell, \int_a^{\cdot} |f'(u)|^p du\right)\right]^{1/p}\left[D\left(\ell, \int_a^{\cdot} |g'(u)|^q du\right)\right]^{1/q}.
\end{aligned}$$

From (2.10) we have

$$D\left(\ell, \int_a^{\cdot} |f'(u)|^p du\right) = \frac{1}{2}(b-a)\int_a^b (b-t)(t-a)|f'(t)|^p dt$$

and

$$D\left(\ell, \int_a^{\cdot} |g'(u)|^q du\right) = \frac{1}{2}(b-a)\int_a^b (b-t)(t-a)|g'(t)|^q dt.$$

Therefore

$$\left[D\left(\ell,\int_a^{\cdot}\left|f'(u)\right|^p du\right)\right]^{1/p}\left[D\left(\ell,\int_a^{\cdot}\left|g'(u)\right|^q du\right)\right]^{1/q}$$
$$=\left[\frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\left|f'(t)\right|^p dt\right]^{1/p}$$
$$\times\left[\frac{1}{2}(b-a)\int_a^b (b-t)(t-a)\left|g'(t)\right|^q dt\right]^{1/q}$$
$$=\frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)\left|f'(t)\right|^p dt\right]^{1/p}$$
$$\times\left[\int_a^b (b-t)(t-a)\left|g'(t)\right|^q dt\right]^{1/q}$$

and the first part of the theorem is proved.

Now, observe that

$$\int_a^b (b-t)(t-a)\left|f'(t)\right|^p dt\le\frac{1}{4}(b-a)^2\int_a^b\left|f'(t)\right|^p dt$$

and

$$\int_a^b (b-t)(t-a)\left|g'(t)\right|^q dt\le\frac{1}{4}(b-a)^2\int_a^b\left|g'(t)\right|^q dt,$$

which gives the last part of (2.11). □

Remark 3. *Assume that $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $\gamma,\ \delta>1$ with $\frac{1}{\gamma}+\frac{1}{\delta}=1$. Then by Hölder's inequality we get*

$$\int_a^b (b-t)(t-a)\left|f'(t)\right|^p dt$$
$$\le(b-a)^{2+1/\beta}\left[B(\beta+1,\beta+1)\right]^{1/\beta}\left(\int_a^b\left|f'(t)\right|^{\alpha p}dt\right)^{1/\alpha}$$

and

$$\int_a^b (b-t)(t-a)\left|g'(t)\right|^q dt$$
$$\le(b-a)^{2+1/\delta}\left[B(\delta+1,\delta+1)\right]^{1/\delta}\left(\int_a^b\left|g'(t)\right|^{\gamma q}dt\right)^{1/\gamma}.$$

Then

$$\left[\int_a^b (b-t)(t-a)\left|f'(t)\right|^p dt\right]^{1/p}$$
$$\le(b-a)^{(2\beta+1)/(\beta p)}\left[B(\beta+1,\beta+1)\right]^{1/(\beta p)}\left(\int_a^b\left|f'(t)\right|^{\alpha p}dt\right)^{1/(\alpha p)}$$

and

$$\left[\int_a^b (b-t)(t-a)|g'(t)|^q\,dt\right]^{1/q}$$
$$\leq (b-a)^{(2\delta+1)/(\delta q)}\left[B(\delta+1,\delta+1)\right]^{1/(\delta q)}\left(\int_a^b |g'(t)|^{\gamma q}\,dt\right)^{1/(\gamma q)}.$$

Therefore

$$\frac{1}{2}(b-a)\left[\int_a^b (b-t)(t-a)|f'(t)|^p\,dt\right]^{1/p}$$
$$\times\left[\int_a^b (b-t)(t-a)|g'(t)|^q\,dt\right]^{1/q}$$
$$\leq \frac{1}{2}(b-a)(b-a)^{(2\beta+1)/(\beta p)}\left[B(\beta+1,\beta+1)\right]^{1/(\beta p)}\left(\int_a^b |f'(t)|^{\alpha p}\,dt\right)^{1/(\alpha p)}$$
$$\times (b-a)^{(2\delta+1)/(\delta q)}\left[B(\delta+1,\delta+1)\right]^{1/(\delta q)}\left(\int_a^b |g'(t)|^{\gamma q}\,dt\right)^{1/(\gamma q)}$$
$$= \frac{1}{2}\left[B(\beta+1,\beta+1)\right]^{1/(\beta p)}\left[B(\delta+1,\delta+1)\right]^{1/(\delta q)}$$
$$\times (b-a)^{1+(2\beta+1)/(\beta p)+(2\delta+1)/(\delta q)}$$
$$\times\left(\int_a^b |f'(t)|^{\alpha p}\,dt\right)^{1/(\alpha p)}\left(\int_a^b |g'(t)|^{\gamma q}\,dt\right)^{1/(\gamma q)}$$

and by (2.11) we get

$$|D(f,g)| \leq \frac{1}{2}\left[B(\beta+1,\beta+1)\right]^{1/(\beta p)}\left[B(\delta+1,\delta+1)\right]^{1/(\delta q)} \times (b-a)^{1+(2\beta+1)/(\beta p)+(2\delta+1)/(\delta q)}\left\|f'\right\|_{[a,b],\alpha p}\left\|g'\right\|_{[a,b],\gamma q}, \tag{2.14}$$

where $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, $\alpha,\ \beta>1$ *with* $\frac{1}{\alpha}+\frac{1}{\beta}=1$ *and* $\gamma,\ \delta>1$ *with* $\frac{1}{\gamma}+\frac{1}{\delta}=1$.

3. Applications for Uni-Dimensional Fourier Transform

The *Fourier Transform* has applications in a wide variety of fields in science and engineering [6, p. xi].

Let $g:[a,b]\to\mathbb{C}$ be a Lebesgue integrable mapping defined on the finite interval $[a,b]$ and $\mathcal{F}(g)$ its *finite Fourier transform*, i.e.,

$$\mathcal{F}(g)(t) := \int_a^b g(s)\,e^{-2\pi its}ds. \tag{3.1}$$

Define E, the exponential mean of two complex numbers, by

$$E(z,w) := \begin{cases} \dfrac{e^z-e^w}{z-w}, & \text{if } z\neq w \\ \exp(w) & \text{if } z=w \end{cases},\quad z,\ w\in\mathbb{C}. \tag{3.2}$$

In [25], Hanna, Dragomir and Roumeliotis obtained the following result:

Theorem 4. *Let $g:[a,b]\to\mathbb{K}$ be a complex-valued integrable function and there exists the constants $\varphi,\ \phi\in\mathbb{C}$ with the property that, either*

$$\left|g(s)-\frac{\phi+\varphi}{2}\right|\leq\frac{1}{2}\left|\phi-\varphi\right|\text{ for }\mu\text{-a.e. }s\in[a,b]\tag{3.3}$$

or, equivalently

$$\operatorname{Re}\left[\left(\phi-g(s)\right)\left(\overline{g(s)}-\overline{\varphi}\right)\right]\geq 0\text{ for a.e. }s\in[a,b]\tag{3.4}$$

holds. Then we have the inequality

$$\begin{aligned}&\left|\mathcal{F}(g)(t)-E(-2\pi ita,-2\pi itb)\int_a^b g(s)\,ds\right|\\&\leq\frac{1}{2}\left|\phi-\varphi\right|(b-a)\left[1-\frac{\sin^2\left[\pi t(b-a)\right]}{\pi^2t^2(b-a)^2}\right]^{\frac{1}{2}},\end{aligned}\tag{3.5}$$

for each $t\in[a,b]$ $(t\neq 0)$, where $E(\cdot,\cdot)$ is the exponential mean defined above.

Let $f(s)=e^{-2\pi its}$, $s,\ t\in[a,b]$ $(t\neq 0)$. Then

$$\int_a^b e^{-2\pi its}ds=(b-a)\,E(-2\pi ita,-2\pi itb),$$

$$\left|e^{2\pi its}\right|^2=1,$$

$$f'(s)=-2\pi ite^{-2\pi its},\quad\left|f'(s)\right|=2\pi\left|t\right|$$

and

$$\left\|f'\right\|_{[a,b],1}=\int_a^b\left|f'(s)\right|ds=2\pi\left|t\right|(b-a)$$

for $s,\ t\in[a,b]$.

If we use inequality (2.1) for $f(s)=e^{-2\pi its}$, $s,\ t\in[a,b]$ $(t\neq 0)$, then

$$\left|\mathcal{F}(g)(t)-E(-2\pi ita,-2\pi itb)\int_a^b g(s)\,ds\right|\leq\frac{1}{4}\pi\left|t\right|(b-a)^3\left\|g'\right\|_{[a,b],1},\tag{3.6}$$

for all absolutely continuous function $g:[a,b]\to\mathbb{C}$.

If we use the inequality (2.7) for $f(s)=e^{-2\pi its}$, $s,\ t\in[a,b]$ $(t\neq 0)$, then

$$\left|\mathcal{F}(g)(t)-E(-2\pi ita,-2\pi itb)\int_a^b g(s)\,ds\right|\leq\frac{1}{6}\pi\left|t\right|\left\|g'\right\|_{[a,b],\infty}(b-a)^4\tag{3.7}$$

for all absolutely continuous function $g:[a,b]\to\mathbb{C}$ such that $\left\|g'\right\|_{[a,b],\infty}<\infty$.

From (2.9) we get for $f(s)=e^{-2\pi its}$, $s,\ t\in[a,b]\ (t\neq 0)$, then for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\left|\mathcal{F}(g)(t)-E(-2\pi ita,-2\pi itb)\int_a^b g(s)\,ds\right| \tag{3.8}$$
$$\leq \pi|t|(b-a)\int_a^b (b-t)(t-a)|g'(t)|\,dt$$
$$\leq \pi|t|(b-a)\times\begin{cases}\frac{1}{4}(b-a)^2\|g'\|_{[a,b],1}\,,\\ \\ (b-a)^{2+1/q}[B(q+1,q+1)]^{1/q}\|g'\|_{[a,b],p}\,,\\ \\ \frac{1}{6}(b-a)^3\|g'\|_{[a,b],\infty}\,,\end{cases}$$

for all absolutely continuous function $g:[a,b]\to\mathbb{C}$ such that the norms in the right hand side are finite.

From (2.11) we get for $f(s)=e^{-2\pi its}$, $s,\ t\in[a,b]\ (t\neq 0)$, then

$$\left|\mathcal{F}(g)(t)-E(-2\pi ita,-2\pi itb)\int_a^b g(s)\,ds\right| \tag{3.9}$$
$$\leq \frac{1}{6}\pi|t|6^{1/q}(b-a)^{1+3/p}\left[\int_a^b (b-t)(t-a)|g'(t)|^q\,dt\right]^{1/q}$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

In particular, we have for $p=q=2$ that

$$\left|\mathcal{F}(g)(t)-E(-2\pi ita,-2\pi itb)\int_a^b g(s)\,ds\right| \tag{3.10}$$
$$\leq \frac{1}{6}\pi|t|6^{1/2}(b-a)^{1+3/2}\left[\int_a^b (b-t)(t-a)|g'(t)|^2\,dt\right]^{1/2}.$$

Finally, by (2.14) for $f(s)=e^{-2\pi its}$, $s,\ t\in[a,b]\ (t\neq 0)$, we get

$$\left|\mathcal{F}(g)(t)-E(-2\pi ita,-2\pi itb)\int_a^b g(s)\,ds\right| \tag{3.11}$$
$$\leq \pi|t|[B(\beta+1,\beta+1)]^{1/(\beta p)}[B(\delta+1,\delta+1)]^{1/(\delta q)}$$
$$\times(b-a)^{1+(2\beta+1)/(\beta p)+(2\delta+1)/(\delta q)}\|g'\|_{[a,b],\gamma q}\,,$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, $\alpha,\ \beta>1$ with $\frac{1}{\alpha}+\frac{1}{\beta}=1$ and $\gamma,\ \delta>1$ with $\frac{1}{\gamma}+\frac{1}{\delta}=1$.

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