INTEGRAL INEQUALITIES FOR THE WEIGHTED CEBYSEV FUNCTIONAL WITH APPLICATIONS

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Abstract. For $w : [a,b] \to \mathbb{R}$ continuous and positive on the interval $[a,b]$ and f, g are Lebesgue integrable on $[a,b]$, we consider the Čebyšev functional

$$D_w(f,g) := \int_a^b w(t)\,dt \int_a^b f(t)\,g(t)\,w(t)\,dt - \int_a^b f(t)\,w(t)\,dt \int_a^b g(t)\,w(t)\,dt.$$

Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$, f is *Lebesgue integrable* and satisfies the condition $m \le f(t) \le M$ for $t \in [a,b]$ and $g : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$ with $\frac{g'}{w}$ is essentially bounded, namely $\frac{g'}{w} \in L_\infty[a,b]$. In this paper we show among others that

$$\begin{aligned} &|D_w(f,g)| \\ &\le \frac{1}{2}\left(\int_a^b w(s)\,ds\right)^3 \left\|\frac{g'}{w}\right\|_{[a,b],\infty} \\ &\times \frac{\left(\frac{1}{\int_a^b w(t)dt}\int_a^b f(t)\,w(t)\,dt - m\right)\left(M - \frac{1}{\int_a^b w(t)dt}\int_a^b f(t)\,w(t)\,dt\right)}{M-m} \\ &\le \frac{1}{8}(M-m)\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds. \end{aligned}$$

Applications for continuous probability density functions supported on infinite intervals are also given.

1. Introduction

For two Lebesgue integrable functions $h, k : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(h,k) := (b-a)\int_a^b h(t)\,k(t)\,dt - \int_a^b h(t)\,dt \int_a^b k(t)\,dt.$$

In 1934, G. Grüss [23] showed that

$$|D(h,k)| \le \frac{1}{4}(b-a)^2 (M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \le h \le M < \infty, \quad -\infty < n \le k \le N < \infty \quad \text{a.e. on } [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

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Another lesser known inequality for $D(h,k)$ was derived in 1882 by Čebyšev [10] under the assumption that h', k' exist and are continuous on $[a,b]$, and is given by

$$|D(h,k)| \le \frac{1}{12} \|h'\|_\infty \|k'\|_\infty (b-a)^4, \tag{1.3}$$

where $\|h'\|_\infty := \sup_{t\in[a,b]} |h'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $h, k : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $h', k' \in L_\infty [a,b]$.

In 1970, A. M. Ostrowski [30] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(h,k)| \le \frac{1}{8} (b-a)^3 (M-m) \|k'\|_\infty, \tag{1.4}$$

provided h is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $k : [a,b] \to \mathbb{R}$ is absolutely continuous and $k' \in L_\infty [a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [26] (see also [27, p. 210]) obtained the following result as well:

$$|D(h,k)| \le \frac{1}{\pi^2} \|h'\|_2 \|k'\|_2 (b-a)^3, \tag{1.5}$$

provided h, k are absolutely continuous and $h', k' \in L_2 [a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [6], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(h,k)| \le (b-a) \times \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|h-\gamma\|_\infty \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt, \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|h-\gamma\|_q \left(\int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right|^p dt \right)^{\frac{1}{p}}, \\ \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(h,k)| \le (b-a) \|h\|_\infty \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \le h \le M$ for a.e. $x \in [a,b]$, then $\left\| h - \frac{m+M}{2} \right\|_\infty \le \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [9]

$$|D(h,k)| \le \frac{1}{2} (b-a)(M-m) \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [7].

The following result holds [15].

Theorem 1. *Let $f : [a,b] \to \mathbb{C}$ be of bounded variation on $[a,b]$ and $g : [a,b] \to \mathbb{C}$ a Lebesgue integrable function on $[a,b]$. Then*

$$|D(f,g)| \le \frac{1}{2} (b-a) \bigvee_a^b (f) \int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| dt$$

where $\bigvee_a^b (f)$ *denotes the total variation of* f *on the interval* $[a,b]$. *The constant* $\frac{1}{2}$ *is best possible in (1.9).*

For more recent upper bounds related to the Čebyšev functional see [1]-[9], [11]-[19] and [22]-[29].

In [12] we obtained the following refinement of Ostrowski's inequality (1.4)

$$\begin{aligned} |D(h,k)| &\le \frac{1}{2}(b-a)^3 \frac{\left(\frac{1}{b-a}\int_a^b h(t)\,dt - m\right)\left(M - \frac{1}{b-a}\int_a^b h(t)\,dt\right)}{M-m}\|k'\|_\infty \\ &\le \frac{1}{8}(b-a)^3 (M-m)\|k'\|_\infty , \end{aligned} \tag{1.9}$$

provided $m \le h \le M$ a.e. on $[a,b]$ and k is absolutely continuous on $[a,b]$.

In [8], Cerone & Dragomir obtained the following result

$$\begin{aligned} |D(p,k)| \le \frac{1}{2}(b-a)\left(\int_a^b (t-a)(b-t)[p'(t)]^2\,dt\right)^{1/2} \\ \times \left(\int_a^b (t-a)(b-t)[k'(t)]^2\,dt\right)^{1/2} \end{aligned} \tag{1.10}$$

where h,k are absolutely continuous on $[a,b]$ where $(\ell - a)(b-\ell)[p']^2, (\ell-a)(b-\ell)[k']^2 \in L[a,b]$ and $\ell(t) = t, t\in[a,b]$. The constant $\frac{1}{2}$ is the best possible constant in (1.10).

Assume that $w:[a,b] \to (0,\infty)$ is continuous on $[a,b]$, f is *Lebesgue integrable* and satisfies the condition $m \le f(t) \le M$ for $t \in [a,b]$ and $g:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ with $\frac{g'}{w}$ is essentially bounded, namely $\frac{g'}{w} \in L_\infty[a,b]$. In this paper we show among others that

$$\begin{aligned} &|D_w(f,g)| \\ &\le \frac{1}{2}\left(\int_a^b w(s)\,ds\right)^3 \left\|\frac{g'}{w}\right\|_{[a,b],\infty} \\ &\times \frac{\left(\frac{1}{\int_a^b w(t)dt}\int_a^b f(t)w(t)\,dt - m\right)\left(M - \frac{1}{\int_a^b w(t)dt}\int_a^b f(t)w(t)\,dt\right)}{M-m} \\ &\le \frac{1}{8}(M-m)\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds. \end{aligned}$$

Applications for continuous probability density functions supported on infinite intervals are also given.

2. Weighted Inequalities

We can define, as above

$$\begin{aligned} &D_{h'}(f,g) \\ &:= [h(b)-h(a)]\int_a^b f(t)g(t)h'(t)\,dt - \int_a^b f(t)h'(t)\,dt\int_a^b g(t)h'(t)\,dt, \end{aligned} \tag{2.1}$$

where h is absolutely continuous and f, g are Lebesgue measurable on $[a,b]$ and such that the above integrals exist.

The following weighted version of Ostrowski's inequality holds:

Theorem 2. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f is Lebesgue integrable and satisfies the condition $m\le f(t)\le M$ for $t\in[a,b]$ and $g:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ and $\frac{g'}{h'}$ is essentially bounded, namely $\frac{g'}{h'}\in L_\infty[a,b]$, then we have*

$$\begin{aligned}
&|D_{h'}(f,g)| \\
&\le \frac{1}{2}[h(b)-h(a)]^3\left\|\frac{g'}{h'}\right\|_{[a,b],\infty} \\
&\times\frac{\left(\frac{1}{h(b)-h(a)}\int_a^b f(t)h'(t)\,dt-m\right)\left(M-\frac{1}{h(b)-h(a)}\int_a^b f(t)h'(t)\,dt\right)}{M-m} \\
&\le \frac{1}{8}[h(b)-h(a)]^3(M-m)\left\|\frac{g'}{h'}\right\|_{[a,b],\infty}.
\end{aligned}\tag{2.2}$$

The constant $\frac{1}{8}$ is best possible.

Proof. Assume that $[c,d]\subset[a,b]$. If $g:[c,d]\to\mathbb{C}$ is absolutely continuous on $[c,d]$, then $g\circ h^{-1}:[h(c),h(d)]\to\mathbb{C}$ is absolutely continuous on $[h(c),h(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(g\circ h^{-1}\right)'(z)=\left(g'\circ h^{-1}\right)(z)\left(h^{-1}\right)'(z)=\frac{\left(g'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\tag{2.3}$$

for almost every (a.e.) $z\in[h(c),h(d)]$.

If $x\in[c,d]$, then by taking $z=h(x)$, we get

$$\left(g\circ h^{-1}\right)'(z)=\frac{\left(g'\circ h^{-1}\right)(h(x))}{\left(h'\circ h^{-1}\right)(h(x))}=\frac{g'(x)}{h'(x)}.$$

Therefore, since $\frac{g'}{h'}\in L_\infty[c,d]$, hence $\left(g\circ h^{-1}\right)'\in L_\infty[h(c),h(d)]$. Also

$$\left\|\left(g\circ h^{-1}\right)'\right\|_{[h(c),h(d)],\infty}=\left\|\frac{g'}{h'}\right\|_{[c,d],\infty}.$$

Now, if we use the inequality (1.9) for the functions $f\circ h^{-1}$ and $g\circ h^{-1}$ on the interval $[h(a),h(b)]$, then we get

$$\begin{aligned}
&\left|[h(b)-h(a)]\int_{h(a)}^{h(b)} f\circ h^{-1}(u)g\circ h^{-1}(u)du\right. \\
&\left.-\int_{h(a)}^{h(b)} f\circ h^{-1}(u)du\int_{h(a)}^{h(b)} g\circ h^{-1}(u)du\right| \\
&\le\frac{1}{2}[h(b)-h(a)]^3\left\|\left(g\circ h^{-1}\right)'\right\|_{[h(a),h(b)],\infty} \\
&\times\frac{\left(\frac{1}{h(b)-h(a)}\int_{h(a)}^{h(b)} f\circ h^{-1}(u)du-m\right)\left(M-\frac{1}{h(b)-h(a)}\int_{h(a)}^{h(b)} g\circ h^{-1}(u)du\right)}{M-m} \\
&\le\frac{1}{8}[h(b)-h(a)]^3(M-m)\left\|\left(g\circ h^{-1}\right)'\right\|_{[h(a),h(b)],\infty}
\end{aligned}\tag{2.4}$$

since $m\le f\circ h^{-1}(u)\le M$ for all $u\in[h(a),h(b)]$.

Observe also that, by the change of variable $t = h^{-1}(u)$, $u \in [h(a), h(b)]$, we have $u = h(t)$ that gives $du = h'(t)\,dt$ and

$$\int_{h(a)}^{h(b)} \left(f \circ h^{-1}\right)(u)\,du = \int_a^b f(t)\,h'(t)\,dt,$$
$$\int_{h(a)}^{h(b)} g \circ h^{-1}(u)du = \int_a^b g(t)\,h'(t)\,dt,$$
$$\int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u)du = \int_a^b f(t)\,g(t)\,h'(t)\,dt$$

and

$$\left\| \left(g \circ h^{-1}\right)' \right\|_{[h(a),h(b)],\infty} = \left\| \frac{g'}{h'} \right\|_{[a,b],\infty}.$$

By making use of (2.4) we then get the desired result (2.2).

The best constant follows by Ostrowski's inequality (1.4). □

If $w : [a,b] \to \mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W : [a,b] \to [0,\infty)$, $W(x) := \int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x) = w(x)$ for any $x \in (a,b)$.

Consider the Čebyšev functional

$$D_w(f,g) := \int_a^b w(t)\,dt \int_a^b f(t)\,g(t)\,w(t)\,dt - \int_a^b f(t)\,w(t)\,dt \int_a^b g(t)\,w(t)\,dt,$$

then we have the weighted inequalities:

Corollary 1. *Assume that* $w : [a,b] \to (0,\infty)$ *is continuous on* $[a,b]$, f *is Lebesgue integrable and satisfies the condition* $m \le f(t) \le M$ *for* $t \in [a,b]$ *and* $g : [a,b] \to \mathbb{R}$ *is absolutely continuous on* $[a,b]$ *with* $\frac{g'}{w}$ *is essentially bounded, namely* $\frac{g'}{w} \in L_\infty[a,b]$, *then we have*

$$\begin{aligned} &|D_w(f,g)| \\ &\le \frac{1}{2}\left(\int_a^b w(s)\,ds\right)^3 \left\|\frac{g'}{w}\right\|_{[a,b],\infty} \\ &\times \frac{\left(\frac{1}{\int_a^b w(t)dt}\int_a^b f(t)\,w(t)\,dt - m\right)\left(M - \frac{1}{\int_a^b w(t)dt}\int_a^b f(t)\,w(t)\,dt\right)}{M-m} \\ &\le \frac{1}{8}(M-m)\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\int_a^b w(s)\,ds. \end{aligned} \tag{2.5}$$

The constant $\frac{1}{8}$ *is best possible.*

Remark 1. *Under the assumptions of Corollary 1 and if there exists a constant* $K>0$ *such that* $|g'(t)| \le K w(t)$ *for a.e.* $t\in[a,b]$, *then by (2.5) we get*

$$\begin{aligned} &|D_w(f,g)| \\ &\le \frac{1}{2}\left(\int_a^b w(s)\,ds\right)^3 K \\ &\times \frac{\left(\frac{1}{\int_a^b w(t)dt}\int_a^b f(t)w(t)\,dt - m\right)\left(M - \frac{1}{\int_a^b w(t)dt}\int_a^b f(t)w(t)\,dt\right)}{M-m} \\ &\le \frac{1}{8}(M-m)K\int_a^b w(s)\,ds. \end{aligned} \tag{2.6}$$

We also have:

Theorem 3. *Let* $h:[a,b]\to[h(a),h(b)]$ *be a continuous strictly increasing function that is differentiable on* (a,b). *If* f *and* g *are absolutely continuous and*

$$\frac{[h(b)-h][h-h(a)]}{h'}[f']^2,\ \frac{[h(b)-h][h-h(a)]}{h'}[g']^2 \in L[a,b],$$

then

$$\begin{aligned} |D_{h'}(f,g)| \le{}& \frac{1}{2}[h(b)-h(a)] \\ &\times\left(\int_a^b \frac{[h(b)-h(t)][h(t)-h(a)]}{h'(t)}[f'(t)]^2\,dt\right)^{1/2} \\ &\times\left(\int_a^b \frac{[h(b)-h(t)][h(t)-h(a)]}{h'(t)}[g'(t)]^2\,dt\right)^{1/2}. \end{aligned} \tag{2.7}$$

The constant $\frac{1}{2}$ *is best possible.*

Proof. I we use the inequality (1.10) for the functions $p=f\circ h^{-1}$ and $k=g\circ h^{-1}$ on the interval $[h(a),h(b)]$, then we get

$$\begin{aligned} &\left|[h(b)-h(a)]\int_{h(a)}^{h(b)} f\circ h^{-1}(u)g\circ h^{-1}(u)du\right. \\ &\left.-\int_{h(a)}^{h(b)} f\circ h^{-1}(u)du\int_{h(a)}^{h(b)} g\circ h^{-1}(u)du\right| \\ &\le \frac{1}{2}[h(b)-h(a)] \\ &\times\left(\int_{h(a)}^{h(b)}(h(b)-u)(u-h(a))\left[\left(f\circ h^{-1}\right)'(u)\right]^2du\right)^{1/2} \\ &\times\left(\int_{h(a)}^{h(b)}(h(b)-u)(u-h(a))\left[\left(g\circ h^{-1}\right)'(u)\right]^2du\right)^{1/2}. \end{aligned} \tag{2.8}$$

As in the proof of Theorem 2,

$$\left(f\circ h^{-1}\right)'(u) = \frac{\left(f'\circ h^{-1}\right)(u)}{\left(h'\circ h^{-1}\right)(u)} \text{ and } \left(g\circ h^{-1}\right)'(u) = \frac{\left(f'\circ h^{-1}\right)(u)}{\left(h'\circ h^{-1}\right)(u)}.$$

Observe also that, by the change of variable $t = h^{-1}(u)$, $u \in [h(a), h(b)]$, we have $u = h(t)$ that gives $du = h'(t)\,dt$ and

$$\begin{aligned}&\int_{h(a)}^{h(b)} (h(b) - u)(u - h(a)) \left[\left(f \circ h^{-1}\right)'(u)\right]^2 du \\ &= \int_a^b (h(b) - h(t))(h(t) - h(a)) \left[\frac{f'(t)}{h'(t)}\right]^2 h'(t)\,dt \\ &= \int_a^b \frac{(h(b) - h(t))(h(t) - h(a))}{h'(t)} [f'(t)]^2\,dt\end{aligned}$$

and

$$\begin{aligned}&\int_{h(a)}^{h(b)} (h(b) - u)(u - h(a)) \left[\left(g \circ h^{-1}\right)'(u)\right]^2 du \\ &= \int_a^b \frac{(h(b) - h(t))(h(t) - h(a))}{h'(t)} [g'(t)]^2\,dt.\end{aligned}$$

By utilising (2.8) and the calculations for the Čebyšev's functional from the proof of Theorem 2 we deduce the desired result (2.7). □

Corollary 2. *Assume that $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$. If f and g are absolutely continuous and*

$$\frac{\int_\cdot^b w(s)\,ds \int_a^\cdot w(s)\,ds}{w} [f']^2, \quad \frac{\int_\cdot^b w(s)\,ds \int_a^\cdot w(s)\,ds}{w} [g']^2 \in L[a, b], \tag{2.9}$$

then

$$\begin{aligned}|D_w(f, g)| \le \frac{1}{2} \int_a^b w(s)\,ds &\left(\int_a^b \frac{\int_t^b w(s)\,ds \int_a^t w(s)\,ds}{w(t)} [f'(t)]^2\,dt\right)^{1/2} \\ &\times \left(\int_a^b \frac{\int_t^b w(s)\,ds \int_a^t w(s)\,ds}{w(t)} [g'(t)]^2\,dt\right)^{1/2}.\end{aligned} \tag{2.10}$$

The constant $\frac{1}{2}$ is best possible.

Remark 2. *Under the assumptions of Corollary 2 and if there exists two constants $L, K > 0$ such that $|f'(t)| \le Lw(t)$, $|g'(t)| \le Kw(t)$ for a.e. $t \in [a, b]$, then by (2.10) we get*

$$|D_w(f, g)| \le \frac{1}{2} KL \int_a^b w(s)\,ds \int_a^b w(t) \left(\int_t^b w(s)\,ds \int_a^t w(s)\,ds\right) dt. \tag{2.11}$$

In [8], Cerone & Dragomir obtained the following result as well

$$|D(p, k)| \le \frac{1}{2}(b - a) \|p'\|_{[a,b],\infty} \int_a^b (u - a)(b - u)\,dk(u) \tag{2.12}$$

provided that k is monotonic nondecreasing on $[a, b]$ and p is absolutely continuous on $[a, b]$ with $p' \in L_\infty[a, b]$. The constant $\frac{1}{2}$ is best possible in (2.12).

By the use of inequality (2.12) we can state the following result as well:

Theorem 4. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f and g are absolutely continuous and g is monotonic nondecreasing, then*

$$|D_{h'}(f,g)|\leq\frac{1}{2}[h(b)-h(a)]\left\|\frac{f'}{h'}\right\|_{[a,b],\infty}\times\int_a^b[h(b)-h(t)][h(t)-h(a)]g'(t)\,dt, \tag{2.13}$$

provided that $\frac{f'}{h'}\in L_\infty[a,b]$. The constant $\frac{1}{2}$ is best possible.

Corollary 3. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$. If f and g are absolutely continuous and g is monotonic nondecreasing, then*

$$|D_w(f,g)|\leq\frac{1}{2}\int_a^b w(s)\,ds\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)g'(t)\,dt, \tag{2.14}$$

provided that $\frac{f'}{w}\in L_\infty[a,b]$. The constant $\frac{1}{2}$ is best possible.

Finally, consider the following inequality obtained in [8]

$$|D(p,k)|\leq\frac{1}{2}(b-a)\left\|\frac{p'}{k'}\right\|_{[a,b],\infty}\int_a^b(u-a)(b-u)[k'(u)]^2\,du \tag{2.15}$$

provided that $p,\ k:[a,b]\to\mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b) with $k'(t)\neq0$ for each $t\in(a,b)$. The constant $\frac{1}{2}$ is best possible in (2.15).

Theorem 5. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f and g are absolutely continuous on $[a,b]$ and differentiable on (a,b) with $g'(t)\neq0$ for each $t\in(a,b)$, then*

$$|D_{h'}(f,g)|\leq\frac{1}{2}[h(b)-h(a)]\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}\times\int_a^b\frac{[h(b)-h(t)][h(t)-h(a)]}{h'(t)}[g'(t)]^2\,dt, \tag{2.16}$$

provided that the norm and the integral in the right term are finite. The constant $\frac{1}{2}$ is best possible.

Finally, we have the weighted inequality:

Corollary 4. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$. If f and g are continuous on $[a,b]$ and differentiable on (a,b) with $g'(t)\neq0$ for each $t\in(a,b)$, then*

$$|D_w(f,g)|\leq\frac{1}{2}\int_a^b w(s)\,ds\left\|\frac{f'}{g'}\right\|_{[a,b],\infty}\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}[g'(t)]^2\,dt, \tag{2.17}$$

provided that the norm and the integral in the right term are finite. The constant $\frac{1}{2}$ is best possible.

3. Some Examples

If we take $h : [a, b] \subset (0, \infty) \to \mathbb{R}$, $h(t) = \ln t$, in (2.2), then we get for $w(t) := \ell^{-1}$, where $\ell(t) := t$, that

$$D_{\ell^{-1}}(f, g) := \ln\left(\frac{b}{a}\right) \int_a^b \frac{f(t) g(t)}{t} dt - \int_a^b \frac{f(t)}{t} dt \int_a^b \frac{g(t)}{t} dt.$$

From (2.5) we get

$$\begin{aligned} |D_{\ell^{-1}}(f, g)| &\le \frac{1}{2} \left[\ln\left(\frac{b}{a}\right)\right]^3 \|\ell g'\|_{[a,b],\infty} \\ &\quad \times \frac{\left(\frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{f(t)}{t} dt - m\right)\left(M - \frac{1}{\ln\left(\frac{b}{a}\right)} \int_a^b \frac{f(t)}{t} dt\right)}{M - m} \\ &\le \frac{1}{8}(M - m) \ln\left(\frac{b}{a}\right) \|\ell g'\|_{[a,b],\infty}, \end{aligned} \tag{3.1}$$

provided that f is *Lebesgue integrable* and satisfies the condition $m \le f(t) \le M$ for $t \in [a, b]$ and $g : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ and $\ell g' \in L_\infty[a, b]$.

From (2.10) we get

$$\begin{aligned} |D_{\ell^{-1}}(f, g)| &\le \frac{1}{2} \ln\left(\frac{b}{a}\right) \left(\int_a^b t \ln\left(\frac{b}{t}\right) \ln\left(\frac{t}{a}\right) [f'(t)]^2 dt\right)^{1/2} \\ &\quad \times \left(\int_a^b t \ln\left(\frac{b}{t}\right) \ln\left(\frac{t}{a}\right) [g'(t)]^2 dt\right)^{1/2}, \end{aligned} \tag{3.2}$$

provided that $\ell \ln\left(\frac{b}{\ell}\right) \ln\left(\frac{\ell}{a}\right) [f']^2$, $\ell \ln\left(\frac{b}{\ell}\right) \ln\left(\frac{\ell}{a}\right) [g']^2 \in L[a, b]$.

From (2.14) we derive

$$|D_{\ell^{-1}}(f, g)| \le \frac{1}{2} \ln\left(\frac{b}{a}\right) \|\ell f'\|_{[a,b],\infty} \int_a^b \ln\left(\frac{b}{t}\right) \ln\left(\frac{t}{a}\right) g'(t) dt, \tag{3.3}$$

provided that $\ell f' \in L_\infty[a, b]$ and g is absolutely continuous and monotonic nondecreasing on $[a, b]$.

If f and g are continuous on $[a, b]$ and differentiable on (a, b) with $g'(t) \ne 0$ for each $t \in (a, b)$, then

$$|D_{\ell^{-1}}(f, g)| \le \frac{1}{2} \ln\left(\frac{b}{a}\right) \left\|\frac{f'}{g'}\right\|_{[a,b],\infty} \int_a^b t \ln\left(\frac{b}{t}\right) \ln\left(\frac{t}{a}\right) [g'(t)]^2 dt, \tag{3.4}$$

provided $\frac{f'}{g'} \in L_\infty[a, b]$ and $\ell \ln\left(\frac{b}{\ell}\right) \ln\left(\frac{\ell}{a}\right) [g']^2 \in L[a, b]$.

The interested reader may also consider the weights $w(t) = \exp t$, $t \in [a, b]$ or $w(t) = r\ell^{r-1}$, $r > 0$, $t \in [a, b]$, see for instance the preprint version of [20].

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R} = (-\infty, \infty)$. Namely, if $I = (-\infty, \infty)$, $f : \mathbb{R} \to \mathbb{C}$ is locally absolutely continuous on $\mathbb{R}$ and $w(s) > 0$ for $s \in \mathbb{R}$ with $\int_{-\infty}^{\infty} w(s) ds = 1$, namely w is a probability density function on $(-\infty, \infty)$, f is *Lebesgue measurable* and satisfies the condition $m \le f(t) \le M$ for $t \in (-\infty, \infty)$ and $g : (-\infty, \infty) \to \mathbb{R}$ is locally absolutely continuous on $(-\infty, \infty)$ with $\frac{g'}{w} \in L_\infty(-\infty, \infty)$, then, by considering the functional

$$D_w(f,g) := \int_{-\infty}^{\infty} w(t) f(t) g(t)\,dt - \int_{-\infty}^{\infty} w(t) f(t)\,dt \int_{-\infty}^{\infty} w(t) g(t)\,dt,$$

we have by (2.5) on letting $a \to -\infty$ and $b \to +\infty$

$$|D_w(f,g)| \le \frac{1}{2}\left\|\frac{g'}{w}\right\|_{(-\infty,\infty),\infty} \times \frac{\left(\int_{-\infty}^{\infty} f(t) w(t)\,dt - m\right)\left(M - \int_{-\infty}^{\infty} f(t) w(t)\,dt\right)}{M-m} \le \frac{1}{8}(M-m)\left\|\frac{g'}{w}\right\|_{(-\infty,\infty),\infty}. \tag{3.5}$$

If f and g are locally absolutely continuous on $(-\infty,\infty)$ and

$$\frac{\int_{\cdot}^{\infty} w(s)\,ds \int_{-\infty}^{\cdot} w(s)\,ds}{w}[f']^2,\ \frac{\int_{\cdot}^{\infty} w(s)\,ds \int_{-\infty}^{\cdot} w(s)\,ds}{w}[g']^2 \in L(-\infty,\infty),$$

then by (2.10)

$$|D_w(f,g)| \le \frac{1}{2}\left(\int_{-\infty}^{\infty} \frac{\int_t^{\infty} w(s)\,ds \int_{-\infty}^{t} w(s)\,ds}{w(t)}[f'(t)]^2\,dt\right)^{1/2} \times \left(\int_{-\infty}^{\infty} \frac{\int_t^{\infty} w(s)\,ds \int_{-\infty}^{t} w(s)\,ds}{w(t)}[g'(t)]^2\,dt\right)^{1/2}. \tag{3.6}$$

If f and g are locally absolutely continuous on $(-\infty,\infty)$ and g is monotonic nondecreasing on $(-\infty,\infty)$, then by (2.14)

$$|D_w(f,g)| \le \frac{1}{2}\left\|\frac{f'}{w}\right\|_{(-\infty,\infty),\infty} \int_{-\infty}^{\infty}\left(\int_t^{\infty} w(s)\,ds \int_{-\infty}^{t} w(s)\,ds\right) g'(t)\,dt, \tag{3.7}$$

provided that $\frac{f'}{w} \in L_\infty(-\infty,\infty)$.

If f and g are continuous on $(-\infty,\infty)$ and differentiable on $(-\infty,\infty)$ with $g'(t) \neq 0$ for each $t \in (-\infty,\infty)$, then

$$|D_w(f,g)| \le \frac{1}{2}\left\|\frac{f'}{g'}\right\|_{(-\infty,\infty),\infty} \int_{-\infty}^{\infty} \frac{\int_t^{\infty} w(s)\,ds \int_{-\infty}^{t} w(s)\,ds}{w(t)}[g'(t)]^2\,dt, \tag{3.8}$$

provided that $\frac{f'}{g'} \in L_\infty(-\infty,\infty)$ and $\frac{\int_{\cdot}^{\infty} w(s)ds \int_{-\infty}^{\cdot} w(s)ds}{w}[g']^2 \in L(-\infty,\infty)$.

In probability theory and statistics, the *beta prime distribution* (also known as *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x > 0$ with two parameters α and β, having the probability density function:

$$w_{\alpha,\beta}(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$$

where B is *Beta function,* $B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}$, $\alpha,\beta > 0$. Consider the functional

$$C_{B,\alpha,\beta}(f,g) := B(\alpha,\beta)\int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t)\,g(t)\,dt - \int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t)\,dt \int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} g(t)\,dt$$

where α, $\beta > 0$. The interested reader may state similar inequalities for $C_{B,\alpha,\beta}(\cdot,\cdot)$, see [20].

The probability density of the *normal distribution* on $(-\infty,\infty)$ is

$$w_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x\in\mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

Consider the functional

$$C_{N,\sigma,\mu}(f,g) := \sqrt{2\pi}\sigma\int_{-\infty}^\infty \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t)\,g(t)\,dt - \int_{-\infty}^\infty \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t)\,dt \int_{-\infty}^\infty \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) g(t)\,dt$$

with the parameters μ and σ as above. One can state similar inequalities for $C_{N,\sigma,\mu}(\cdot,\cdot)$, see [20].

References

[1] G. A. Anastassiou, Complex multivariate Fink type identity applied to complex multivariate Ostrowski and Grüss inequalities. *Indian J. Math.* **61** (2019), no. 2, 199–237.

[2] G. A. Anastassiou, Selfadjoint operator Chebyshev-Grüss type inequalities. *Appl. Math.* (Warsaw) **46** (2019), no. 1, 99–114.

[3] N. S. Barnett, P. Cerone, S. S. Dragomir and C. Buşe, Some Grüss type inequalities for vector-valued functions in Banach spaces and applications. *Tamsui Oxf. J. Math. Sci.* **23** (2007), no. 1, 91–103. Preprint *RGMIA Res. Rep. Coll.* **8** (2005), No. 2, Art. 12 [Online https://rgmia.org/papers/v8n2/GTIVVFBSApp.pdf].

[4] H. Budak and M. Z. Sarikaya, On weighted Grüss type inequalities for double integrals. *Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat.* **66** (2017), no. 2, 53–61.

[5] H. Budak and M. Z. Sarikaya, An inequality of Ostrowski-Grüss type for double integrals. *Stud. Univ. Babeş-Bolyai Math.* **62** (2017), no. 2, 163–173.

[6] P. Cerone and S. S. Dragomir, New bounds for the Čebyšev functional, *Appl. Math. Lett.*, **18** (2005), 603-611.

[7] P. Cerone and S. S. Dragomir, A refinement of the Grüss inequality and applications, *Tamkang J. Math.* **38**(2007), No. 1, 37-49. Preprint available at *RGMIA Res. Rep. Coll.*, **5**(2) (2002), Art. 14. [ONLINE http://rgmia.vu.edu.au/v5n2.html].

[8] P. Cerone and S. S. Dragomir, Some new Ostrowski-type bounds for the Čebyšev functional and applications, *J. Math. Ineq.* **8** (2014), No. 1, 159-170.

[9] X.-L. Cheng and J. Sun, Note on the perturbed trapezoid inequality, *J. Inequal. Pure Appl. Math.,* **3**(2) (2002), Art. 29. [ONLINEhttp://jipam.vu.edu.au/article.php?sid=181

[10] P. L. Chebyshev, Sur les expressions approximatives des intègrals dèfinis par les outres prises entre les même limites, *Proc. Math. Soc. Charkov,* **2** (1882), 93-98.

[11] S. S. Dragomir, Improvements of Ostrowski and generalised trapezoid inequality in terms of the upper and lower bounds of the first derivative. *Tamkang J. Math.* **34** (2003), No. 3, 213–222.

[12] S. S. Dragomir, A refinement of Ostrowski's inequality for the Čebyšev functional and applications. *Analysis* (Munich) **23** (2003), no. 4, 287–297.

[13] S. S. Dragomir, Inequalities of Grüss type for the Stieltjes integral and applications, *Kragujevac J. Math.*, **26** (2004), 89-112.

[14] S. S. Dragomir, Inequalities for Stieltjes integrals with convex integrators and applications, *Appl. Math. Lett.,* **20** (2007), 123-130.

[15] S. S. Dragomir, New Grüss' type inequalities for functions of bounded variation and applications. *Appl. Math. Lett.* **25** (2012), no. 10, 1475–1479.

[16] S. S. Dragomir, Inequalities of Lipschitz type for power series in Banach algebras. *Ann. Math. Sil.* No. **29** (2015), 61–83.

[17] S. S. Dragomir, General Lebesgue integral inequalities of Jensen and Ostrowski type for differentiable functions whose derivatives in absolute value are convex and applications. *Ann. Univ. Mariae Curie-Skłodowska Sect.* A **69** (2015), no. 2, 17–45.

[18] S. S. Dragomir, Integral inequalities for Lipschitzian mappings between two Banach spaces and applications. *Kodai Math. J.* **39** (2016), no. 1, 227–251.

[19] S. S. Dragomir, On some Grüss' type inequalities for the complex integral. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **113** (2019), no. 4, 3531–3543.

[20] S. S. Dragomir, Weighted integral inequalities of Ostrowski, Čebyšev and Lupaş type with applications. *Bull. Aust. Math. Soc.* **98** (2018), no. 3, 439–447. Preprint *RGMIA Res. Rep. Coll.* **21** (2018), Art. 44, 12 pp. [https://rgmia.org/papers/v21/v21a44.pdf].

[21] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. (ONLINE: http://rgmia.vu.edu.au/monographs)

[22] J. V. da C. Sousa, D. S. Oliveira and E. de Oliveira, Capelas Grüss-type inequalities by means of generalized fractional integrals. *Bull. Braz. Math. Soc.* (N.S.) **50** (2019), no. 4, 1029–1047.

[23] G. Grüss, Über das maximum des absoluten Betrages von $\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \frac{1}{(b-a)^2}\int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$, *Math. Z.,* **39** (1934), 215-226.

[24] S. Joshi, E. Mittal, R. M. Pandey and S. D. Purohit, Some Grüss type inequalities involving generalized fractional integral operator. *Bull. Transilv. Univ. Braşov Ser. III* **12** (**61**) (2019), no. 1, 41–52.

[25] S. Kermausuor and E. R. Nwaeze, New Ostrowski and Ostrowski-Grüss type inequalities for double integrals on time scales involving a combination of Δ-integral means. *Tamkang J. Math.* **49** (2018), no. 4, 277–289.

[26] A. Lupaş, The best constant in an integral inequality, *Mathematica* (Cluj), **15** (**38**) (1973), No. 2, 219-222.

[27] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, *Classical and New Inequalities in Analysis,* Kluwer Academic Publishers, Dordrecht, 1993.

[28] E. R. Nwaeze, Generalized weighted trapezoid and Grüss type inequalities on time scales. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 4, 13 pp.

[29] E. R. Nwaeze, N. Kaplan, F. G. Tuna, and A. Tuna, Some new inequalities of the Ostrowski-Grüss, Čebyšev, and trapezoid types on time scales. *J. Nonlinear Sci. Appl.* **12** (2019), no. 4, 192–205.

[30] A. M. Ostrowski, On an integral inequality, *Aequationes Math.,* **4** (1970), 358-373.

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