INTEGRAL INEQUALITIES FOR ČEBYŠEV FUNCTIONAL IN GENERAL LEBESGUE SPACES

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Abstract. In this paper we establish several inequalities for the Čebyšev functional associated to S-synchronous dominated functions and strongly (p,q)-H-dominated functions defined on a measurable space. Some inequalities related to Grüss and Čebyšev integral inequalities are also given.

1. Introduction

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$.

For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\geq 0$ for μ-a.e. $x\in\Omega$, consider the Lebesgue space $L_w(\Omega,\mu):=\{f:\Omega\to\mathbb{R},\ f$ is μ-measurable and $\int_\Omega w(x)|f(x)|d\mu(x)<\infty\}$. Assume $\int_\Omega w(x)d\mu(x)>0$. In order to simplify the notation for the integrals, we do not write the variable, namely, instead of $\int_\Omega w(x)d\mu(x)$ we simply write $\int_\Omega wd\mu$.

If $f,\ g:\Omega\to\mathbb{R}$ are μ-measurable functions and $f,\ g,\ fg\in L_w(\Omega,\mu)$, then we may consider the weighted Čebyšev functional in the following form

$$C_w(f,g):=\int_\Omega wd\mu\int_\Omega wfgd\mu-\int_\Omega wfd\mu\int_\Omega wgd\mu. \tag{1.1}$$

The following result is known in the literature as the Grüss inequality, see for instance

$$|T_w(f,g)|\leq\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)\left(\int_\Omega wd\mu\right)^2, \tag{1.2}$$

provided

$$-\infty<\gamma\leq f\leq\Gamma<\infty,\ \ -\infty<\delta\leq g\leq\Delta<\infty \tag{1.3}$$

for μ-a.e. on Ω. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Note that if $\Omega=\{1,\ldots,n\}$ and μ is the discrete measure on Ω, then we obtain the discrete Grüss inequality

$$\left|W_n\sum_{i=1}^n w_ix_iy_i-\sum_{i=1}^n w_ix_i\sum_{i=1}^n w_iy_i\right|\leq\frac{1}{4}(\Gamma-\gamma)(\Delta-\delta)W_n^2, \tag{1.4}$$

provided $\gamma\leq x_i\leq\Gamma$, $\delta\leq y_i\leq\Delta$ for each $i\in\{1,\ldots,n\}$ and $w_i\geq 0$ with $W_n:=\sum_{i=1}^n w_i>0$.

1991 *Mathematics Subject Classification.* 26D15; 26D10.

Key words and phrases. Synchronous Functions, Lipschitzian functions, Grüss' inequality, Čebyšev inequality, Cauchy-Bunyakovsky-Schwarz inequality, Hölder's inequality.

RGMIA Res. Rep. Coll. 24 (2021), Art. 32, 14 pp. Received 16/03/21

For $f \in L_w(\Omega, \mu)$, we may define

$$D_{w,1}(f) := \frac{1}{\int_\Omega w d\mu} \int_\Omega w \left| f - \frac{1}{\int_\Omega w d\mu} \int_\Omega w f d\mu \right| d\mu. \tag{1.5}$$

The following result was obtained by Cerone and Dragomir in 2002, see [7]:

Theorem 1. *Let w, f, $g : \Omega \to \mathbb{R}$ be μ-measurable functions with $w \geq 0$ μ-a.e. on Ω and $\int_\Omega w d\mu > 0$. If f, g, $fg \in L_w(\Omega, \mu)$ and there exists the constants δ, Δ such that*

$$-\infty < \delta \leq g \leq \Delta < \infty \quad \text{for } \mu\text{-a.e. on } \Omega, \tag{1.6}$$

then we have the inequality

$$|C_w(f,g)| \leq \frac{1}{2}(\Delta - \delta)\left(\int_\Omega w d\mu\right)^2 D_{w,1}(f). \tag{1.7}$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller quantity.

We can define for $f \in L_{2,w}(\Omega, \mu) := \left\{ f : \Omega \to \mathbb{R}, \int_\Omega w |f|^2 d\mu < \infty \right\}$ the quantity

$$D_{w,2}(f) = \left[\frac{\int_\Omega w f^2 d\mu}{\int_\Omega w d\mu} - \left(\frac{\int_\Omega w f d\mu}{\int_\Omega w d\mu} \right)^2 \right]^{\frac{1}{2}}. \tag{1.8}$$

In the same paper [7] we established the following *refinement of Grüss' inequality:*

Corollary 1. *For f, $g : \Omega \to \mathbb{R}$, μ-measurable functions and so that $-\infty < \gamma \leq f < \Gamma < \infty$, $-\infty < \delta \leq \Delta < \infty$ for μ-a.e. on Ω,*

$$\begin{aligned} |C_w(f,g)| &\leq \frac{1}{2}(\Delta - \delta)\left(\int_\Omega w d\mu\right)^2 D_{w,1}(f) \\ &\leq \frac{1}{2}(\Delta - \delta)\left(\int_\Omega w d\mu\right)^2 D_{w,2}(f) \\ &\leq \frac{1}{4}(\Delta - \delta)(\Gamma - \gamma)\left(\int_\Omega w d\mu\right)^2. \end{aligned} \tag{1.9}$$

We can introduce the following concept that extends the monotonicity in the same sense for two functions of a real variable:

Definition 1. *The μ-measurable functions f, $g : \Omega \to \mathbb{R}$ are called synchronous (asynchronous) on Ω, if*

$$[f(x) - f(y)][g(x) - g(y)] \geq 0 \tag{1.10}$$

for μ-a.e. x, $y \in \Omega$,

By making use of the following *Korkine's identity* for Lebesgue integrable functions f, g,

$$C_w(f,g) = \frac{1}{2} \int_\Omega \int_\Omega w(x) w(y) [f(x) - f(y)][g(x) - g(y)] d\mu(x) d\mu(y) \tag{1.11}$$

we get the general Čebyšev's inequality for *synchronous functions* f, g such that f, g, $fg \in L_w(\Omega, \mu)$,

$$C_w(f,g) \geq 0. \tag{1.12}$$

If $\Phi,\ \Psi : [m, M] \to \mathbb{R}$ have the same monotonicity on $[m, M]$ and ℓ is measurable with $-\infty < m \le \ell \le M < \infty$ for μ-a.e. on Ω, then $\Phi \circ \ell,\ \Psi \circ \ell$ are synchronous on Ω, then we have the Čebyšev's inequality

$$C_w (\Phi \circ \ell, \Psi \circ \ell) \ge 0, \tag{1.13}$$

provided that $\Phi \circ \ell,\ \Psi \circ \ell,\ (\Phi \circ \ell)(\Psi \circ \ell) \in L_w(\Omega, \mu)$.

For more recent upper bounds related to the Čebyšev functional see [1]-[9], [11]-[21] and [24]-[31].

2. Inequalities for Generalized Lipschitz Type Condition

We have the following result:

Theorem 2. *Assume that $f,\ g,\ \ell$ are μ-measurable on Ω and satisfying the following conditions for some positive constants $L,\ K > 0$*

$$|f(x) - f(y)| \le L |\ell(x) - \ell(y)| \tag{2.1}$$

and

$$|g(x) - g(y)| \le K |\ell(x) - \ell(y)| \tag{2.2}$$

for μ-a.e. $x,\ y \in \Omega$. If $f,\ g,\ fg,\ \ell,\ \ell^2 \in L_w(\Omega, \mu)$, then

$$|C_w(f, g)| \le LK \left(\int_\Omega w d\mu\right)^2 D^2_{w,2}(\ell). \tag{2.3}$$

Proof. By the identity (1.11) we have

$$\begin{aligned}
|C_w(f, g)| &\le \frac{1}{2}\int_\Omega\int_\Omega w(x)\, w(y)\, |[f(x) - f(y)][g(x) - g(y)]|\, d\mu(x)\, d\mu(y) \\
&= \frac{1}{2}\int_\Omega\int_\Omega w(x)\, w(y)\, |f(x) - f(y)|\, |g(x) - g(y)|\, d\mu(x)\, d\mu(y) \\
&\le \frac{1}{2} LK \int_\Omega\int_\Omega w(x)\, w(y)\, [\ell(x) - \ell(y)]^2\, d\mu(x)\, d\mu(y) \\
&= LKC_w(\ell, \ell) = LK\left[\int_\Omega w d\mu \int_\Omega w\ell^2 d\mu - \left(\int_\Omega w\ell d\mu\right)^2\right] \\
&= LK\left(\int_\Omega w d\mu\right)^2 D^2_{w,2}(\ell)
\end{aligned}$$

and the inequality (2.3) is thus proved. □

Corollary 2. *With the assumptions of Theorem 2 for f and ℓ, we have*

$$|C_w(f, \ell)| \le L\left(\int_\Omega w d\mu\right)^2 D^2_{w,2}(\ell). \tag{2.4}$$

The proof follows by (2.3) on taking $g = \ell$ and $K = 1$.

Corollary 3. *With the assumptions of Theorem 2 and if $-\infty < m \le \ell \le M < \infty$ for μ-a.e. on Ω, then*

$$|C_w(f, g)| \le \frac{1}{4} LK \left(\int_\Omega w d\mu\right)^2 (M - m)^2. \tag{2.5}$$

In particular,

$$|C_w(f,\ell)| \leq \frac{1}{4} L \left(\int_\Omega w d\mu \right)^2 (M-m)^2 . \tag{2.6}$$

The proof follows by (2.3) and by observing that

$$D_{w,2}^2(\ell) \leq \frac{1}{4}(M-m)^2 .$$

We note that similar results for isotonic linear functionals were obtained in 1994 by Pečarić, Dragomir and Sándor, see [33].

We observe that if $\Phi : [m, M] \to \mathbb{R}$ is Lipschitzian with the constant $L > 0$, then

$$|\Phi(u) - \Phi(v)| \leq L |u - v|$$

for all $u, v \in [m, M]$. If $-\infty < m \leq \ell \leq M < \infty$ for μ-a.e. on Ω, then

$$|\Phi(\ell(x)) - \Phi(\ell(y))| \leq L |\ell(x) - \ell(y)|$$

for μ-a.e. $x, y \in \Omega$. Therefore, by choosing $f = \Phi \circ \ell$ we obtain that f satisfies the condition (2.1).

Corollary 4. *If Φ, $\Psi : [m, M] \to \mathbb{R}$ are Lipschitzian with the constants $L > 0$ and $K > 0$ and ℓ is measurable with $-\infty < m \leq \ell \leq M < \infty$ for μ-a.e. on Ω, then*

$$|C_w(\Phi \circ \ell, \Psi \circ \ell)| \leq \frac{1}{4} LK \left(\int_\Omega w d\mu \right)^2 (M-m)^2 . \tag{2.7}$$

In particular,

$$|C_w(\Phi \circ \ell, \ell)| \leq \frac{1}{4} L \left(\int_\Omega w d\mu \right)^2 (M-m)^2 . \tag{2.8}$$

Remark 1. *If Φ, $\Psi : [m, M] \to \mathbb{R}$ are absolutely continuous on $[m, M]$ and $\|\Phi'\|_{[m,M],\infty} := \operatorname{esssup}_{t\in[m,M]} |\Phi'| < \infty$, $\|\Psi'\|_{[m,M],\infty} < \infty$, then from Corollary 4 we derive*

$$|C_w(\Phi \circ \ell, \Psi \circ \ell)| \leq \frac{1}{4} \|\Phi'\|_{[m,M],\infty} \|\Psi'\|_{[m,M],\infty} \left(\int_\Omega w d\mu \right)^2 (M-m)^2 . \tag{2.9}$$

In particular,

$$|C_w(\Phi \circ \ell, \ell)| \leq \frac{1}{4} \|\Phi'\|_{[m,M],\infty} \left(\int_\Omega w d\mu \right)^2 (M-m)^2 . \tag{2.10}$$

Assume that ℓ is measurable with $-\infty < m \leq \ell \leq M < \infty$ for μ-a.e. on Ω. We consider the functions $\Phi(t) = t^r$, $\Psi(t) = t^s$ for $r, s > 0$ and $t \in [m, M] \subset (0, \infty)$. Then

$$\begin{aligned} \|\Phi'\|_{[m,M],\infty} &= r \sup_{t\in[m,M]} t^{r-1} = r \times \begin{cases} m^{r-1} \text{ if } r \in (0,1), \\ M^{r-1} \text{ if } r \in [1,\infty) \end{cases} \\ &= : r B_{m,M,r} \end{aligned}$$

and

$$\begin{aligned} \|\Psi'\|_{[m,M],\infty} &= s \sup_{t\in[m,M]} t^{s-1} = s \times \begin{cases} m^{s-1} \text{ if } s \in (0,1), \\ M^{s-1} \text{ if } s \in [1,\infty). \end{cases} \\ &= : s B_{m,M,s} \end{aligned}$$

If we apply inequality (2.9) for this selection of functions, we get

$$0 \le \int_\Omega w d\mu \int_\Omega w\ell^{r+s} d\mu - \int_\Omega w\ell^r d\mu \int_\Omega w\ell^s d\mu \le \frac{1}{4} rs B_{m,M,r} B_{m,M,s} \left(\int_\Omega w d\mu\right)^2 (M-m)^2, \tag{2.11}$$

provided that ℓ^{r+s}, ℓ^r, $\ell^s \in L_w(\Omega,\mu)$.

Consider the function $\Psi(t) = \ln t$ and $t \in [m,M] \subset (0,\infty)$. Then $\Psi'(t) = \frac{1}{t}$ which gives that $\|\Psi'\|_{[m,M],\infty} = \frac{1}{m}$. Then by (2.9) we get

$$0 \le \int_\Omega w d\mu \int_\Omega w\ell^r \ln \ell d\mu - \int_\Omega w\ell^r d\mu \int_\Omega w \ln \ell d\mu \le \frac{1}{4m} r B_{m,M,r} \left(\int_\Omega w d\mu\right)^2 (M-m)^2, \tag{2.12}$$

provided that $\ell^r \ln \ell$, ℓ^r, $\ln \ell \in L_w(\Omega,\mu)$.

3. Inequalities for S-Dominated Functions

Assume that α, $\beta : [a,b] \to \mathbb{R}$ are *monotonic nondecreasing* on the interval $[a,b]$. We say that the real-valued function $\Phi : [a,b] \to \mathbb{R}$ is *S-dominated* by the pair (α,β) if [18]

$$|\Phi(y) - \Phi(x)|^2 \le [\alpha(y) - \alpha(x)][\beta(y) - \beta(x)] \tag{S}$$

for any x, $y \in [a,b]$.

We observe that by the monotonicity of the functions u and v and by the symmetry of the inequality (S) over x and y we can assume that (S) is satisfied only for $y > x$ with $x, y \in [a,b]$.

We can give numerous examples of such functions.

For instance, if we take f, $g \in L_2[a,b]$ the Hilbert space of all complex-valued functions that are square-Lebesgue integrable and denote

$$\Phi(x) := \int_a^x f(t) g(t) dt,\ \alpha(x) := \int_a^x |f(t)|^2 dt \text{ and } \beta(x) := \int_a^x |g(t)|^2 dt,$$

then we observe that α and β are monotonic nondecreasing on $[a,b]$ and by Cauchy-Bunyakovsky-Schwarz integral inequality we have for any $y > x$ with x, $y \in [a,b]$ that

$$|\Phi(y) - \Phi(x)|^2 = \left|\int_x^y f(t) g(t) dt\right|^2 \le \int_x^y |f(t)|^2 dt \int_x^y |g(t)|^2 dt \le [\alpha(y) - \alpha(x)][\beta(y) - \beta(x)].$$

Now, for r, $s > 0$ if we consider $f(t) := t^r$ and $g(t) := t^s$ for $t \ge 0$, then

$$\Phi_{r,s}(x) := \int_0^x t^{r+s} dt = \frac{1}{r+s+1} x^{r+s+1}$$

and

$$\alpha_r(x) := \int_0^x t^{2r} dt = \frac{1}{2r+1} x^{2r+1},\ \beta_s(x) := \int_0^x t^{2s} dt = \frac{1}{2s+1} x^{2s+1}.$$

Taking into account the above comments we observe that the function $\Phi_{r,s}$ is *S-dominated* by the pair (α_r, β_s) on any subinterval of $[0,\infty)$.

We can generalize the above concept as follows for measurable real-valued functions h, u, v on Ω.

Definition 2. *Assume that* u, v *and* h *are* μ*-measurable synchronous functions on* Ω. *If they satisfy the condition*

$$|h(y)-h(x)|^2 \le [u(y)-u(x)][v(y)-v(x)] \tag{3.1}$$

for μ*-a.e.* $x, y\in\Omega$, *then we say that* h *is S-synchronous-dominated by the pair* (u,v).

Theorem 3. *If* h *is S-synchronous-dominated by the pair* (u,v), *then*

$$0 \le C_w(h,h) \le C_w(u,v), \tag{3.2}$$

provided that u, v, uv, h, $h^2 \in L_w(\Omega,\mu)$.

Proof. By (3.1) we get

$$\begin{aligned}\frac{1}{2}\int_\Omega\int_\Omega w(x)w(y)|h(y)-h(x)|^2\,d\mu(x)\,d\mu(y) \\ \le \frac{1}{2}\int_\Omega\int_\Omega w(x)w(y)[u(y)-u(x)][v(y)-v(x)]\,d\mu(x)\,d\mu(y).\end{aligned} \tag{3.3}$$

Since

$$\frac{1}{2}\int_\Omega\int_\Omega w(x)w(y)|h(y)-h(x)|^2\,d\mu(x)\,d\mu(y) = C_w(h,h)$$

and

$$\frac{1}{2}\int_\Omega\int_\Omega w(x)w(y)[u(y)-u(x)][v(y)-v(x)]\,d\mu(x)\,d\mu(y) = C_w(u,v),$$

hence by (3.3) we derive (3.2). □

In the same spirit we also have:

Theorem 4. *If* h *is S-synchronous-dominated by the synchronous pair* (u,v) *and* k *is synchronous with* u, *then* k *is synchronous with* v *and*

$$|C_w(k,h)|^2 \le C_w(k,u)\,C_w(k,v) \tag{3.4}$$

provided that k, h, u, v, kh, ku, $kv \in L_w(\Omega,\mu)$.

Proof. Since $u(x)-u(y)\neq 0$ for μ-a.e. $x, y\in\Omega$ (otherwise the *S-dominated property becomes* trivial), then for those $x, y\in\Omega$

$$[k(y)-k(x)][v(y)-v(x)] = [k(y)-k(x)][u(x)-u(y)]\left[\frac{v(y)-v(x)}{u(x)-u(y)}\right]. \tag{3.5}$$

Since k is synchronous with u, then

$$[k(y)-k(x)][u(x)-u(y)] \ge 0.$$

Since v is synchronous with u, then

$$[u(y)-u(x)][v(y)-v(x)] \ge 0$$

and by dividing with $(u(x)-u(y))^2\neq 0$, we get

$$\frac{v(y)-v(x)}{u(x)-u(y)} \ge 0$$

and by (3.5) we derive that

$$[k(y)-k(x)][v(y)-v(x)]\geq 0$$

for μ-a.e. $x,\, y\in\Omega$, which shows that (k,v) are synchronous.

From (3.1) we have

$$|h(y)-h(x)|\leq |u(y)-u(x)|^{1/2}\,|v(y)-v(x)|^{1/2}$$

and by multiplying with $|k(y)-k(x)|\geq 0$, we derive

$$|(h(y)-h(x))(k(y)-k(x))|\leq |k(y)-k(x)|\,|u(y)-u(x)|^{1/2}\,|v(y)-v(x)|^{1/2},$$

for μ-a.e. $x,\, y\in\Omega$.

If we multiply this inequality by $w(x)\,w(y)\geq 0$ and integrate, then

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(h(y)-h(x))(k(y)-k(x))|\,d\mu(x)\,d\mu(y)\\
&\leq \frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|k(y)-k(x)|\\
&\times |u(y)-u(x)|^{1/2}\,|v(y)-v(x)|^{1/2}\,d\mu(x)\,d\mu(y).
\end{aligned}$$

By using the properties of the integral and modulus, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|(h(y)-h(x))(k(y)-k(x))|\,d\mu(x)\,d\mu(y)\\
&\geq \left|\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,[(h(y)-h(x))(k(y)-k(x))]\,d\mu(x)\,d\mu(y)\right|\\
&= |C_w(h,k)|.
\end{aligned}$$

By the weighted double integral Cauchy-Bunyakovsky-Schwarz inequality, we get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|k(y)-k(x)|\,|u(y)-u(x)|^{1/2}\,|v(y)-v(x)|^{1/2}\\
&\times d\mu(x)\,d\mu(y)\\
&\leq \frac{1}{2}\left(\int_\Omega\int_\Omega w(x)\,w(y)\,|k(y)-k(x)|\left(|u(y)-u(x)|^{1/2}\right)^2 d\mu(x)\,d\mu(y)\right)^{1/2}\\
&\times\left(\int_\Omega\int_\Omega w(x)\,w(y)\,|k(y)-k(x)|\left(|v(y)-v(x)|^{1/2}\right)^2 d\mu(x)\,d\mu(y)\right)^{1/2}\\
&=\left(\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|k(y)-k(x)|\,|u(y)-u(x)|\,d\mu(x)\,d\mu(y)\right)^{1/2}\\
&\times\left(\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|k(y)-k(x)|\,|v(y)-v(x)|\,d\mu(x)\,d\mu(y)\right)^{1/2}\\
&=\left(\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,[k(y)-k(x)]\,[u(y)-u(x)]\,d\mu(x)\,d\mu(y)\right)^{1/2}\\
&\times\left(\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,[k(y)-k(x)]\,[v(y)-v(x)]\,d\mu(x)\,d\mu(y)\right)^{1/2}\\
&=[C_w(k,u)]^{1/2}\,[C_w(k,v)]^{1/2},
\end{aligned}$$

which proves the desired inequality (3.4). □

Remark 2. *Assume that α, $\beta : [m, M] \to \mathbb{R}$ are monotonic nondecreasing on the interval $[m, M]$ and that the real-valued function $\Phi : [m, M] \to \mathbb{R}$ is S-dominated by the pair (α, β). If ℓ is measurable with $-\infty < m \le \ell \le M < \infty$ for μ-a.e. on Ω, then*

$$|\Phi(\ell(y)) - \Phi(\ell(x))|^2 \le [\alpha(\ell(y)) - \alpha(\ell(x))][\beta(\ell(y)) - \beta(\ell(x))]$$

for μ-a.e. $x, y \in \Omega$. This implies that the function $\Phi \circ \ell$ is S-synchronous-dominated by the pair $(\alpha \circ \ell, \beta \circ \ell)$.

By the inequality (3.2) we derive

$$0 \le C_w(\Phi \circ \ell, \Phi \circ \ell) \le C_w(\alpha \circ \ell, \beta \circ \ell), \tag{3.6}$$

provided that $\alpha \circ \ell$, $\beta \circ \ell$, $(\alpha \circ \ell)(\beta \circ \ell)$, $\Phi \circ \ell$, $(\Phi \circ \ell)^2 \in L_w(\Omega, \mu)$.

If $k : \Omega \to \mathbb{R}$ is synchronous with $\alpha \circ \ell$, then k is synchronous with $\beta \circ \ell$ and

$$|C_w(k, \Phi \circ \ell)|^2 \le C_w(k, \alpha \circ \ell) C_w(k, \beta \circ \ell) \tag{3.7}$$

provided that k, $\Phi \circ \ell$, $\alpha \circ \ell$, $\beta \circ \ell$, $k(\Phi \circ \ell)$, $k(\alpha \circ \ell)$, $k(\beta \circ \ell) \in L_w(\Omega, \mu)$.

If $\Psi : [m, M] \to \mathbb{R}$ is monotonic nondecreasing, then $k = \Psi \circ \ell$ is synchronous with $\alpha \circ \ell$ and $\beta \circ \ell$, and by (3.7) we e get the more practical result

$$|C_w(\Psi \circ \ell, \Phi \circ \ell)|^2 \le C_w(\Psi \circ \ell, \alpha \circ \ell) C_w(\Psi \circ \ell, \beta \circ \ell) \tag{3.8}$$

provided that $\Psi \circ \ell$, $\Phi \circ \ell$, $\alpha \circ \ell$, $\beta \circ \ell$, $(\Psi \circ \ell)(\Phi \circ \ell)$, $(\Psi \circ \ell)(\alpha \circ \ell)$, $(\Psi \circ \ell)(\beta \circ \ell) \in L_w(\Omega, \mu)$.

For r, $s > 0$, then by taking

$$\Phi_{r,s}(x) := \frac{1}{r + s + 1} x^{r+s+1}$$

and

$$\alpha_r(x) := \frac{1}{2r + 1} x^{2r+1},\ \beta_s(x) := \frac{1}{2s + 1} x^{2s+1},$$

we observe that the function $\Phi_{r,s}$ is S-*dominated* by the pair (α_r, β_s) on any subinterval of $[0, \infty)$.

Assume that ℓ is measurable with $-\infty < m \le \ell \le M < \infty$ for μ-a.e. on Ω. If we use inequality (3.6), then we get

$$\begin{aligned} 0 &\le \int_\Omega w d\mu \int_\Omega w \ell^{2(r+s+1)} d\mu - \left(\int_\Omega w \ell^{(r+s+1)} d\mu \right)^2 \\ &\le \frac{(r + s + 1)^2}{(2r + 1)(2s + 1)} \\ &\quad \times \left[\int_\Omega w d\mu \int_\Omega w \ell^{2(r+s+1)} d\mu - \int_\Omega w \ell^{2r+1} d\mu \int_\Omega w \ell^{2s+1} d\mu \right] \end{aligned} \tag{3.9}$$

for r, $s > 0$, provided $\ell^{2(r+s+1)}$, $\ell^{(r+s+1)}$, ℓ^{2r+1}, $\ell^{2s+1} \in L_w(\Omega, \mu)$.

If $\Psi : [m, M] \to \mathbb{R}$ is monotonic nondecreasing, then by (3.8) we derive

$$0 \le \int_\Omega w d\mu \int_\Omega w\ell^{r+s+1} (\Psi \circ \ell) d\mu - \int_\Omega w\ell^{r+s+1} d\mu \int_\Omega w (\Psi \circ \ell) d\mu \le \frac{r+s+1}{(2r+1)(2s+1)} \times \left[\int_\Omega w d\mu \int_\Omega w\ell^{2r+1} (\Psi \circ \ell) d\mu - \int_\Omega w\ell^{2r+1} d\mu \int_\Omega w (\Psi \circ \ell) d\mu\right] \times \left[\int_\Omega w d\mu \int_\Omega w\ell^{2s+1} (\Psi \circ \ell) d\mu - \int_\Omega w\ell^{2s+1} d\mu \int_\Omega w (\Psi \circ \ell) d\mu\right] \tag{3.10}$$

for $r, s > 0$, provided $\ell^{r+s+1} (\Psi \circ \ell), \ell^{r+s+1}, \Psi \circ \ell, \ell^{2r+1} (\Psi \circ \ell), \ell^{2r+1}, \ell^{2s+1} (\Psi \circ \ell), \ell^{2s+1} \in L_w (\Omega, \mu)$.

If in this inequality we take $\Psi (t) = t^\tau$ with $\tau > 0$, then we get

$$0 \le \int_\Omega w d\mu \int_\Omega w\ell^{r+s+\tau+1} d\mu - \int_\Omega w\ell^{r+s+1} d\mu \int_\Omega w\ell^\tau d\mu \le \frac{r+s+1}{(2r+1)(2s+1)} \times \left[\int_\Omega w d\mu \int_\Omega w\ell^{2r+\tau+1} d\mu - \int_\Omega w\ell^{2r+1} d\mu \int_\Omega w\ell^\tau d\mu\right] \times \left[\int_\Omega w d\mu \int_\Omega w\ell^{2s+\tau+1} d\mu - \int_\Omega w\ell^{2s+1} d\mu \int_\Omega w\ell^\tau d\mu\right] \tag{3.11}$$

for $r, s, \tau > 0$, provided $\ell^{r+s+\tau+1}, \ell^{r+s+1}, \ell^\tau, \ell^{2r+\tau+1}, \ell^{2r+1}, \ell^\tau, \ell^{2s+\tau+1}, \ell^{2s+1} \in L_w (\Omega, \mu)$.

4. Inequalities for (p, q)-H-Dominated Functions

Assume that $\alpha, \beta : [a, b] \to \mathbb{R}$ are *monotonic nondecreasing* on the interval $[a, b]$. We say that the real-valued function $\Phi : [a, b] \to \mathbb{R}$ is (p, q)-H-*dominated* by the pair (α, β) if [19]

$$|\Phi (y) - \Phi (x)| \le [\alpha (y) - \alpha (x)]^{1/p} [\beta (y) - \beta (x)]^{1/q} \tag{S}$$

for any $x, y \in [a, b]$ with $y \ge x$.

We can give numerous examples of such functions.

For instance, if we take f, g two measurable real-valued functions such that $|f|^p$ and $|g|^q$ are Lebesgue integrable and denote

$$\Phi (x) := \int_a^x f (t) g (t) dt, \ \alpha (x) := \int_a^x |f (t)|^p dt \text{ and } v (x) := \int_a^x |g (t)|^q dt, \tag{4.1}$$

then we observe that u and v are monotonic nondecreasing on $[a, b]$ and by *Hölder integral inequality* we have for any $y \ge x$ with $x, y \in [a, b]$ that

$$|\Phi (y) - \Phi (x)| = \left|\int_x^y f (t) g (t) dt\right| \le \left(\int_x^y |f (t)|^p dt\right)^{1/p} \left(\int_x^y |g (t)|^q dt\right)^{1/q} \le [\alpha (y) - \alpha (x)]^{1/p} [\beta (y) - \beta (x)]^{1/q}.$$

Now, for $m, n > 0$ if we consider $f (t) := t^m$ and $g (t) := t^n$ for $t \ge 0$, then

$$\Phi_{m,n} (x) := \int_0^x t^{m+n} dt = \frac{1}{m+n+1} x^{m+n+1}$$

and

$$\alpha_{m,p}(x) := \int_0^x t^{pm} dt = \frac{1}{pm+1} x^{pm+1}, \ \beta_{n,q}(x) := \int_0^x t^{qn} dt = \frac{1}{qn+1} x^{qn+1},$$

for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1.$

We can introduce a similar concept for measurable functions on Ω :

Definition 3. *Assume that u, v and h are μ-measurable real-valued functions on Ω and $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If they satisfy the condition*

$$|h(y) - h(x)| \le |u(y) - u(x)|^{1/p} |v(y) - v(x)|^{1/q} \tag{4.2}$$

for μ-a.e. $x,\ y \in \Omega$, then we say that h is strongly (p,q)-H-dominated by the pair (u,v). If $p = q = 2$, we say that h is strongly S-dominated by the pair (u,v).

We observe that the examples in (4.1) are also valid for *strongly* (p,q)-H-*dominated functions* on a real interval $[a,b]$.

Theorem 5. *Let $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If h is strongly (p,q)-H-dominated by the pair (u,v) and k is synchronous with u and v, then*

$$|C_w(k,h)| \le [C_w(k,u)]^{1/p} [C_w(k,v)]^{1/q} \tag{4.3}$$

provided that $k,\ h,\ u,\ v,\ kh,\ ku,\ kv \in L_w(\Omega,\mu)$.

In particular, if h is strongly S-dominated by the pair (u,v) and k is synchronous with u and v, then

$$|C_w(k,h)|^2 \le C_w(k,u)\, C_w(k,v). \tag{4.4}$$

Proof. From (4.2) we have

$$|h(y) - h(x)| \le |u(y) - u(x)|^{1/p} |v(y) - v(x)|^{1/q}$$

and by multiplying with $|k(y) - k(x)| \ge 0$, we derive

$$\begin{aligned} &|(h(y) - h(x))(k(y) - k(x))| \\ &\le |k(y) - k(x)|\, |u(y) - u(x)|^{1/p} |v(y) - v(x)|^{1/q}, \end{aligned}$$

for μ-a.e. $x,\ y \in \Omega$.

If we multiply this inequality by $w(x)\,w(y) \ge 0$ and integrate, then

$$\begin{aligned} &\frac{1}{2}\int_\Omega \int_\Omega w(x)\,w(y)\,|(h(y) - h(x))(k(y) - k(x))|\, d\mu(x)\, d\mu(y) \\ &\le \frac{1}{2}\int_\Omega \int_\Omega w(x)\,w(y)\,|k(y) - k(x)| \\ &\times |u(y) - u(x)|^{1/p} |v(y) - v(x)|^{1/q} d\mu(x)\, d\mu(y). \end{aligned}$$

By using the properties of the integral and modulus, we have

$$\begin{aligned} &\frac{1}{2}\int_\Omega \int_\Omega w(x)\,w(y)\,|(h(y) - h(x))(k(y) - k(x))|\, d\mu(x)\, d\mu(y) \\ &\ge \left| \frac{1}{2}\int_\Omega \int_\Omega w(x)\,w(y)\,[(h(y) - h(x))(k(y) - k(x))]\, d\mu(x)\, d\mu(y) \right| \\ &= |C_w(h,k)|. \end{aligned}$$

By the weighted double integral Hölder's inequality for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we get

$$\begin{aligned}
&\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|k(y)-k(x)|\,|u(y)-u(x)|^{1/p}\,|v(y)-v(x)|^{1/q}\\
&\times d\mu(x)\,d\mu(y)\\
&\leq \frac{1}{2}\left(\int_\Omega\int_\Omega w(x)\,w(y)\,|k(y)-k(x)|\left(|u(y)-u(x)|^{1/p}\right)^p d\mu(x)\,d\mu(y)\right)^{1/p}\\
&\times\left(\int_\Omega\int_\Omega w(x)\,w(y)\,|k(y)-k(x)|\left(|v(y)-v(x)|^{1/q}\right)^q d\mu(x)\,d\mu(y)\right)^{1/q}\\
&=\left(\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|k(y)-k(x)|\,|u(y)-u(x)|\,d\mu(x)\,d\mu(y)\right)^{1/p}\\
&\times\left(\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,|k(y)-k(x)|\,|v(y)-v(x)|\,d\mu(x)\,d\mu(y)\right)^{1/q}\\
&=\left(\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,[k(y)-k(x)]\,[u(y)-u(x)]\,d\mu(x)\,d\mu(y)\right)^{1/p}\\
&\times\left(\frac{1}{2}\int_\Omega\int_\Omega w(x)\,w(y)\,[k(y)-k(x)]\,[v(y)-v(x)]\,d\mu(x)\,d\mu(y)\right)^{1/q}\\
&=[C_w(k,u)]^{1/p}\,[C_w(k,v)]^{1/q},
\end{aligned}$$

which proves the desired inequality (3.4). □

Remark 3. *Assume that α, $\beta:[m,M]\to\mathbb{R}$ are monotonic nondecreasing on the interval $[m,M]$ and that the real-valued function $\Phi:[m,M]\to\mathbb{R}$ is (p,q)-H-dominated by the pair (α,β), where p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If ℓ is measurable with $-\infty<m\leq\ell\leq M<\infty$ for μ-a.e. on Ω, then*

$$|\Phi(\ell(y))-\Phi(\ell(x))|\leq|\alpha(\ell(y))-\alpha(\ell(x))|^{1/p}\,|\beta(\ell(y))-\beta(\ell(x))|^{1/q}$$

for μ-a.e. x, $y\in\Omega$. This implies that the function $\Phi\circ\ell$ is strongly (p,q)-H-dominated by the pair $(\alpha\circ\ell,\beta\circ\ell)$.

If $k:\Omega$ is synchronous with $\alpha\circ\ell$ and $\beta\circ\ell$, then

$$|C_w(k,\Phi\circ\ell)|\leq[C_w(k,\alpha\circ\ell)]^{1/p}\,[C_w(k,\beta\circ\ell)]^{1/q} \tag{4.5}$$

provided that k, $\Phi\circ\ell$, $\alpha\circ\ell$, $\beta\circ\ell$, $k(\Phi\circ\ell)$, $k(\alpha\circ\ell)$, $k(\beta\circ\ell)\in L_w(\Omega,\mu)$.

If $\Psi:[m,M]\to\mathbb{R}$ is monotonic nondecreasing, then $\Psi\circ\ell$ is synchronous with $\alpha\circ\ell$ and $\beta\circ\ell$ and by (4.5) we derive

$$|C_w(\Psi\circ\ell,\Phi\circ\ell)|\leq[C_w(\Psi\circ\ell,\alpha\circ\ell)]^{1/p}\,[C_w(\Psi\circ\ell,\beta\circ\ell)]^{1/q}. \tag{4.6}$$

For m, $n>0$, consider the functions

$$\Phi_{m,n}(x):=\frac{1}{m+n+1}x^{m+n+1}$$

and

$$\alpha_{m,p}(x):=\frac{1}{pm+1}x^{pm+1},\ \beta_{n,q}(x):=\frac{1}{qn+1}x^{qn+1},$$

for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

We observe that $\Phi_{m,n}$ is (p,q)-H-*dominated* by the pair $\left(\alpha_{m,p},\beta_{n,q}\right)$ on any subinterval of $[0,\infty)$.

Assume that ℓ is measurable with $-\infty<m\leq\ell\leq M<\infty$ for μ-a.e. on Ω. If $\Psi:[m,M]\to\mathbb{R}$ is monotonic nondecreasing, then by (4.6) we have

$$\begin{aligned}(4.7)\qquad 0&\leq\int_\Omega wd\mu\int_\Omega w\ell^{m+n+1}\left(\Psi\circ\ell\right)d\mu-\int_\Omega w\ell^{m+n+1}d\mu\int_\Omega w\left(\Psi\circ\ell\right)d\mu\\&\leq\frac{m+n+1}{\left(pm+1\right)^{1/p}\left(qn+1\right)^{1/q}}\\&\times\left[\int_\Omega wd\mu\int_\Omega w\ell^{pm+1}\left(\Psi\circ\ell\right)d\mu-\int_\Omega w\ell^{pm+1}d\mu\int_\Omega w\left(\Psi\circ\ell\right)d\mu\right]^{1/p}\\&\times\left[\int_\Omega wd\mu\int_\Omega w\ell^{qn+1}\left(\Psi\circ\ell\right)d\mu-\int_\Omega w\ell^{qn+1}d\mu\int_\Omega w\left(\Psi\circ\ell\right)d\mu\right]^{1/q}\end{aligned}$$

provided that $\ell^{m+n+1}, \left(\Psi\circ\ell\right), \ell^{m+n+1}\left(\Psi\circ\ell\right), \ell^{pm+1}, \ell^{pm+1}\left(\Psi\circ\ell\right), \ell^{qn+1}, \ell^{qn+1}\left(\Psi\circ\ell\right)$ $\in L_w\left(\Omega,\mu\right).$

For $p=q=2$, we derive

$$\begin{aligned}(4.8)\qquad 0&\leq\int_\Omega wd\mu\int_\Omega w\ell^{m+n+1}\left(\Psi\circ\ell\right)d\mu-\int_\Omega w\ell^{m+n+1}d\mu\int_\Omega w\left(\Psi\circ\ell\right)d\mu\\&\leq\frac{m+n+1}{\left(2m+1\right)^{1/2}\left(2n+1\right)^{1/2}}\\&\times\left[\int_\Omega wd\mu\int_\Omega w\ell^{2m+1}\left(\Psi\circ\ell\right)d\mu-\int_\Omega w\ell^{2m+1}d\mu\int_\Omega w\left(\Psi\circ\ell\right)d\mu\right]^{1/2}\\&\times\left[\int_\Omega wd\mu\int_\Omega w\ell^{2n+1}\left(\Psi\circ\ell\right)d\mu-\int_\Omega w\ell^{2n+1}d\mu\int_\Omega w\left(\Psi\circ\ell\right)d\mu\right]^{1/2},\end{aligned}$$

provided that $\ell^{m+n+1}, \left(\Psi\circ\ell\right), \ell^{m+n+1}\left(\Psi\circ\ell\right), \ell^{2m+1}, \ell^{2m+1}\left(\Psi\circ\ell\right), \ell^{2n+1}, \ell^{2n+1}\left(\Psi\circ\ell\right)$ $\in L_w\left(\Omega,\mu\right).$

If in (4.7) we take $\Psi\left(t\right)=t^\tau$ with $\tau>0$, then we get

$$\begin{aligned}(4.9)\qquad 0&\leq\int_\Omega wd\mu\int_\Omega w\ell^{m+n+\tau+1}d\mu-\int_\Omega w\ell^{m+n+1}d\mu\int_\Omega w\ell^\tau d\mu\\&\leq\frac{m+n+1}{\left(pm+1\right)^{1/p}\left(qn+1\right)^{1/q}}\\&\times\left[\int_\Omega wd\mu\int_\Omega w\ell^{pm+\tau+1}d\mu-\int_\Omega w\ell^{pm+1}d\mu\int_\Omega w\ell^\tau d\mu\right]^{1/p}\\&\times\left[\int_\Omega wd\mu\int_\Omega w\ell^{qn+\tau+1}d\mu-\int_\Omega w\ell^{qn+1}d\mu\int_\Omega w\ell^\tau d\mu\right]^{1/q}\end{aligned}$$

provided that $\ell^{m+n+1}, \ell^\tau, \ell^{m+n+\tau+1}, \ell^{pm+1}, \ell^{pm+\tau+1}, \ell^{qn+1}, \ell^{qn+\tau+1}\in L_w\left(\Omega,\mu\right).$

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