SOME INEQUALITIES FOR THE WEIGHTED ČEBYŠEV FUNCTIONAL IN TERMS OF INTEGRAL MEANS DIFFERENCE WITH APPLICATIONS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. For $w:[a,b]\to\mathbb{R}$ continuous and positive on the interval $[a,b]$ and f,g are Lebesgue integrable on $[a,b]$, we consider the Čebyšev functional

$$D_w(f,g):=\int_a^b w(t)\,dt\int_a^b f(t)\,g(t)\,w(t)\,dt-\int_a^b f(t)\,w(t)\,dt\int_a^b g(t)\,w(t)\,dt.$$

In this paper we show among other that, if f is of bounded variation on $[a,b]$ and g is integrable on $[a,b]$, then

$$|D_w(f,g)|\le\frac{1}{4}\left(\int_a^b w(s)\,ds\right)^2\|\Delta_{g,w}\|_{[a,b],\infty}\bigvee_a^b(f),$$

where

$$\Delta_{g,w}(t):=\frac{1}{\int_t^b w(s)\,ds}\int_t^b g(s)\,w(s)\,ds-\frac{1}{\int_a^t w(s)\,ds}\int_a^t g(s)\,w(s)\,ds$$

for $t\in(a,b)$. Applications for continuous probability density functions supported on infinite intervals are also given.

1. Introduction

In 1998, S. S. Dragomir and I. Fedotov [21], in order to approximate the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ with the simpler expression

$$\frac{1}{b-a}[u(b)-u(a)]\int_a^b f(t)\,dt$$

introduced the following error functional

$$C(f,u):=\int_a^b f(t)\,du(t)-\frac{1}{b-a}[u(b)-u(a)]\int_a^b f(t)\,dt \tag{1.1}$$

provided that both the Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ and the Riemann integral $\int_a^b f(t)\,dt$ exist.

If $u(t)=\int_a^t g(s)\,ds$, $t\in[a,b]$, with g integrable on $[a,b]$, then

$$\begin{aligned}C\left(f,\int_a^{\cdot} g(s)\,ds\right)&=\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\int_a^b g(t)\,dt\\&=(b-a)^{-1}D(f,g),\end{aligned}\tag{1.2}$$

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where

$$D(f,g) = (b-a)\int_a^b f(t)\,g(t)\,dt - \int_a^b f(t)\,dt \int_a^b g(t)\,dt$$

is the well known *Čebyšev functional.* Therefore $C(f,u)$ can be see as a generalised *Čebyšev type functional.*

The natural connection provided by the equality (1.2) also motivates the study of the functional $D(\cdot,\cdot;a,b)$ since there are numerous results in the literature concerning bounds for the Čebyšev functional for which we only mention the following ones:

$$|D(f,g)| \le \frac{1}{4}(b-a)^2(\Phi-\phi)(\Gamma-\gamma) \qquad \text{(Grüss 1935, [23])} \tag{1.3}$$

provided $\varphi \le f(x) \le \Phi$, $\gamma \le g(x) \le \Gamma$ for each $x \in [a,b]$;

$$|D(f,g)| \le \frac{1}{12}\cdot(b-a)^4 \|f'\|_\infty \|g'\|_\infty \qquad \text{(Čebyšev 1882, [7])} \tag{1.4}$$

if f, g are absolutely continuous on $[a,b]$ and f', $g' \in L_\infty[a,b]$;

$$|D(f,g)| \le \frac{1}{8}(b-a)^3(\Phi-\phi)\|g'\|_\infty \qquad \text{(Ostrowski 1970, [26])} \tag{1.5}$$

provided $\varphi \le f(x) \le \Phi$ for any $x \in [a,b]$ and $g' \in L_\infty[a,b]$, and

$$|D(f,g)| \le \frac{1}{\pi^2}(b-a)^3\|f'\|_2\|g'\|_2 \qquad \text{(Lupaş 1973, [25])} \tag{1.6}$$

provided f', $g' \in L_2[a,b]$. The multiplicative constants $\frac{1}{4}, \frac{1}{12}, \frac{1}{8}$ and $\frac{1}{\pi^2}$ are best possible in the sense that they cannot be replaced by smaller quantities.

Recently, Cerone and Dragomir [3], proved the following result:

$$|D(f,g)| \le \inf_{\gamma\in\mathbb{R}} \|g-\gamma\|_\infty \cdot (b-a)\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \tag{1.7}$$

provided $f \in L[a,b]$ and $g \in L_\infty[a,b]$.

As particular cases of (1.7), we can state the results:

$$|D(f,g)| \le \|g\|_\infty (b-a)\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt \tag{1.8}$$

if $g \in L_\infty[a,b]$ and $f \in L[a,b]$, and

$$|D(f,g)| \le \frac{1}{2}(M-m)(b-a)\int_a^b \left| f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right| dt, \tag{1.9}$$

where $m \le g(x) \le M$ for $x \in [a,b]$. The constants 1 in (1.8) and $\frac{1}{2}$ in (1.9) are best possible. The inequality (1.9) has been obtained before in a different way by Cheng & Sun in [8]. However they did not considered the problem of sharpness.

For generalizations of (1.9) in abstract Lebesgue spaces, best constants and discrete versions, see [4] in both preprint and final form.

In this paper we show among others that, if f is of bounded variation on $[a,b]$ and g is integrable on $[a,b]$, then

$$|D_w(f,g)| \le \frac{1}{4}\left(\int_a^b w(s)\,ds\right)^2 \|\Delta_{g,w}\|_{[a,b],\infty} \bigvee_a^b (f),$$

where

$$\Delta_{g,w}(t) := \frac{1}{\int_t^b w(s)\,ds}\int_t^b g(s)\,w(s)\,ds - \frac{1}{\int_a^t w(s)\,ds}\int_a^t g(s)\,w(s)\,ds$$

for $t \in (a,b)$. Applications for continuous probability density functions supported on infinite intervals are also given.

2. Preliminary Facts

For the integrator $u : [a,b] \to \mathbb{R}$ consider the following auxiliary mappings Φ_u, Γ_u and Δ_u that have been introduced in [15] (see also [16] and [17]):

$$\Phi_u(t) := \frac{(t-a)\,u(b) + (b-t)\,u(a)}{b-a} - u(t), \qquad t \in [a,b]; \tag{2.1}$$

$$\Gamma_u(t) := (t-a)\,[u(b) - u(t)] - (b-t)\,[u(t) - u(a)], \qquad t \in [a,b] \tag{2.2}$$

and

$$\Delta_u(t) := \frac{u(b) - u(t)}{b-t} - \frac{u(t) - u(a)}{t-a}, \qquad t \in (a,b). \tag{2.3}$$

The following representation result was essentially established in [15], (see also [16]).

Theorem 1 (Dragomir 2004, [15])**.** *Let $k, u : [a,b] \to \mathbb{R}$ be such that the Riemann-Stieltjes integral $\int_a^b k(t)\,du(t)$ and the Riemann integral $\int_a^b k(t)\,dt$ exist. Then*

$$\begin{aligned} C(k,u) &= \int_a^b \Phi_u(t)\,dk(t) = \frac{1}{b-a}\int_a^b \Gamma_u(t)\,dk(t) \\ &= \frac{1}{b-a}\int_a^b (t-a)(b-t)\,\Delta_u(t)\,dk(t). \end{aligned} \tag{2.4}$$

The following bounds for the functional $C(k,u)$ can then be stated:

Theorem 2 (Dragomir 2004, [15])**.** *Assume that $k, u : [a,b] \to \mathbb{R}$.*

(i) *If k is of bounded variation and u is continuous on $[a,b]$, then*

$$|C(k,u)| \le \begin{cases} \sup_{t\in[a,b)} |\Phi_u(t)| \bigvee_a^b(k), \\ \\ \frac{1}{b-a}\sup_{t\in[a,b]} |\Gamma_u(t)| \bigvee_a^b(k), \\ \\ \frac{1}{b-a}\sup_{t\in(a,b)} \left[(t-a)(b-t)\,|\Delta_u(t)|\right] \bigvee_a^b(k). \end{cases} \tag{2.5}$$

(ii) *If k is L-Lipschitzian and u is Riemann integrable on $[a,b]$, then*

$$|C(k,u)| \le \begin{cases} L\int_a^b |\Phi_u(t)|\,dt, \\ \\ \frac{L}{b-a}\int_a^b |\Gamma_u(t)|\,dt, \\ \\ \frac{L}{b-a}\int_a^b (t-a)(b-t)\,|\Delta_u(t)|\,dt. \end{cases} \tag{2.6}$$

(iii) *If k is monotonic nondecreasing on $[a,b]$ and u is continuous on $[a,b]$, then*

$$|C(k,u)| \leq \begin{cases} \int_a^b |\Phi_u(t)|\,dk(t), \\ \\ \frac{1}{b-a}\int_a^b |\Gamma_u(t)|\,dk(t), \\ \\ \frac{1}{b-a}\int_a^b (t-a)(b-t)|\Delta_u(t)|\,dk(t). \end{cases} \tag{2.7}$$

Corollary 1 (Dragomir 2004, [15])**.** *Let $k,u:[a,b]\to\mathbb{R}$.*

(i) *If k is of bounded variation and u is continuous, then*

$$|C(k,u)| \leq \frac{1}{4}(b-a)\|\Delta_u\|_{[a,b],\infty}\bigvee_a^b(k); \tag{2.8}$$

(ii) *If k is L-Lipschitzian and u is Riemann integrable on $[a,b]$, then*

$$|C(k,u)| \leq \begin{cases} \frac{1}{6}L(b-a)^2\|\Delta_u\|_{[a,b],\infty}, \\ \\ L(b-a)^{1+\frac{1}{q}}\left[B(q+1,q+1)\right]^{\frac{1}{q}}\|\Delta_u\|_{[a,b],p}, \\ \text{if } \Delta_u\in L_p[a,b] \quad \text{and } \; p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{4}L(b-a)\|\Delta_u\|_{[a,b],1}, \end{cases} \tag{2.9}$$

where $B(\cdot,\cdot)$ is Euler's Beta function;

(iii) *If k is monotonic nondecreasing on $[a,b]$ and u is continuous, then*

$$|C(k,u)| \leq \begin{cases} \frac{1}{4}(b-a)\int_a^b |\Delta_u(t)|\,dk(t), \\ \\ \frac{1}{b-a}\left(\int_a^b \left[(b-t)(t-a)\right]^q dk(t)\right)^{\frac{1}{q}}\left(\int_a^b |\Delta_u(t)|^p dk(t)\right)^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{b-a}\|\Delta_u\|_{[a,b],\infty}\int_a^b (t-a)(b-t)\,dk(t). \end{cases} \tag{2.10}$$

If $u(t)=\int_a^t v(s)\,ds$, $t\in[a,b]$, with v integrable on $[a,b]$, then

$$\tilde{\Delta}_v(t) := \Delta_{\int_a^{\cdot} v(s)ds}(t) = \frac{1}{b-t}\int_t^b v(s)\,ds - \frac{1}{t-a}\int_a^t v(s)\,ds,\ t\in(a,b). \tag{2.11}$$

On making use of the identity (2.4) we have the following representation of the Cebysev's functional

$$D(k,v) = \int_a^b (t-a)(b-t)\tilde{\Delta}_v(t)\,dk(t). \tag{2.12}$$

By the use of Corollary 1, we can state now the following result for *Čebyšev functional:*

Proposition 1. *Assume that $v:[a,b]\to\mathbb{C}$ is Lebesgue integrable on $[a,b]$.*

(i) *If k is of bounded variation on $[a,b]$, then*

$$|D(k,v)| \leq \frac{1}{4}(b-a)^2 \left\|\tilde{\Delta}_v\right\|_{[a,b],\infty} \bigvee_a^b (k). \tag{2.13}$$

(ii) *If k is L-Lipschitzian on $[a,b]$, then*

$$|D(k,v)| \leq L(b-a)^2 \begin{cases} \frac{1}{6}(b-a)\left\|\tilde{\Delta}_v\right\|_{[a,b],\infty}, \\ \\ (b-a)^{\frac{1}{q}}\left[B(q+1,q+1)\right]^{\frac{1}{q}}\left\|\tilde{\Delta}_v\right\|_{[a,b],p}, \\ \text{if } \tilde{\Delta}_v \in L_p[a,b] \text{ and } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \frac{1}{4}\left\|\tilde{\Delta}_v\right\|_{[a,b],1}. \end{cases} \tag{2.14}$$

(iii) *If k is monotonic nondecreasing on $[a,b]$, then*

$$|D(k,v)| \leq \begin{cases} \frac{1}{4}(b-a)^2\int_a^b\left|\tilde{\Delta}_v(t)\right|dk(t), \\ \\ \left(\int_a^b\left[(b-t)(t-a)\right]^q dk(t)\right)^{\frac{1}{q}}\left(\int_a^b\left|\tilde{\Delta}_v(t)\right|^p dk(t)\right)^{\frac{1}{p}}, \\ \qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|\tilde{\Delta}_v\right\|_{[a,b],\infty}\int_a^b(t-a)(b-t)\,dk(t). \end{cases} \tag{2.15}$$

3. Main Results

We can define the functional:

$$D_{h'}(f,g) := [h(b)-h(a)]\int_a^b f(t)g(t)h'(t)\,dt - \int_a^b f(t)h'(t)\,dt\int_a^b g(t)h'(t)\,dt, \tag{3.1}$$

where h is absolutely continuous and f, g are Lebesgue measurable on $[a,b]$ and such that the above integrals exist.

Theorem 3. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f is of bounded variation on $[a,b]$ and g is integrable on $[a,b]$, then*

$$|D_{h'}(f,g)| \leq \frac{1}{4}[h(b)-h(a)]^2\left\|\Delta_{g,h}\right\|_{[a,b],\infty}\bigvee_a^b(f), \tag{3.2}$$

where

$$\Delta_{g,h}(t) := \frac{1}{h(b)-h(t)}\int_t^b g(s)h'(s)\,ds - \frac{1}{h(t)-h(a)}\int_a^t g(s)h'(s)\,ds \tag{3.3}$$

for $t\in(a,b)$.

Proof. If we use the inequality (2.13) for the functions $k = f \circ h^{-1}$ and $v = g \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$\begin{aligned} &\left| [h(b) - h(a)] \int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u) du \right. \\ &\left. - \int_{h(a)}^{h(b)} f \circ h^{-1}(u) du \int_{h(a)}^{h(b)} g \circ h^{-1}(u) du \right| \\ &\leq \frac{1}{4} [h(b) - h(a)]^2 \left\| \tilde{\Delta}_{g \circ h^{-1}} \right\|_{[h(a), h(b)], \infty} \bigvee_{h(a)}^{h(b)} \left(f \circ h^{-1} \right). \end{aligned} \tag{3.4}$$

Observe also that, by the change of variable $t = h^{-1}(u)$, $u \in [h(a), h(b)]$, we have $u = h(t)$ that gives $du = h'(t) dt$ and

$$\int_{h(a)}^{h(b)} \left(f \circ h^{-1} \right)(u) du = \int_a^b f(t) h'(t) dt,$$

$$\int_{h(a)}^{h(b)} g \circ h^{-1}(u) du = \int_a^b g(t) h'(t) dt,$$

$$\int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u) du = \int_a^b f(t) g(t) h'(t) dt.$$

We also have for $u \in [h(a), h(b)]$

$$\begin{aligned} &\tilde{\Delta}_{g \circ h^{-1}}(u) \\ &= \frac{1}{h(b) - u} \int_u^{h(b)} g \circ h^{-1}(z) dz - \frac{1}{u - h(a)} \int_{h(a)}^u g \circ h^{-1}(z) dz. \end{aligned}$$

For $t = h^{-1}(u)$, $u \in [h(a), h(b)]$ we get

$$\begin{aligned} &\tilde{\Delta}_{g \circ h^{-1}}(h(t)) \\ &= \frac{1}{h(b) - h(t)} \int_{h(t)}^{h(b)} g \circ h^{-1}(z) dz - \frac{1}{h(t) - h(a)} \int_{h(a)}^{h(t)} g \circ h^{-1}(z) dz \\ &= \frac{1}{h(b) - h(t)} \int_t^b g(s) h'(s) ds - \frac{1}{h(t) - h(a)} \int_a^t g(s) h'(s) ds \\ &= \Delta_{g,h}(t) \end{aligned}$$

for $t \in [a, b]$.

Therefore

$$\begin{aligned} \left\| \tilde{\Delta}_{g \circ h^{-1}} \right\|_{[h(a), h(b)], \infty} &= \sup_{u \in [h(a), h(b)]} \left| \tilde{\Delta}_{g \circ h^{-1}}(u) \right| \\ &= \sup_{t \in [a,b]} \left| \Delta_{g,h}(t) \right| = \left\| \Delta_{g,h} \right\|_{[a,b], \infty}. \end{aligned}$$

If $h(a) = u_0 \leq u_1 \leq ... \leq u_n = h(b)$ is a division of $[h(a), h(b)]$, then $a = t_0 := h^{-1}(u_0) \leq t_1 := h^{-1}(u_1) \leq ... \leq t_n := h^{-1}(u_n) = b$ is a division of $[a, b]$ and

$$\sum_{i=0}^{n} \left| f \circ h^{-1}(u_{i+1}) - f \circ h^{-1}(u_i) \right| = \sum_{i=0}^{n} \left| f(t_{i+1}) - f(t_i) \right|.$$

Therefore

$$\sup_{Div[h(a),h(b)]}\sum_{i=0}^{n}\left|f\circ h^{-1}(u_{i+1})-f\circ h^{-1}(u_i)\right|=\sup_{Div[a,b]}\sum_{i=0}^{n}\left|f(t_{i+1})-f(t_i)\right|,$$

which shows that $\bigvee_{h(a)}^{h(b)}\left(f\circ h^{-1}\right)=\bigvee_a^b(f)$.

By making use of the inequality (3.4) we derive the desired result (3.2). □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Consider the weighted Čebyšev functional

(3.5)
$$D_w(f,g):=\int_a^b w(t)\,dt\int_a^b f(t)\,g(t)\,w(t)\,dt-\int_a^b f(t)\,w(t)\,dt\int_a^b g(t)\,w(t)\,dt,$$

then we have the weighted inequalities:

Corollary 2. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$. If f is of bounded variation on $[a,b]$ and g is integrable on $[a,b]$, then*

$$|D_w(f,g)|\le\frac{1}{4}\left(\int_a^b w(s)\,ds\right)^2\|\Delta_{g,w}\|_{[a,b],\infty}\bigvee_a^b(f),\tag{3.6}$$

where

$$\Delta_{g,w}(t):=\frac{1}{\int_t^b w(s)\,ds}\int_t^b g(s)\,w(s)\,ds-\frac{1}{\int_a^t w(s)\,ds}\int_a^t g(s)\,w(s)\,ds\tag{3.7}$$

for $t\in(a,b)$.

In the case when f satisfies a Lipschitz type condition, then we can state:

Theorem 4. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f satisfies the condition*

$$|f(t)-f(s)|\le K\,|h(t)-h(s)|\tag{3.8}$$

for all $t,\ s\in[a,b]$ and g is integrable on $[a,b]$, then

$$|D_{h'}(f,g)|\le K\,[h(b)-h(a)]^2\tag{3.9}$$
$$\times\begin{cases}\frac{1}{6}[h(b)-h(a)]\,\|\Delta_{g,h}(t)\|_{[a,b],\infty}, \\ \\ [h(b)-h(a)]^{\frac{1}{q}}\,[B(q+1,q+1)]^{\frac{1}{q}} \\ \times\left(\int_a^b|\Delta_{g,h}(t)|^p\,h'(t)\,dt\right)^{1/p} \text{ if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{1}{4}\int_a^b|\Delta_{g,h}(t)|\,h'(t)\,dt,\end{cases}$$

where $\Delta_{g,h}$ is given by (3.3).

If f is differentiable and

$$|f'(t)|\le Kh'(t)\ \text{ for all } t\in(a,b),\tag{3.10}$$

then (3.9) also holds.

Proof. The fact that $f\circ h^{-1}$ is K-Lipschitzian on $[h(a),h(b)]$, namely

$$\left|f\circ h^{-1}(u)-f\circ h^{-1}(z)\right|\leq K|u-z|$$

for all u, $z\in[h(a),h(b)]$ is equivalent, via $t=h^{-1}(u)$, $s=h^{-1}(z)$, to (3.8) for all t, $s\in[a,b]$.

We have for $v=g\circ h^{-1}$ that

$$\begin{aligned}
&\left\|\tilde{\Delta}_{g\circ h^{-1}}\right\|_{[h(a),h(b)],p}^{p}\\
&=\int_{h(a)}^{h(b)}\left|\frac{1}{h(b)-u}\int_{u}^{h(b)}g\circ h^{-1}(z)\,dz-\frac{1}{u-h(a)}\int_{h(a)}^{u}g\circ h^{-1}(z)\,dz\right|^{p}du\\
&=\int_{a}^{b}\left|\frac{1}{h(b)-h(t)}\int_{t}^{b}g(s)\,h'(s)\,ds-\frac{1}{h(t)-h(a)}\int_{a}^{t}g(s)\,h'(s)\,ds\right|^{p}h'(t)\,dt\\
&=\int_{a}^{b}\left|\Delta_{g,h}(t)\right|^{p}h'(t)\,dt.
\end{aligned}$$

By utilising the calculations in the proof of Theorem 3 and employing the inequality (2.14) for $k=g\circ h^{-1}$ and $v=g\circ h^{-1}$ on the interval $[h(a),h(b)]$, then we get the desired result (3.9).

If f is differentiable, then $k=f\circ h^{-1}$ is K-Lipschitzian on $[h(a),h(b)]$ is equivalent to $|k'(u)|\leq K$ for all $u\in(h(a),h(b))$. Now

$$\left(f\circ h^{-1}\right)'(u)=\left(f'\circ h^{-1}\right)(u)\left(h^{-1}\right)'(u)=\frac{\left(f'\circ h^{-1}\right)(u)}{\left(h'\circ h^{-1}\right)(u)}$$

and

$$\left|\frac{\left(f'\circ h^{-1}\right)(u)}{\left(h'\circ h^{-1}\right)(u)}\right|\leq K$$

for all $u\in(h(a),h(b))$ that is equivalent to (3.10). □

Corollary 3. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$. *If* f *satisfies the condition*

$$|f(t)-f(s)|\leq K\left|\int_{s}^{t}w(s)\,ds\right| \tag{3.11}$$

for all t, $s\in[a,b]$ *with* $K>0$ *and* g *is integrable on* $[a,b]$, *then*

$$|D_{w}(f,g)|\leq K\left(\int_{a}^{b}w(s)\,ds\right)^{2} \tag{3.12}$$

$$\times\begin{cases}\frac{1}{6}\left(\int_{a}^{b}w(s)\,ds\right)\left\|\Delta_{g,w}(t)\right\|_{[a,b],\infty}, & \\[2ex] \left(\int_{a}^{b}w(s)\,ds\right)^{\frac{1}{q}}\left[B(q+1,q+1)\right]^{\frac{1}{q}} & \\ \times\left(\int_{a}^{b}\left|\Delta_{g,w}(t)\right|^{p}w(t)\,dt\right)^{1/p} & \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex] \frac{1}{4}\int_{a}^{b}\left|\Delta_{g,w}(t)\right|w(t)\,dt, & \end{cases}$$

where $\Delta_{g,w}$ is given by (3.7).

If f is differentiable and

$$|f'(t)| \leq Kw(t) \text{ for all } t \in (a,b), \tag{3.13}$$

then (3.12) also holds.

If k is monotonic nondecreasing and differentiable on (a,b), then by (2.15) for $v : [a,b] \to \mathbb{C}$ Lebesgue integrable on $[a,b]$ we have

$$|D(k,v)| \leq \begin{cases} \frac{1}{4}(b-a)^2 \int_a^b \left|\tilde{\Delta}_v(t)\right| k'(t)\,dt, \\ \\ \left(\int_a^b [(h(b)-h(t))(h(t)-h(a))]^q f'(t)\,dt\right)^{1/q} \\ \times \left(\int_a^b |\Delta_{g,h}(t)|^p f'(t)\,dt\right)^{1/p}, \ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \left\|\tilde{\Delta}_v\right\|_{[a,b],\infty} \int_a^b (t-a)(b-t)k'(t)\,dt. \end{cases} \tag{3.14}$$

Theorem 5. *Let $h : [a,b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f is monotonic nondecreasing and differentiable on (a,b) and g is integrable on $[a,b]$, then*

$$|D_{h'}(f,g)| \leq \begin{cases} \frac{1}{4}[h(b)-h(a)]^2 \int_a^b |\Delta_{g,h}(t)| f'(t)\,dt \\ \\ \left(\int_a^b [(h(b)-h(t))(h(t)-h(a))]^q f'(t)\,dt\right)^{\frac{1}{q}} \\ \times \left(\int_a^b |\Delta_{g,h}(t)|^p f'(t)\,dt\right)^{1/p}, \ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \|\Delta_{g,h}(t)\|_{[a,b],\infty} \int_a^b (h(b)-h(t))(h(t)-h(a)) f'(t)\,dt, \end{cases} \tag{3.15}$$

where $\Delta_{g,h}$ is given by (3.3).

Proof. If we use the inequality (3.14) for the functions $k = f \circ h^{-1}$ and $v = g \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$\begin{aligned} &\left|[h(b)-h(a)] \int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u)du \right. \\ &\left. - \int_{h(a)}^{h(b)} f \circ h^{-1}(u)du \int_{h(a)}^{h(b)} g \circ h^{-1}(u)du \right| \end{aligned} \tag{3.16}$$

$$\leq \begin{cases} \frac{1}{4}[h(b)-h(a)]^2 \int_{h(a)}^{h(b)} \left|\tilde{\Delta}_{g\circ h^{-1}}(u)\right| \left(f\circ h^{-1}\right)'(u)\,du, \\ \\ \left(\int_{h(a)}^{h(b)} [(h(b)-u)(u-h(a))]^q \left(f\circ h^{-1}\right)'(u)\,du\right)^{\frac{1}{q}} \\ \times \left(\int_{h(a)}^{h(b)} \left|\tilde{\Delta}_{g\circ h^{-1}}(u)\right|^p \left(f\circ h^{-1}\right)'(u)\,du\right)^{\frac{1}{p}}, \\ \qquad\qquad p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|\tilde{\Delta}_{g\circ h^{-1}}\right\|_{[h(a),h(b)],\infty} \int_{h(a)}^{h(b)} (h(b)-u)(u-h(a)) \left(f\circ h^{-1}\right)'(u)\,du. \end{cases}$$

We have for $u\in[h(a),h(b)]$

$$\left(f\circ h^{-1}\right)'(u)=\left(f'\circ h^{-1}\right)(u)\left(h^{-1}\right)'(u)=\frac{\left(f'\circ h^{-1}\right)(u)}{\left(h'\circ h^{-1}\right)(u)}.$$

Observe that for by the change of variable $t=h^{-1}(u)$, $u\in[h(a),h(b)]$, we have $u=h(t)$ that gives $du=h'(t)\,dt$ and

$$\begin{aligned}&\int_{h(a)}^{h(b)}\left|\tilde{\Delta}_{g\circ h^{-1}}(u)\right|\left(f\circ h^{-1}\right)'(u)\,du\\&=\int_{h(a)}^{h(b)}\left|\tilde{\Delta}_{g\circ h^{-1}}(u)\right|\frac{\left(f'\circ h^{-1}\right)(u)}{\left(h'\circ h^{-1}\right)(u)}du\\&=\int_a^b\left|\Delta_{g,h}(t)\right|\frac{f'(t)}{h'(t)}h'(t)\,dt=\int_a^b\left|\Delta_{g,h}(t)\right|f'(t)\,dt,\end{aligned}$$

$$\begin{aligned}&\left(\int_{h(a)}^{h(b)}\left[\left(h(b)-u\right)\left(u-h(a)\right)\right]^q\left(f\circ h^{-1}\right)'(u)\,du\right)^{\frac{1}{q}}\\&=\left(\int_a^b\left[\left(h(b)-h(t)\right)\left(h(t)-h(a)\right)\right]^q f'(t)\,dt\right)^{1/q},\end{aligned}$$

$$\left(\int_{h(a)}^{h(b)}\left|\tilde{\Delta}_{g\circ h^{-1}}(u)\right|^p\left(f\circ h^{-1}\right)'(u)\,du\right)^{\frac{1}{p}}=\left(\int_a^b\left|\Delta_{g,h}(t)\right|^p f'(t)\,dt\right)^{1/p}$$

and

$$\begin{aligned}&\int_{h(a)}^{h(b)}\left(h(b)-u\right)\left(u-h(a)\right)\left(f\circ h^{-1}\right)'(u)\,du\\&=\int_a^b\left(h(b)-h(t)\right)\left(h(t)-h(a)\right)f'(t)\,dt.\end{aligned}$$

By utilising (3.16) we deduce (3.15). □

Corollary 4. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$. If f is monotonic nondecreasing and differentiable on (a,b) and g is integrable on $[a,b]$, then*

$$(3.17)\quad \left|D_w(f,g)\right|\le\begin{cases}\frac{1}{4}\left(\int_a^b w(s)\,ds\right)^2\int_a^b\left|\Delta_{g,w}(t)\right|f'(t)\,dt\\[2ex]\left(\int_a^b\left[\left(\int_t^b w(s)\,ds\right)\left(\int_a^t w(s)\,ds\right)\right]^q f'(t)\,dt\right)^{\frac{1}{q}}\\\times\left(\int_a^b\left|\Delta_{g,w}(t)\right|^p f'(t)\,dt\right)^{1/p},\ p>1,\ \frac{1}{p}+\frac{1}{q}=1,\\[2ex]\left\|\Delta_{g,w}(t)\right\|_{[a,b],\infty}\int_a^b\left(\int_t^b w(s)\,ds\right)\left(\int_a^t w(s)\,ds\right)f'(t)\,dt,\end{cases}$$

where $\Delta_{g,w}$ is given by (3.7).

4. Some Examples

For $\alpha \in \mathbb{R}$, $\alpha \neq 0$, we consider the weight $w(t) = \exp(\alpha t)$ and the functional

$$D_{\exp(\alpha\cdot)}(f,g) := \frac{1}{\alpha}\left[\exp(\alpha b) - \exp(\alpha a)\right]\int_a^b f(t)\,g(t)\exp(\alpha t)\,dt \\ - \int_a^b f(t)\exp(\alpha t)\,dt \int_a^b g(t)\exp(\alpha t)\,dt,$$

on the real interval $[a,b]$.

For this weight we have

$$\Delta_{g,\exp(\alpha\cdot)}(t) := \frac{\alpha}{\exp(\alpha b) - \exp(\alpha t)}\int_t^b g(s)\exp(\alpha s)\,ds \\ - \frac{\alpha}{\exp(\alpha t) - \exp(\alpha a)}\int_a^t g(s)\,w(s)\,ds.$$

By employing inequality (3.6) we derive If f is of bounded variation on $[a,b]$ and g is integrable on $[a,b]$, then

$$\left|D_{\exp(\alpha\cdot)}(f,g)\right| \leq \frac{1}{4\alpha^2}\left[\exp(\alpha b) - \exp(\alpha a)\right]^2 \left\|\Delta_{g,\exp(\alpha\cdot)}\right\|_{[a,b],\infty} \bigvee_a^b (f). \tag{4.1}$$

If we assume that f is differentiable on (a,b) and satisfies the condition

$$|f'(t)| \leq K\exp(\alpha t) \text{ for all } t \in (a,b),$$

then by (3.12) we get

$$\left|D_{\exp(\alpha\cdot)}(f,g)\right| \leq \frac{K}{\alpha^2}\left(\exp(\alpha b) - \exp(\alpha a)\right)^2 \times \begin{cases} \frac{1}{6\alpha}\left[\exp(\alpha b) - \exp(\alpha a)\right]\left\|\Delta_{g,\exp(\alpha\cdot)}\right\|_{[a,b],\infty}, \\ \\ \left(\frac{1}{\alpha}\left[\exp(\alpha b) - \exp(\alpha a)\right]\right)^{\frac{1}{q}}\left[B(q+1,q+1)\right]^{\frac{1}{q}} \\ \times\left(\int_a^b\left|\Delta_{g,\exp(\alpha\cdot)}\right|^p \exp(\alpha t)\,dt\right)^{1/p} \\ \text{ if } p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{1}{4}\int_a^b\left|\Delta_{g,\exp(\alpha\cdot)}\right|\exp(\alpha t)\,dt. \end{cases} \tag{4.2}$$

The interested reader may also consider the weights $w(t) = \frac{1}{t}$, $t \in [a,b]$ or $w(t) = r\ell^{r-1}$, $r > 0$, $t \in [a,b]$, see for instance the preprint version of [**?**].

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R} = (-\infty,\infty)$. Namely, if $I = (-\infty,\infty)$, $w(s) > 0$ for $s \in \mathbb{R}$ with $\int_{-\infty}^{\infty} w(s)\,ds = 1$, i.e., w is a probability density function on $(-\infty,\infty)$, f and g are *Lebesgue measurable* with wf, wg, $wfg \in L(-\infty,\infty)$, then we can consider the functional

$$D_{w,\mathbb{R}}(f,g) := \int_{-\infty}^{\infty} w(t)\,f(t)\,g(t)\,dt - \int_{-\infty}^{\infty} w(t)\,f(t)\,dt\int_{-\infty}^{\infty} w(t)\,g(t)\,dt,$$

provided that all integrals are convergent.

If f is of locally bounded variation on $(-\infty,\infty)$ with $\bigvee_{-\infty}^{\infty}(f) := \lim_{a\to-\infty,b\to\infty}\bigvee_a^b < \infty$ and g is integrable on $(-\infty,\infty)$, then

$$|D_{w,\mathbb{R}}(f,g)| \leq \frac{1}{4}\|\Delta_{g,w,\mathbb{R}}\|_{[a,b],\infty}\bigvee_{-\infty}^{\infty}(f), \tag{4.3}$$

where

$$\Delta_{g,w,\mathbb{R}}(t) := \frac{1}{\int_t^{\infty} w(s)\,ds}\int_t^{\infty} g(s)\,w(s)\,ds - \frac{1}{\int_{-\infty}^{t} w(s)\,ds}\int_{-\infty}^{t} g(s)\,w(s)\,ds \tag{4.4}$$

for $t\in(-\infty,\infty)$.

It is know that, if f is differentiable on (a,b), then

$$\bigvee_a^b(f) = \int_a^b |f'(t)|\,dt.$$

Threfore, if $\int_{-\infty}^{\infty}|f'(t)|\,dt$ is convergent, then

$$\bigvee_{-\infty}^{\infty}(f) = \int_{-\infty}^{\infty}|f'(t)|\,dt,$$

providing many examples of such functions.

If f is differentiable and

$$|f'(t)| \leq Kw(t) \text{ for all } t\in(-\infty,\infty), \tag{4.5}$$

then from (3.12) we get

$$|D_{w,\mathbb{R}}(f,g)| \leq K\times\begin{cases} \frac{1}{6}\|\Delta_{g,w,\mathbb{R}}\|_{[a,b],\infty}, \\ \\ [B(q+1,q+1)]^{\frac{1}{q}}\left(\int_a^b|\Delta_{g,w,\mathbb{R}}(t)|^p w(t)\,dt\right)^{1/p} \\ \text{if } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \\ \frac{1}{4}\int_a^b|\Delta_{g,w,\mathbb{R}}(t)|\,w(t)\,dt. \end{cases} \tag{4.6}$$

In probability theory and statistics, the *beta prime distribution* (also known as *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x>0$ with two parameters α and β, having the probability density function:

$$w_{\alpha,\beta}(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$$

where B is *Beta function,* $B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}$, $\alpha,\beta>0$. Consider the functional

$$\begin{aligned}C_{B,\alpha,\beta}(f,g) &:= B(\alpha,\beta)\int_0^{\infty} t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t)\,g(t)\,dt \\ &\quad - \int_0^{\infty} t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t)\,dt\int_0^{\infty} t^{\alpha-1}(1+t)^{-\alpha-\beta} g(t)\,dt\end{aligned}$$

where $\alpha, \beta > 0$. The interested reader may state similar inequalities for $C_{B,\alpha,\beta}(\cdot,\cdot)$, see [?].

The probability density of the *normal distribution* on $(-\infty, \infty)$ is

$$w_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \; x \in \mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

Consider the functional

$$C_{N,\sigma,\mu}(f,g) := \sqrt{2\pi}\sigma \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t) g(t) \, dt - \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t) \, dt \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) g(t) \, dt$$

with the parameters μ and σ as above. One can state similar inequalities for $C_{N,\sigma,\mu}(\cdot,\cdot)$, see [?].

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa