TWO NEW INTEGRAL INEQUALITIES FOR THE WEIGHTED ČEBYŠEV FUNCTIONAL WITH APPLICATIONS

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Abstract. For $w : [a,b] \to \mathbb{R}$ continuous and positive on the interval $[a,b]$ and f, g are Lebesgue integrable on $[a,b]$, we consider the Čebyšev functional

$$D_w(f,g) := \int_a^b w(t)\,dt \int_a^b f(t)\,g(t)\,w(t)\,dt - \int_a^b f(t)\,w(t)\,dt \int_a^b g(t)\,w(t)\,dt.$$

Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$, g is *Lebesgue integrable* and satisfies the condition $m \le g(t) \le M$ for $t \in [a,b]$. If $f : [a,b] \to \mathbb{C}$ is of bounded variation, then

$$|D_w(f,g)| \le \frac{1}{4}(M-m)\left(\int_a^b w(s)\,ds\right)^2 \bigvee_a^b (f).$$

Moreover, if $f : [a,b] \to \mathbb{R}$ is nondecreasing on $[a,b]$, then

$$\begin{aligned}
&|D_w(f,g)| \\
&\le (M-m)\int_a^b w(s)\,ds \int_a^b \left(\int_a^t w(s)\,ds - \int_t^b w(s)\,ds\right) f(t)\,w(t)\,dt \\
&\le (M-m)\int_a^b w(s)\,ds \times \begin{cases} \frac{1}{2}\left(\int_a^b w(s)\,ds\right)\max\{|f(a)|,|f(b)|\} \\[2ex] \frac{1}{(q+1)^{1/q}}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b |f(t)|^p\,w(t)\,dt\right)^{1/p} \\[2ex] \int_a^b |f(t)|\,w(t)\,dt, \end{cases}
\end{aligned}$$

where $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and the integrals in the right side are finite.

Applications for continuous probability density functions supported on infinite intervals are also given.

1. Introduction

For two Lebesgue integrable functions $h, k : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(h,k) := (b-a)\int_a^b h(t)\,k(t)\,dt - \int_a^b h(t)\,dt \int_a^b k(t)\,dt.$$

In 1934, G. Grüss [25] showed that

$$|D(h,k)| \le \frac{1}{4}(b-a)^2 (M-m)(N-n), \tag{1.1}$$

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provided m, M, n, N are real numbers with the property that

$$-\infty < m \le h \le M < \infty, \quad -\infty < n \le k \le N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $D(h,k)$ was derived in 1882 by Čebyšev [10] under the assumption that h', k' exist and are continuous on $[a,b]$, and is given by

$$|D(h,k)| \le \frac{1}{12} \|h'\|_\infty \|k'\|_\infty (b-a)^4, \tag{1.3}$$

where $\|h'\|_\infty := \sup_{t\in[a,b]} |h'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $h, k : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $h', k' \in L_\infty[a,b]$.

In 1970, A. M. Ostrowski [32] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(h,k)| \le \frac{1}{8} (b-a)^3 (M-m) \|k'\|_\infty, \tag{1.4}$$

provided h is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $k : [a,b] \to \mathbb{R}$ is absolutely continuous and $k' \in L_\infty[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [28] (see also [29, p. 210]) obtained the following result as well:

$$|D(h,k)| \le \frac{1}{\pi^2} \|h'\|_2 \|k'\|_2 (b-a)^3, \tag{1.5}$$

provided h, k are absolutely continuous and $h', k' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [6], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(h,k)| \le (b-a) \times \begin{cases} \inf_{\gamma\in\mathbb{R}} \|h-\gamma\|_\infty \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt, \\ \\ \inf_{\gamma\in\mathbb{R}} \|h-\gamma\|_q \left(\int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right|^p dt \right)^{\frac{1}{p}}, \\ \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(h,k)| \le (b-a) \|h\|_\infty \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \le h \le M$ for a.e. $x \in [a,b]$, then $\left\| h - \frac{m+M}{2} \right\|_\infty \le \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [9]

$$|D(h,k)| \le \frac{1}{2} (b-a)(M-m) \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [7].

For more recent upper bounds related to the Čebyšev functional see [1]-[9], [11]-[21] and [24]-[31].

In [12] we obtained the following refinement of Ostrowski's inequality (1.4)

$$|D(h,k)| \leq \frac{1}{2}(b-a)^3 \frac{\left(\frac{1}{b-a}\int_a^b h(t)\,dt - m\right)\left(M - \frac{1}{b-a}\int_a^b h(t)\,dt\right)}{M-m}\|k'\|_\infty \leq \frac{1}{8}(b-a)^3 (M-m)\|k'\|_\infty , \tag{1.9}$$

provided $m \leq h \leq M$ a.e. on $[a,b]$ and k is absolutely continuous on $[a,b]$.

In [8], Cerone & Dragomir obtained the following result

$$|D(p,k)| \leq \frac{1}{2}(b-a)\left(\int_a^b (t-a)(b-t)[p'(t)]^2\,dt\right)^{1/2} \times \left(\int_a^b (t-a)(b-t)[k'(t)]^2\,dt\right)^{1/2} \tag{1.10}$$

where h,k are absolutely continuous on $[a,b]$ where $(\ell-a)(b-\ell)[p']^2, (\ell-a)(b-\ell)[k']^2 \in L[a,b]$ and $\ell(t)=t, t\in[a,b]$. The constant $\frac{1}{2}$ is the best possible constant in (1.10).

In 2002, the author obtained in [15] the following inequality for the Čebyšev's functional:

Theorem 1. *Let $f:[a,b]\to\mathbb{C}$ be an absolutely continuous function with $f'\in L_\infty[a,b]$ and $g:[a,b]\to\mathbb{C}$ measurable on $[a,b]$. Then one has the inequality*

$$\begin{aligned} &|D(f,g)| \\ &\leq (b-a)\|f'\|_{[a,b],\infty}\int_a^b \left|x-\frac{a+b}{2}\right|\left|g(x)-\frac{1}{b-a}\int_a^b g(y)\,dy\right|dx \\ &\leq \frac{1}{2}(b-a)^2\|f'\|_{[a,b],\infty}\begin{cases}\frac{1}{2}(b-a)\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],\infty}, \\ \frac{(b-a)^{1/q}}{(q+1)^{1/q}}\left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],p}, \\ \left\|g-\frac{1}{b-a}\int_a^b g(y)\,dy\right\|_{[a,b],1},\end{cases}\end{aligned} \tag{1.11}$$

provided that the norms on the right side are finite.

In 2007 we obtained the following result as well [16]:

Theorem 2. *If $f:[a,b]\to\mathbb{C}$ is of bounded variation and $g:[a,b]\to\mathbb{R}$ is Lebesgue integrable and satisfies the bounds*

$$-\infty < m \leq g \leq M < \infty \text{ a.e. on } [a,b],$$

then

$$|D(f,g)| \leq \frac{1}{4}(M-m)(b-a)^2\bigvee_a^b(f). \tag{1.12}$$

The constant $1/4$ is the best possible.

Moreover, if $f:[a,b]\to\mathbb{R}$ *is nondecreasing on* $[a,b]$, *then*

$$|D(f,g)| \leq 2(b-a)(M-m)\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt \tag{1.13}$$
$$\leq \begin{cases} \frac{1}{2}(M-m)(b-a)^2\max\{|f(a)|,|f(b)|\} \\ \frac{1}{(q+1)^{1/q}}(b-a)^{1+1/q}(M-m)\|f\|_{[a,b],p} \\ (b-a)(M-m)\|f\|_{[a,b],1}. \end{cases}$$

The constants 2 *and* 1/2 *are the best possible.*

In this paper we show among others that, if $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, g is *Lebesgue integrable* and satisfies the condition $m\leq g(t)\leq M$ for $t\in[a,b]$, while $f:[a,b]\to\mathbb{C}$ is of bounded variation, then

$$|D_w(f,g)|\leq\frac{1}{4}(M-m)\left(\int_a^b w(s)\,ds\right)^2\bigvee_a^b(f).$$

Moreover, if $f:[a,b]\to\mathbb{R}$ is nondecreasing on $[a,b]$, then

$$\begin{aligned}&|D_w(f,g)|\\ &\leq (M-m)\int_a^b w(s)\,ds\int_a^b\left(\int_a^t w(s)\,ds-\int_t^b w(s)\,ds\right)f(t)\,w(t)\,dt\\ &\leq (M-m)\int_a^b w(s)\,ds\times\begin{cases}\frac{1}{2}\left(\int_a^b w(s)\,ds\right)\max\{|f(a)|,|f(b)|\}\\ \frac{1}{(q+1)^{1/q}}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b|f(t)|^p w(t)\,dt\right)^{1/p}\\ \int_a^b|f(t)|\,w(t)\,dt,\end{cases}\end{aligned}$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and the integrals in the right side are finite.

2. Main Results

We can define, as above

$$\begin{aligned}&D_{h'}(f,g) \\ &:= [h(b)-h(a)]\int_a^b f(t)\,g(t)\,h'(t)\,dt-\int_a^b f(t)\,h'(t)\,dt\int_a^b g(t)\,h'(t)\,dt,\end{aligned}\tag{2.1}$$

where h is absolutely continuous and $f,\ g$ are Lebesgue measurable on $[a,b]$ and such that the above integrals exist.

Theorem 3. *Let* $h:[a,b]\to[h(a),h(b)]$ *be a continuous strictly increasing function that is differentiable on* (a,b). *If* $f:[a,b]\to\mathbb{R}$ *is absolutely continuous on* $[a,b]$

and $\frac{f'}{h'}$ *is essentially bounded, namely* $\frac{f'}{h'} \in L_{\infty}[a,b]$*, then we have*

$$\begin{aligned}
&|D_{h'}(f,g)| \\
&\le [h(b)-h(a)] \left\| \frac{f'}{h'} \right\|_{[a,b],\infty} \\
&\times \int_a^b \left| h(t) - \frac{h(a)+h(b)}{2} \right| \left| g(t) - \frac{1}{h(b)-h(a)} \int_a^b g(s)\,h'(s)\,ds \right| h'(t)\,dt \\
&\le \frac{1}{2}[h(b)-h(a)]^2 \left\| \frac{f'}{h'} \right\|_{[a,b],\infty} \\
&\times \begin{cases} \frac{1}{2}[h(b)-h(a)] \operatorname{esssup}_{t\in[a,b]} \left| g(t) - \frac{1}{h(b)-h(a)} \int_a^b g(s)\,h'(s)\,ds \right|, \\ \frac{[h(b)-h(a)]^{1/q}}{(q+1)^{1/q}} \left(\int_a^b \left| g(t) - \frac{1}{h(b)-h(a)} \int_a^b g(s)\,h'(s)\,ds \right|^p h'(t)\,dt \right)^{1/p}, \\ \int_a^b \left| g(t) - \frac{1}{h(b)-h(a)} \int_a^b g(s)\,h'(s)\,ds \right| h'(t)\,dt, \end{cases}
\end{aligned} \tag{2.2}$$

where $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and* $g:[a,b]\to\mathbb{C}$ *is measurable and such that the integrals in the right side are finite.*

Proof. Assume that $[c,d]\subset[a,b]$. If $g:[c,d]\to\mathbb{C}$ is absolutely continuous on $[c,d]$, then $g\circ h^{-1}:[h(c),h(d)]\to\mathbb{C}$ is absolutely continuous on $[h(c),h(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(g\circ h^{-1}\right)'(z) = \left(g'\circ h^{-1}\right)(z)\left(h^{-1}\right)'(z) = \frac{\left(g'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)} \tag{2.3}$$

for almost every (a.e.) $z\in[h(c),h(d)]$.

If $x\in[c,d]$, then by taking $z=h(x)$, we get

$$\left(g\circ h^{-1}\right)'(z) = \frac{\left(g'\circ h^{-1}\right)(h(x))}{\left(h'\circ h^{-1}\right)(h(x))} = \frac{g'(x)}{h'(x)}.$$

Now, if we use the inequality (1.11) for the functions $f\circ h^{-1}$ and $g\circ h^{-1}$ on the interval $[h(a),h(b)]$, then we get

$$\begin{aligned}
&\Bigg| [h(b)-h(a)] \int_{h(a)}^{h(b)} f\circ h^{-1}(u) g\circ h^{-1}(u)du \\
&- \int_{h(a)}^{h(b)} f\circ h^{-1}(u)du \int_{h(a)}^{h(b)} g\circ h^{-1}(u)du \Bigg| \\
&\le [h(b)-h(a)] \left\| \left(f\circ h^{-1}\right)' \right\|_{[h(a),h(b)],\infty} \\
&\times \int_{h(a)}^{h(b)} \left| u - \frac{h(a)+h(b)}{2} \right| \\
&\times \left| g\circ h^{-1}(u) - \frac{1}{h(b)-h(a)} \int_{h(a)}^{h(b)} g\circ h^{-1}(y)\,dy \right| du
\end{aligned} \tag{2.4}$$

$$
\leq \frac{1}{2}\left[h\left(b\right)-h\left(a\right)\right]^{2}\left\|\left(f\circ h^{-1}\right)'\right\|_{[h(a),h(b)],\infty}
$$
$$
\times\left\{\begin{array}{l}
\frac{1}{2}\left[h\left(b\right)-h\left(a\right)\right]\left\|g\circ h^{-1}-\frac{1}{h(b)-h(a)}\int_{h(a)}^{h(b)}g\circ h^{-1}\left(y\right)dy\right\|_{[h(a),h(b)],\infty},\\
\\
\frac{[h(b)-h(a)]^{1/q}}{(q+1)^{1/q}}\left\|g\circ h^{-1}-\frac{1}{h(b)-h(a)}\int_{h(a)}^{h(b)}g\circ h^{-1}\left(y\right)dy\right\|_{[h(a),h(b)],p},\\
\\
\left\|g\circ h^{-1}-\frac{1}{h(b)-h(a)}\int_{h(a)}^{h(b)}g\circ h^{-1}\left(y\right)dy\right\|_{[h(a),h(b)],1}.
\end{array}\right.
$$

Observe also that, by the change of variable $t=h^{-1}\left(u\right),\ u\in\left[h\left(a\right),h\left(b\right)\right],$ we have $u=h\left(t\right)$ that gives $du=h'\left(t\right)dt$ and

$$
\int_{h(a)}^{h(b)}\left(f\circ h^{-1}\right)\left(u\right)du=\int_{a}^{b}f\left(t\right)h'\left(t\right)dt,
$$
$$
\int_{h(a)}^{h(b)}g\circ h^{-1}(u)du=\int_{a}^{b}g\left(t\right)h'\left(t\right)dt,
$$
$$
\int_{h(a)}^{h(b)}f\circ h^{-1}(u)g\circ h^{-1}(u)du=\int_{a}^{b}f\left(t\right)g\left(t\right)h'\left(t\right)dt
$$

and

$$
\left\|\left(f\circ h^{-1}\right)'\right\|_{[h(a),h(b)],\infty}=\left\|\frac{f'}{h'}\right\|_{[a,b],\infty}.
$$

Also

$$
\left\|g\circ h^{-1}-\frac{1}{h\left(b\right)-h\left(a\right)}\int_{h(a)}^{h(b)}g\circ h^{-1}\left(y\right)dy\right\|_{[h(a),h(b)],\infty}
$$
$$
=\underset{u\in[h(a),h(b)]}{\operatorname{esssup}}\left|g\circ h^{-1}\left(u\right)-\frac{1}{h\left(b\right)-h\left(a\right)}\int_{a}^{b}g\left(t\right)h'\left(t\right)dt\right|
$$
$$
=\underset{t\in[a,b]}{\operatorname{esssup}}\left|g\left(t\right)-\frac{1}{h\left(b\right)-h\left(a\right)}\int_{a}^{b}g\left(s\right)h'\left(s\right)ds\right|,
$$

$$
\left\|g\circ h^{-1}-\frac{1}{h\left(b\right)-h\left(a\right)}\int_{h(a)}^{h(b)}g\circ h^{-1}\left(y\right)dy\right\|_{[h(a),h(b)],p}
$$
$$
=\left(\int_{h(a)}^{h(b)}\left|g\circ h^{-1}\left(u\right)-\frac{1}{h\left(b\right)-h\left(a\right)}\int_{a}^{b}g\left(s\right)h'\left(s\right)ds\right|^{p}du\right)^{1/p}
$$
$$
=\left(\int_{a}^{b}\left|g\left(t\right)-\frac{1}{h\left(b\right)-h\left(a\right)}\int_{a}^{b}g\left(s\right)h'\left(s\right)ds\right|^{p}h'\left(t\right)dt\right)^{1/p}
$$

and

$$\begin{aligned}&\left\|g\circ h^{-1}-\frac{1}{h(b)-h(a)}\int_{h(a)}^{h(b)}g\circ h^{-1}(y)\,dy\right\|_{[h(a),h(b)],1}\\&=\int_a^b\left|g(t)-\frac{1}{h(b)-h(a)}\int_a^b g(s)\,h'(s)\,ds\right|h'(t)\,dt\end{aligned}$$

and by (2.4) we deduce (2.2). □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Consider the weighted Čebyšev functional

$$D_w(f,g):=\int_a^b w(t)\,dt\int_a^b f(t)\,g(t)\,w(t)\,dt-\int_a^b f(t)\,w(t)\,dt\int_a^b g(t)\,w(t)\,dt,$$

then we have the weighted inequalities:

Corollary 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, f is Lebesgue integrable and satisfies the condition $m\le f(t)\le M$ for $t\in[a,b]$ and $g:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ with $\frac{g'}{w}$ is essentially bounded, namely $\frac{g'}{w}\in L_\infty[a,b]$, then we have*

$$\begin{aligned}&|D_w(f,g)| \qquad (2.5)\\&\le\frac{1}{2}\int_a^b w(t)\,dt\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\int_a^b\left|\int_a^t w(s)\,ds-\int_t^b w(s)\,ds\right|\\&\times\left|g(t)-\frac{1}{\int_a^b w(s)\,ds}\int_a^b g(s)\,w(s)\,ds\right|w(t)\,dt\\&\le\frac{1}{2}\left(\int_a^b w(t)\,dt\right)^2\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\\&\times\begin{cases}\frac{1}{2}\left(\int_a^b w(t)\,dt\right)\operatorname{esssup}_{t\in[a,b]}\left|g(t)-\frac{1}{\int_a^b w(t)dt}\int_a^b g(s)\,w(s)\,ds\right|,\\[2mm]\frac{\left(\int_a^b w(t)dt\right)^{1/q}}{(q+1)^{1/q}}\left(\int_a^b\left|g(t)-\frac{1}{\int_a^b w(t)dt}\int_a^b g(s)\,w(s)\,ds\right|^p w(t)\,dt\right)^{1/p},\\[2mm]\int_a^b\left|g(t)-\frac{1}{\int_a^b w(t)dt}\int_a^b g(s)\,w(s)\,ds\right|w(t)\,dt,\end{cases}\end{aligned}$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and the integrals in the right side are finite.

We also have:

Theorem 4. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If $f:[a,b]\to\mathbb{C}$ is of bounded variation and $g:[a,b]\to\mathbb{R}$ is Lebesgue integrable and satisfies the bounds*

$$-\infty<m\le g\le M<\infty\ \text{a.e. on}\ [a,b],$$

then

$$|D_{h'}(f,g)| \le \frac{1}{4}(M-m)[h(b)-h(a)]^2 \bigvee_a^b (f). \tag{2.6}$$

Moreover, if $f:[a,b]\to\mathbb{R}$ *is nondecreasing on* $[a,b]$, *then*

$$\begin{aligned} &|D_{h'}(f,g)| \\ &\le 2[h(b)-h(a)](M-m)\int_a^b \left(h(t)-\frac{h(a)+h(b)}{2}\right) f(t)\,h'(t)\,dt \\ &\le [h(b)-h(a)] \\ &\times \begin{cases} \frac{1}{2}(M-m)[h(b)-h(a)]\max\{|f(a)|,|f(b)|\} \\ \\ \frac{1}{(q+1)^{1/q}}[h(b)-h(a)]^{1/q}(M-m)\left(\int_a^b |f(t)|^p h'(t)\,dt\right)^{1/p} \\ \\ (M-m)\int_a^b |f(t)|\,h'(t)\,dt, \end{cases} \end{aligned} \tag{2.7}$$

where $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$ *and the integrals in the right side are finite.*

Proof. Now, if we use the inequality (1.11) for the functions $f\circ h^{-1}$ and $g\circ h^{-1}$ on the interval $[h(a),h(b)]$, then we get

$$\begin{aligned} &\Bigg|[h(b)-h(a)]\int_{h(a)}^{h(b)} f\circ h^{-1}(u)g\circ h^{-1}(u)du \\ &-\int_{h(a)}^{h(b)} f\circ h^{-1}(u)du\int_{h(a)}^{h(b)} g\circ h^{-1}(u)du\Bigg| \\ &\le \frac{1}{4}(M-m)(h(b)-h(a))^2 \bigvee_{h(a)}^{h(b)}\left(f\circ h^{-1}\right) \end{aligned} \tag{2.8}$$

since $m\le g\circ h^{-1}(u)\le M$ for a.e. $u\in[h(a),h(b)]$.

If $h(a)=u_0\le u_1\le ...\le u_n=h(b)$ is a division of $[h(a),h(b)]$, then $a=t_0:=h^{-1}(u_0)\le t_1:=h^{-1}(u_1)\le ...\le t_n:=h^{-1}(u_n)=b$ is a division of $[a,b]$ and

$$\sum_{i=0}^{n}\left|f\circ h^{-1}(u_{i+1})-f\circ h^{-1}(u_i)\right| = \sum_{i=0}^{n}|f(t_{i+1})-f(t_i)|.$$

Therefore

$$\sup_{Div[h(a),h(b)]}\sum_{i=0}^{n}\left|f\circ h^{-1}(u_{i+1})-f\circ h^{-1}(u_i)\right| = \sup_{Div[a,b]}\sum_{i=0}^{n}|f(t_{i+1})-f(t_i)|,$$

which shows that $\bigvee_{h(a)}^{h(b)}\left(f\circ h^{-1}\right)=\bigvee_a^b(f)$.

By utilising (2.8) and the calculations from the proof of Theorem 3, we get (2.6).

If $f:[a,b]\to\mathbb{R}$ is nondecreasing on $[a,b]$, then $f\circ h^{-1}$ is nondecreasing on $[h(a),h(b)]$ and by (1.13) we get

$$\begin{aligned}
&|D_{h'}(f,g)| \qquad (2.9)\\
&\le 2\left[h(b)-h(a)\right](M-m)\int_{h(a)}^{h(b)}\left(u-\frac{h(a)+h(b)}{2}\right)f\circ h^{-1}(u)\,du\\
&\le \left[h(b)-h(a)\right]\\
&\times\begin{cases}\frac{1}{2}(M-m)\left[h(b)-h(a)\right]\max\left\{\left|f\circ h^{-1}(h(a))\right|,\left|f\circ h^{-1}(h(b))\right|\right\}\\[2mm]\frac{1}{(q+1)^{1/q}}\left[h(b)-h(a)\right]^{1/q}(M-m)\left\|f\circ h^{-1}\right\|_{[h(a),h(b)],p}\\[2mm](M-m)\left\|f\circ h^{-1}\right\|_{[h(a),h(b)],1}.\end{cases}
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{h(a)}^{h(b)}\left(u-\frac{h(a)+h(b)}{2}\right)f\circ h^{-1}(u)\,du\\
&=\int_a^b\left(h(t)-\frac{h(a)+h(b)}{2}\right)f(t)\,h'(t)\,dt
\end{aligned}$$

and

$$\left\|f\circ h^{-1}\right\|_{[h(a),h(b)],p}=\left(\int_{h(a)}^{h(b)}\left|f\circ h^{-1}(u)\right|^p du\right)^{1/p}=\left(\int_a^b|f(t)|^p\,h'(t)\,dt\right)^{1/p}$$

for $p\ge1$, hence by (2.9) we derive (2.7). □

Corollary 2. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$, g is Lebesgue integrable and satisfies the condition $m\le g(t)\le M$ for $t\in[a,b]$. If $f:[a,b]\to\mathbb{C}$ is of bounded variation, then*

$$|D_w(f,g)|\le\frac{1}{4}(M-m)\left(\int_a^b w(s)\,ds\right)^2\bigvee_a^b(f). \tag{2.10}$$

Moreover, if $f:[a,b]\to\mathbb{R}$ is nondecreasing on $[a,b]$, then

$$\begin{aligned}
&|D_w(f,g)| \qquad (2.11)\\
&\le(M-m)\int_a^b w(s)\,ds\int_a^b\left(\int_a^t w(s)\,ds-\int_t^b w(s)\,ds\right)f(t)\,w(t)\,dt\\
&\le(M-m)\int_a^b w(s)\,ds\\
&\times\begin{cases}\frac{1}{2}\left(\int_a^b w(s)\,ds\right)\max\{|f(a)|,|f(b)|\}\\[2mm]\frac{1}{(q+1)^{1/q}}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b|f(t)|^p w(t)\,dt\right)^{1/p}\\[2mm]\int_a^b|f(t)|\,w(t)\,dt,\end{cases}
\end{aligned}$$

where p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and the integrals in the right side are finite.

3. Some Examples

If we take $h:[a,b]\subset(0,\infty)\to\mathbb{R}$, $h(t)=\ln t$, in (2.2), then we get for $w(t):=\ell^{-1}$, where $\ell(t):=t$, that

$$D_{\ell^{-1}}(f,g):=\ln\left(\frac{b}{a}\right)\int_a^b\frac{f(t)\,g(t)}{t}dt-\int_a^b\frac{f(t)}{t}dt\int_a^b\frac{g(t)}{t}dt.$$

From (2.5) we get

$$\begin{aligned}(3.1)\quad &|D_{\ell^{-1}}(f,g)|\\ &\le\frac{1}{2}\left[\ln\left(\frac{b}{a}\right)\right]\|f'\ell\|_{[a,b],\infty}\int_a^b\frac{1}{t}\left|\ln\left(\frac{t^2}{ab}\right)\right|\left|g(t)-\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{g(s)}{s}ds\right|dt\\ &\le\frac{1}{2}\left[\ln\left(\frac{b}{a}\right)\right]^2\|f'\ell\|_{[a,b],\infty}\\ &\times\begin{cases}\frac{1}{2}\left[\ln\left(\frac{b}{a}\right)\right]\operatorname{esssup}_{t\in[a,b]}\left|g(t)-\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{g(s)}{s}ds\right|,\\ \frac{\left[\ln\left(\frac{b}{a}\right)\right]^{1/q}}{(q+1)^{1/q}}\left(\int_a^b\left|g(t)-\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{g(s)}{s}ds\right|^p\frac{1}{t}dt\right)^{1/p},\\ \int_a^b\left|g(t)-\frac{1}{\ln\left(\frac{b}{a}\right)}\int_a^b\frac{g(s)}{s}ds\right|\frac{1}{t}dt,\end{cases}\end{aligned}$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, provided the integrals in the right side are finite.

Assume that g is *Lebesgue integrable* and satisfies the condition $m\le g(t)\le M$ for $t\in[a,b]$. If $f:[a,b]\to\mathbb{C}$ is of bounded variation, then

$$|D_{\ell^{-1}}(f,g)|\le\frac{1}{4}(M-m)\left[\ln\left(\frac{b}{a}\right)\right]^2\bigvee_a^b(f).\tag{3.2}$$

Moreover, if $f:[a,b]\to\mathbb{R}$ is nondecreasing on $[a,b]$, then

$$\begin{aligned}(3.3)\quad &|D_{\ell^{-1}}(f,g)|\\ &\le(M-m)\left[\ln\left(\frac{b}{a}\right)\right]\int_a^b\left[\ln\left(\frac{t^2}{ab}\right)\right]\frac{f(t)}{t}dt\\ &\le(M-m)\left[\ln\left(\frac{b}{a}\right)\right]\times\begin{cases}\frac{1}{2}\left[\ln\left(\frac{b}{a}\right)\right]\max\{|f(a)|,|f(b)|\}\\ \frac{1}{(q+1)^{1/q}}\left[\ln\left(\frac{b}{a}\right)\right]^{1/q}\left(\int_a^b\frac{|f(t)|^p}{t}dt\right)^{1/p}\\ \int_a^b\frac{|f(t)|}{t}dt.\end{cases}\end{aligned}$$

The interested reader may also consider the weights $w(t)=\exp t$, $t\in[a,b]$ or $w(t)=r\ell^{r-1}$, $r>0$, $t\in[a,b]$, see for instance the preprint version of [22].

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R}=(-\infty,\infty)$. Namely, if $I=(-\infty,\infty)$, $w(s)>0$ for $s\in\mathbb{R}$ with $\int_{-\infty}^{\infty}w(s)\,ds=1$, i.e., w is a probability density function on $(-\infty,\infty)$, f and g are *Lebesgue measurable* with wf, wg, $wfg\in L(-\infty,\infty)$, then we can consider the functional

$$D_w(f,g) := \int_{-\infty}^{\infty} w(t) f(t) g(t)\, dt - \int_{-\infty}^{\infty} w(t) f(t)\, dt \int_{-\infty}^{\infty} w(t) g(t)\, dt.$$

Assume that $w : (-\infty,\infty) \to (0,\infty)$ is continuous on $(-\infty,\infty)$, f is *Lebesgue integrable* and satisfies the condition $m \le f(t) \le M$ for $t \in (-\infty,\infty)$ and $g : [a,b] \to \mathbb{R}$ is locally absolutely continuous on $(-\infty,\infty)$ with $\frac{g'}{w}$ is essentially bounded, namely $\frac{g'}{w} \in L_\infty(-\infty,\infty)$, then we have by (2.5) for $a \to -\infty$ and $b \to \infty$ that

$$\begin{aligned}
(3.4)\qquad & |D_w(f,g)| \\
& \le \frac{1}{2}\left\|\frac{f'}{w}\right\|_{(-\infty,\infty),\infty} \int_{-\infty}^{\infty} \left|\int_{-\infty}^{t} w(s)\, ds - \int_{t}^{\infty} w(s)\, ds\right| \\
& \times \left|g(t) - \int_{-\infty}^{\infty} g(s) w(s)\, ds\right| w(t)\, dt \\
& \le \frac{1}{2}\left\|\frac{f'}{w}\right\|_{(-\infty,\infty),\infty} \\
& \times \begin{cases} \frac{1}{2}\operatorname{esssup}_{t\in(-\infty,\infty)} \left|g(t) - \int_{-\infty}^{\infty} g(s) w(s)\, ds\right|, \\[2ex] \frac{1}{(q+1)^{1/q}} \left(\int_{-\infty}^{\infty} \left|g(t) - \int_{-\infty}^{\infty} g(s) w(s)\, ds\right|^p w(t)\, dt\right)^{1/p}, \\[2ex] \int_{-\infty}^{\infty} \left|g(t) - \int_{-\infty}^{\infty} g(s) w(s)\, ds\right| w(t)\, dt, \end{cases}
\end{aligned}$$

where $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and the integrals in the right side are finite.

If g is *Lebesgue integrable* and satisfies the condition $m \le g(t) \le M$ for $t \in (-\infty,\infty)$. If $f : (-\infty,\infty) \to \mathbb{R}$ is nondecreasing on $(-\infty,\infty)$ with $f(-\infty) := \lim_{t\to-\infty} f(t)$ and $f(\infty) := \lim_{t\to\infty} f(t)$ are finite, then

$$\begin{aligned}
(3.5)\qquad |D_w(f,g)| & \le (M-m)\int_{-\infty}^{\infty} \left(\int_{-\infty}^{t} w(s)\, ds - \int_{t}^{\infty} w(s)\, ds\right) f(t) w(t)\, dt \\
& \le (M-m) \times \begin{cases} \frac{1}{2}\max\{|f(-\infty)|, |f(\infty)|\} \\[2ex] \frac{1}{(q+1)^{1/q}} \left(\int_{-\infty}^{\infty} |f(t)|^p w(t)\, dt\right)^{1/p} \\[2ex] \int_{-\infty}^{\infty} |f(t)| w(t)\, dt, \end{cases}
\end{aligned}$$

where $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and the integrals in the right side are finite.

In probability theory and statistics, the *beta prime distribution* (also known as *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x > 0$ with two parameters α and β, having the probability density function:

$$w_{\alpha,\beta}(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$$

where B is *Beta function,* $B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}$, $\alpha,\beta > 0$. Consider the functional

$$C_{B,\alpha,\beta}(f,g) := B(\alpha,\beta)\int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t)\,g(t)\,dt - \int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t)\,dt \int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} g(t)\,dt$$

where α, $\beta > 0$. The interested reader may state similar inequalities for $C_{B,\alpha,\beta}(\cdot,\cdot)$, see [22].

The probability density of the *normal distribution* on $(-\infty,\infty)$ is

$$w_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x\in\mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance.*

Consider the functional

$$C_{N,\sigma,\mu}(f,g) := \sqrt{2\pi}\sigma\int_{-\infty}^\infty \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t)\,g(t)\,dt - \int_{-\infty}^\infty \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t)\,dt \int_{-\infty}^\infty \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) g(t)\,dt$$

with the parameters μ and σ as above. One can state similar inequalities for $C_{N,\sigma,\mu}(\cdot,\cdot)$, see [22].

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