INTEGRAL INEQUALITIES FOR THE WEIGHTED ČEBYŠEV FUNCTIONAL UNDER SOME LIPSCHITZ TYPE CONDITIONS WITH APPLICATIONS

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Abstract. For $w:[a,b]\to\mathbb{R}$ continuous and positive on the interval $[a,b]$ and f,g are Lebesgue integrable on $[a,b]$, we consider the Čebyšev functional

$$D_w(f,g):=\int_a^b w(t)\,dt\int_a^b f(t)\,g(t)\,w(t)\,dt-\int_a^b f(t)\,w(t)\,dt\int_a^b g(t)\,w(t)\,dt.$$

In this paper we show among other that, if f satisfies the condition

$$|f(t)-f(s)|\le K\left|\int_s^t w(\tau)\,d\tau\right|$$

for all $t,\ s\in[a,b]$ and g is continuous on $[a,b]$, then

$$|D_w(f,g)|\le\frac{1}{2}K\int_a^b w(t)\,dt\int_a^b |g(t)|\,w(t)\,dt.$$

Applications for continuous probability density functions supported on infinite intervals are also given.

1. Introduction

For two Lebesgue integrable functions $h,\ k:[a,b]\to\mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(h,k):=(b-a)\int_a^b h(t)\,k(t)\,dt-\int_a^b h(t)\,dt\int_a^b k(t)\,dt.$$

In 1934, G. Grüss [25] showed that

$$|D(h,k)|\le\frac{1}{4}(b-a)^2(M-m)(N-n),\tag{1.1}$$

provided $m,\ M,\ n,\ N$ are real numbers with the property that

$$-\infty<m\le h\le M<\infty,\quad -\infty<n\le k\le N<\infty\quad\text{a.e. on}\quad [a,b].\tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $D(h,k)$ was derived in 1882 by Čebyšev [10] under the assumption that h',k' exist and are continuous on $[a,b]$, and is given by

$$|D(h,k)|\le\frac{1}{12}\|h'\|_\infty\|k'\|_\infty(b-a)^4,\tag{1.3}$$

where $\|h'\|_\infty:=\sup_{t\in[a,b]}|h'(t)|<\infty$.

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The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

Čebyšev's inequality (1.3) also holds if $h,\ k : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $h',\ k' \in L_\infty [a,b]$.

In 1970, A. M. Ostrowski [33] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(h,k)| \le \frac{1}{8}(b-a)^3 (M-m) \|k'\|_\infty , \tag{1.4}$$

provided h is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $k : [a,b] \to \mathbb{R}$ is absolutely continuous and $k' \in L_\infty [a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [29] (see also [30, p. 210]) obtained the following result as well:

$$|D(h,k)| \le \frac{1}{\pi^2} \|h'\|_2 \|k'\|_2 (b-a)^3 , \tag{1.5}$$

provided $h,\ k$ are absolutely continuous and $h',\ k' \in L_2 [a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [6], P. Cerone and S. S. Dragomir proved the following inequalities:

$$\begin{aligned} &|D(h,k)| \\ &\le (b-a) \times \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|h-\gamma\|_\infty \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt, \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|h-\gamma\|_q \left(\int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right|^p dt \right)^{\frac{1}{p}}, \\ \text{where } p>1,\ 1/p+1/q=1. \end{cases} \end{aligned} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(h,k)| \le (b-a) \|h\|_\infty \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \le h \le M$ for a.e. $x \in [a,b]$, then $\left\| h - \frac{m+M}{2} \right\|_\infty \le \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [9]

$$|D(h,k)| \le \frac{1}{2}(b-a)(M-m) \int_a^b \left| k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds \right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [7].

In [12] we obtained the following refinement of Ostrowski's inequality (1.4)

$$\begin{aligned} |D(h,k)| &\le \frac{1}{2}(b-a)^3 \frac{\left(\frac{1}{b-a}\int_a^b h(t)\,dt - m \right)\left(M - \frac{1}{b-a}\int_a^b h(t)\,dt \right)}{M-m} \|k'\|_\infty \\ &\le \frac{1}{8}(b-a)^3 (M-m) \|k'\|_\infty , \end{aligned} \tag{1.9}$$

provided $m \le h \le M$ a.e. on $[a,b]$ and k is absolutely continuous on $[a,b]$.

In [8], Cerone & Dragomir obtained the following result

$$|D(p,k)| \leq \frac{1}{2}(b-a)\left(\int_a^b (t-a)(b-t)[p'(t)]^2 dt\right)^{1/2} \times \left(\int_a^b (t-a)(b-t)[k'(t)]^2 dt\right)^{1/2} \tag{1.10}$$

where h, k are absolutely continuous on $[a,b]$ where $(\ell - a)(b-\ell)[p']^2, (\ell-a)(b-\ell)[k']^2 \in L[a,b]$ and $\ell(t) = t, t \in [a,b]$. The constant $\frac{1}{2}$ is the best possible constant in (1.10).

For more recent upper bounds related to the Čebyšev functional see [1]-[9], [11]-[20] and [24]-[32].

We say that a function $v : [a,b] \to \mathbb{R}$ is K-Lipschitzian with $K > 0$ if $|v(t) - v(s)| \leq K|t-s|$ for any $t, s \in [a,b]$.

In 2001, Dragomir and Fedotov [22] obtained the following result for approximating the Riemann-Stieltjes integral:

Theorem 1. *If u is of bounded variation on $[a,b]$ and k is continuous on $[a,b]$, then*

$$\left|\int_a^b k(t)\,du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b k(t)\,dt\right| \leq \bigvee_a^b(u) \max_{t\in[a,b]}\left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right|. \tag{1.11}$$

The inequality (1.11) is sharp.

Moreover, if k is Lipschitzian with the constant K, then

$$\left|\int_a^b k(t)\,du(t) - \frac{u(b)-u(a)}{b-a}\int_a^b k(t)\,dt\right| \leq \frac{1}{2}K(b-a)\bigvee_a^b(u). \tag{1.12}$$

The constant $\frac{1}{2}$ is best possible in (1.12).

By taking in Theorem 1 $u(t) = \int_a^t v(s)\,ds$ we derive the following result for the Čebyšev functional:

Corollary 1. *If v is continuous on $[a,b]$ and k is continuous on $[a,b]$, then*

$$|D(v,k)| \leq (b-a)\int_a^b |v(s)|\,ds \max_{t\in[a,b]}\left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right|. \tag{1.13}$$

Moreover, if k is Lipschitzian with the constant K, then

$$|D(v,k)| \leq \frac{1}{2}K(b-a)\int_a^b |v(s)|\,ds. \tag{1.14}$$

The constant $\frac{1}{2}$ is best possible in (1.14).

The following lemma may be stated:

Lemma 1. *Let $u : [a,b] \to \mathbb{R}$ and $l, L \in \mathbb{R}$ with $L > l$. The following statements are equivalent:*

(i) *The function $u - \frac{l+L}{2}e$, where $e(t) = t$, $t \in [a,b]$ is $\frac{1}{2}(L-l)$-Lipschitzian;*

(ii) *We have the inequalities*

$$l \le \frac{u(t)-u(s)}{t-s} \le L \quad \text{for each } t,s \in [a,b] \quad \text{with } t \ne s; \tag{1.15}$$

(iii) *We have the inequalities*

$$l(t-s) \le u(t)-u(s) \le L(t-s) \quad \text{for each } t,s \in [a,b] \quad \text{with } t > s. \tag{1.16}$$

Following [28], we can introduce the definition of (l, L)-Lipschitzian functions:

Definition 1. *The function $u : [a,b] \to \mathbb{R}$ which satisfies one of the equivalent conditions (i)–(iii) from Lemma 1 is said to be (l, L)-Lipschitzian on $[a,b]$.*

If $L > 0$ and $l = -L$, then $(-L, L)$-Lipschitzian means L-Lipschitzian in the classical sense.

Utilising *Lagrange's mean value theorem*, we can state the following result that provides examples of (l, L)-Lipschitzian functions.

Proposition 1. *Let $u : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $-\infty < l = \inf_{t\in[a,b]} u'(t)$ and $\sup_{t\in[a,b]} u'(t) = L < \infty$, then u is (l, L)-Lipschitzian on $[a,b]$.*

Observe that

$$D\left(v, k - \frac{l+L}{2}\cdot e\right) = D(v,k) - \frac{l+L}{2} D(v,e)$$

and by Corollary 1 we derive the following perturbed version of (1.14):

Corollary 2. *If k is (l, L)-Lipschitzian and v is continuous on $[a,b]$, then*

$$\left| D(v,k) - \frac{l+L}{2} D(v,e) \right| \le \frac{1}{4}(L-l)(b-a)\int_a^b |v(s)|\, ds. \tag{1.17}$$

The constant $\frac{1}{4}$ is best possible in (1.17).

Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$. In this paper we show among other that, if f satisfies the condition

$$|f(t) - f(s)| \le K \left| \int_s^t w(\tau)\, d\tau \right|$$

for all t, $s \in [a,b]$ and g is continuous on $[a,b]$, then

$$|D_w(f,g)| \le \frac{1}{2} K \int_a^b w(t)\, dt \int_a^b |g(t)|\, w(t)\, dt.$$

Applications for continuous probability density functions supported on infinite intervals are also given.

2. Main Results

We can define the functional:

$$D_{h'}(f,g) := [h(b)-h(a)] \int_a^b f(t)\, g(t)\, h'(t)\, dt - \int_a^b f(t)\, h'(t)\, dt \int_a^b g(t)\, h'(t)\, dt, \tag{2.1}$$

where h is absolutely continuous and f, g are Lebesgue measurable on $[a,b]$ and such that the above integrals exist.

Theorem 2. *Let $h : [a, b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is differentiable on (a, b). If f and g are complex-valued continuous functions on $[a, b]$, then*

$$|D_{h'}(f, g)| \le [h(b) - h(a)] \int_a^b |f(t)| h'(t)\, dt \times \max_{t \in [a,b]} \left| g(t) - \frac{1}{h(b) - h(a)} \int_a^b g(t) h'(t)\, dt \right|. \tag{2.2}$$

Proof. If we use the inequality (1.13) for the functions $f \circ h^{-1}$ and $g \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$\begin{aligned} &\left| [h(b) - h(a)] \int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u) du \right. \\ &\left. - \int_{h(a)}^{h(b)} f \circ h^{-1}(u) du \int_{h(a)}^{h(b)} g \circ h^{-1}(u) du \right| \\ &\le [h(b) - h(a)] \int_{h(a)}^{h(b)} \left| f \circ h^{-1}(u) \right| du \\ &\times \max_{u \in [h(a), h(b)]} \left| g \circ h^{-1}(u) - \frac{1}{h(b) - h(a)} \int_{h(a)}^{h(b)} g \circ h^{-1}(z)\, dz \right|. \end{aligned} \tag{2.3}$$

Observe also that, by the change of variable $t = h^{-1}(u)$, $u \in [h(a), h(b)]$, we have $u = h(t)$ that gives $du = h'(t)\, dt$ and

$$\int_{h(a)}^{h(b)} \left(f \circ h^{-1} \right)(u)\, du = \int_a^b f(t) h'(t)\, dt,$$

$$\int_{h(a)}^{h(b)} g \circ h^{-1}(u) du = \int_a^b g(t) h'(t)\, dt,$$

$$\int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u) du = \int_a^b f(t) g(t) h'(t)\, dt.$$

Also,

$$\int_{h(a)}^{h(b)} \left| f \circ h^{-1}(u) \right| du = \int_a^b |f(t)| h'(t)\, dt$$

and

$$\begin{aligned} &\max_{u \in [h(a), h(b)]} \left| g \circ h^{-1}(u) - \frac{1}{h(b) - h(a)} \int_{h(a)}^{h(b)} g \circ h^{-1}(z)\, dz \right| \\ &= \max_{t \in [a,b]} \left| g(t) - \frac{1}{h(b) - h(a)} \int_a^b g(t) h'(t)\, dt \right| \end{aligned}$$

and by (2.3) we derive (2.2). □

If $w : [a, b] \to \mathbb{R}$ is continuous and positive on the interval $[a, b]$, then the function $W : [a, b] \to [0, \infty)$, $W(x) := \int_a^x w(s)\, ds$ is strictly increasing and differentiable on (a, b). We have $W'(x) = w(x)$ for any $x \in (a, b)$.

Consider the weighted Čebyšev functional

$$D_w(f,g) := \int_a^b w(t)\,dt \int_a^b f(t)\,g(t)\,w(t)\,dt - \int_a^b f(t)\,w(t)\,dt \int_a^b g(t)\,w(t)\,dt,$$

then we have the weighted inequalities:

Corollary 3. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$. If f and g are continuous on $[a,b]$ with complex values, then*

$$\begin{aligned} |D_w(f,g)| &\le \int_a^b w(t)\,dt \int_a^b |f(t)|\,w(t)\,dt \\ &\quad \times \max_{t\in[a,b]} \left| g(t) - \frac{1}{\int_a^b w(t)\,dt} \int_a^b g(t)\,w(t)\,dt \right|. \end{aligned} \tag{2.4}$$

We also have:

Theorem 3. *Let $h : [a,b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f satisfies the condition*

$$|f(t) - f(s)| \le K\,|h(t) - h(s)| \tag{2.5}$$

for all t, $s \in [a,b]$ and g is continuous on $[a,b]$, then

$$|D_{h'}(f,g)| \le \frac{1}{2} K\,[h(b) - h(a)] \int_a^b |g(t)|\,h'(t)\,dt. \tag{2.6}$$

Proof. The fact that $f \circ h^{-1}$ is K-Lipschitzian on $[h(a), h(b)]$, namely

$$\left| f \circ h^{-1}(u) - f \circ h^{-1}(z) \right| \le K\,|u - z|$$

for all u, $z \in [h(a), h(b)]$ is equivalent, via $t = h^{-1}(u)$, $s = h^{-1}(z)$, to

$$|f(t) - f(s)| \le K\,|h(t) - h(s)|$$

for all t, $s \in [a,b]$.

Now, if we write the inequality (1.14) for $k = f \circ h^{-1}$ and $v = g \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$\begin{aligned} &\left| [h(b) - h(a)] \int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u) du \right. \\ &\qquad \left. - \int_{h(a)}^{h(b)} f \circ h^{-1}(u) du \int_{h(a)}^{h(b)} g \circ h^{-1}(u) du \right| \\ &\le \frac{1}{2} K\,[h(b) - h(a)] \int_{h(a)}^{h(b)} \left| g \circ h^{-1}(u) \right| du. \end{aligned} \tag{2.7}$$

By the change of variable $t = h^{-1}(u)$, $u \in [h(a), h(b)]$ we then obtain from (2.7) the desired inequality (2.6). □

Corollary 4. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$. If f satisfies the condition*

$$|f(t) - f(s)| \le K \left| \int_s^t w(\tau)\,d\tau \right| \tag{2.8}$$

for all $t,\ s\in[a,b]$ *and* g *is continuous on* $[a,b]$ *, then*

$$|D_w(f,g)|\le\frac{1}{2}K\int_a^b w(t)\,dt\int_a^b|g(t)|\,w(t)\,dt. \tag{2.9}$$

We also have the perturbed inequality:

Theorem 4. *Let* $h:[a,b]\to[h(a),h(b)]$ *be a continuous strictly increasing function that is differentiable on* (a,b)*. If* g *is continuous and* f *satisfies the condition*

$$\left|f(t)-f(s)-\frac{l+L}{2}[h(t)-h(s)]\right|\le\frac{1}{2}(L-l)|h(t)-h(s)| \tag{2.10}$$

for all $t,\ s\in[a,b]$ *, then*

$$\left|D_{h'}(f,g)-\frac{l+L}{2}\tilde{D}_{h'}(g)\right|\le\frac{1}{4}(L-l)[h(b)-h(a)]\int_a^b|g(t)|\,h'(t)\,dt, \tag{2.11}$$

where

$$\begin{aligned}\tilde{D}_{h'}(g)&:=[h(b)-h(a)]\\&\times\left[\int_a^b h(t)\,g(t)h'(t)\,dt-\frac{h(a)+h(b)}{2}\int_a^b g(t)h'(t)\,dt\right].\end{aligned}$$

If f *is differentiable on* (a,b) *and satisfies the condition*

$$l\le\frac{f'(t)}{h'(t)}\le L \text{ for all } t\in(a,b), \tag{2.12}$$

then (2.11) also holds.

Proof. The fact that $k=f\circ h^{-1}$ is (l,L)-Lipschitzian on $[h(a),h(b)]$ is equivalent to the fact that $f\circ h^{-1}-\frac{l+L}{2}e$ is $\frac{1}{2}(L-l)$-Lipschitzian on $[h(a),h(b)]$, namely

$$\begin{aligned}&\left|\left(f\circ h^{-1}\right)(u)-\frac{l+L}{2}u-\left(f\circ h^{-1}\right)(z)+\frac{l+L}{2}z\right|\\&\le\frac{1}{2}(L-l)|u-z|\end{aligned} \tag{2.13}$$

for $u,\ z\in[h(a),h(b)]$.

Via the change of variable $t=h^{-1}(u)$, $s=h^{-1}(z)$, (2.13) becomes (2.10).

Observe that

$$\begin{aligned}&D\left(g\circ h^{-1},e\right)\\&=[h(b)-h(a)]\int_{h(a)}^{h(b)}u\left(g\circ h^{-1}\right)(u)du-\int_{h(a)}^{h(b)}udu\int_{h(a)}^{h(b)}g\circ h^{-1}(u)du\\&=[h(b)-h(a)]\\&\times\left[\int_{h(a)}^{h(b)}u\left(g\circ h^{-1}\right)(u)du-\frac{h(a)+h(b)}{2}\int_{h(a)}^{h(b)}g\circ h^{-1}(u)du\right]\\&=[h(b)-h(a)]\\&\times\left[\int_a^b h(t)\,g(t)h'(t)\,dt-\frac{h(a)+h(b)}{2}\int_a^b g(t)h'(t)\,dt\right]\\&=D_{h'}(g)\end{aligned} \tag{2.14}$$

and by (1.17) we get (2.11).

If f is differentiable, then $k = f \circ h^{-1}$ is (l, L)-Lipschitzian on $[h(a), h(b)]$ is equivalent to $l \le k'(u) \le L$ for all $u \in (h(a), h(b))$. Now

$$\left(f \circ h^{-1}\right)'(u) = \left(f' \circ h^{-1}\right)(u) \left(h^{-1}\right)'(u) = \frac{\left(f' \circ h^{-1}\right)(u)}{\left(h' \circ h^{-1}\right)(u)}$$

and $l \le \frac{\left(f' \circ h^{-1}\right)(u)}{\left(h' \circ h^{-1}\right)(u)} \le L$ for all $u \in (h(a), h(b))$ is equivalent to (2.12). □

Corollary 5. *Assume that $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$. If g is continuous and f satisfies the condition*

$$l \le \frac{f'(t)}{w(t)} \le L \text{ for all } t \in (a, b), \tag{2.15}$$

then

$$\left| D_w(f, g) - \frac{l + L}{2} \tilde{D}_w(g) \right| \le \frac{1}{4}(L - l) \int_a^b w(s)\, ds \int_a^b |g(t)|\, w(t)\, dt, \tag{2.16}$$

where

$$\begin{aligned} \tilde{D}_w(g) &:= \int_a^b w(s)\, ds \left[\int_a^b g(t) \left(\int_a^t w(s)\, ds - \frac{1}{2} \int_a^b w(s)\, ds \right) w(t)\, dt \right] \\ &= \frac{1}{2} \int_a^b w(s)\, ds \left[\int_a^b g(t) \left(\int_a^t w(s)\, ds - \int_t^b w(s)\, ds \right) w(t)\, dt \right]. \end{aligned}$$

3. Some Examples

For $\alpha \in \mathbb{R}$, $\alpha \neq 0$, we consider the weight $w(t) = \exp(\alpha t)$ and the functional

$$\begin{aligned} D_{\exp(\alpha \cdot)}(f, g) &:= \frac{1}{\alpha} [\exp(\alpha b) - \exp(\alpha a)] \int_a^b f(t)\, g(t) \exp(\alpha t)\, dt \\ &\quad - \int_a^b f(t) \exp(\alpha t)\, dt \int_a^b g(t) \exp(\alpha t)\, dt, \end{aligned}$$

on the real interval $[a, b]$, then by (2.4) we derive

$$\begin{aligned} &\left| D_{\exp(\alpha \cdot)}(f, g) \right| \\ &\le \frac{1}{\alpha} [\exp(\alpha b) - \exp(\alpha a)] \int_a^b |f(t)| \exp(\alpha t)\, dt \\ &\times \max_{t \in [a,b]} \left| g(t) - \frac{\alpha}{\exp(\alpha b) - \exp(\alpha a)} \int_a^b g(t) \exp(\alpha t)\, dt \right| \end{aligned} \tag{3.1}$$

where f and g are continuous on $[a, b]$.

If g is continuous and f satisfies the condition

$$l \le \frac{f'(t)}{\exp(\alpha t)} \le L \text{ for all } t \in (a, b),$$

then

$$\left|D_{\exp(\alpha\cdot)}(f,g)-\frac{l+L}{2}\tilde{D}_{\exp(\alpha\cdot)}(g)\right| \tag{3.2}$$
$$\leq\frac{1}{4}(L-l)\frac{1}{\alpha}\left[\exp(\alpha b)-\exp(\alpha a)\right]\int_a^b|g(t)|\exp(\alpha t)\,dt,$$

where

$$\tilde{D}_{\exp(\alpha\cdot)}(g):=\frac{1}{\alpha^2}\left[\exp(\alpha b)-\exp(\alpha a)\right]$$
$$\times\int_a^b g(t)\left(\exp(\alpha t)-\frac{\exp(\alpha a)+\exp(\alpha b)}{2}\right)\exp(\alpha t)\,dt.$$

If $|f'(t)|\leq L\exp(\alpha t)$ for all $t\in(a,b)$, where $L>0$, then by (3.2) we derive

$$\left|D_{\exp(\alpha\cdot)}(f,g)\right|\leq\frac{1}{2}\frac{L}{\alpha}\left[\exp(\alpha b)-\exp(\alpha a)\right]\int_a^b|g(t)|\exp(\alpha t)\,dt. \tag{3.3}$$

The interested reader may also consider the weights $w(t)=\frac{1}{t}$, $t\in[a,b]$ or $w(t)=r\ell^{r-1}$, $r>0$, $t\in[a,b]$, see for instance the preprint version of [21].

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R}=(-\infty,\infty)$. Namely, if $I=(-\infty,\infty)$, $w(s)>0$ for $s\in\mathbb{R}$ with $\int_{-\infty}^{\infty}w(s)\,ds=1$, i.e., w is a probability density function on $(-\infty,\infty)$, f and g are *Lebesgue measurable* with wf, wg, $wfg\in L(-\infty,\infty)$, then we can consider the functional

$$D_{w,\mathbb{R}}(f,g):=\int_{-\infty}^{\infty}w(t)f(t)g(t)\,dt-\int_{-\infty}^{\infty}w(t)f(t)\,dt\int_{-\infty}^{\infty}w(t)g(t)\,dt.$$

If f and g are continuous on $(-\infty,\infty)$ with complex values, then

$$|D_{w,\mathbb{R}}(f,g)|\leq\int_{-\infty}^{\infty}|f(t)|\,w(t)\,dt\max_{t\in[a,b]}\left|g(t)-\int_{-\infty}^{\infty}g(t)\,w(t)\,dt\right|, \tag{3.4}$$

provided that the integrals from the right side are finite.

If g is continuous and f satisfies the condition

$$l\leq\frac{f'(t)}{w(t)}\leq L\text{ for all }t\in(-\infty,\infty),$$

then

$$\left|D_{w,\mathbb{R}}(f,g)-\frac{l+L}{2}\tilde{D}_{w,\mathbb{R}}(g)\right|\leq\frac{1}{4}(L-l)\int_{-\infty}^{\infty}|g(t)|\,w(t)\,dt, \tag{3.5}$$

provided that the integral from the right side is finite, where

$$\tilde{D}_w(g):=\frac{1}{2}\left[\int_{-\infty}^{\infty}g(t)\left(\int_{-\infty}^{t}w(s)\,ds-\int_t^{\infty}w(s)\,ds\right)w(t)\,dt\right].$$

If $|f'(t)|\leq Lw(t)$ for all $t\in(-\infty,\infty)$ where $L>0$ is given, then by (3.5) for $l=-L$ we derive:

$$|D_{w,\mathbb{R}}(f,g)|\leq\frac{1}{2}L\int_{-\infty}^{\infty}|g(t)|\,w(t)\,dt \tag{3.6}$$

provided that the integral from the right side is finite.

In probability theory and statistics, the *beta prime distribution* (also known as *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x > 0$ with two parameters α and β, having the probability density function:

$$w_{\alpha,\beta}(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}$$

where B is *Beta function,* $B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}$, $\alpha,\beta > 0$. Consider the functional

$$C_{B,\alpha,\beta}(f,g) := B(\alpha,\beta)\int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t) g(t) dt - \int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} f(t) dt \int_0^\infty t^{\alpha-1}(1+t)^{-\alpha-\beta} g(t) dt$$

where α, $\beta > 0$. The interested reader may state similar inequalities for $C_{B,\alpha,\beta}(\cdot,\cdot)$, see [21].

The probability density of the *normal distribution* on $(-\infty,\infty)$ is

$$w_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x \in \mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

Consider the functional

$$C_{N,\sigma,\mu}(f,g) := \sqrt{2\pi}\sigma\int_{-\infty}^\infty \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t) g(t) dt - \int_{-\infty}^\infty \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t) dt \int_{-\infty}^\infty \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) g(t) dt$$

with the parameters μ and σ as above. One can state similar inequalities for $C_{N,\sigma,\mu}(\cdot,\cdot)$, see [21].

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