SOME INTEGRAL INEQUALITIES FOR THE WEIGHTED ČEBYŠEV FUNCTIONAL OF TWO FUNCTIONS OF BOUNDED VARIATION WITH APPLICATIONS

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Abstract. For $w : [a,b] \to \mathbb{R}$ continuous and positive on the interval $[a,b]$ and f,g are Lebesgue integrable on $[a,b]$, we consider the Čebyšev functional

$$D_w(f,g) := \int_a^b w(t)\,dt \int_a^b f(t)\,g(t)\,w(t)\,dt - \int_a^b f(t)\,w(t)\,dt \int_a^b g(t)\,w(t)\,dt.$$

In this paper we show among others that, if f and g are complex-valued functions of bounded variation on $[a,b]$, then

$$|D_w(f,g)| \le \frac{1}{4}\left(\int_a^b w(s)\,ds\right)^2 \bigvee_a^b(f) \bigvee_a^b(g).$$

Applications for continuous probability density functions supported on infinite intervals are also given.

1. Introduction

For two Lebesgue integrable functions $h,\ k : [a,b] \to \mathbb{C}$, in order to compare the integral mean of the product with the product of the integral means, we consider the *Čebyšev functional* defined by

$$D(h,k) := (b-a)\int_a^b h(t)\,k(t)\,dt - \int_a^b h(t)\,dt \int_a^b k(t)\,dt.$$

In 1934, G. Grüss [25] showed that

$$|D(h,k)| \le \frac{1}{4}(b-a)^2 (M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property that

$$-\infty < m \le h \le M < \infty, \quad -\infty < n \le k \le N < \infty \quad \text{a.e. on} \quad [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

Another lesser known inequality for $D(h,k)$ was derived in 1882 by Čebyšev [10] under the assumption that h', k' exist and are continuous on $[a,b]$, and is given by

$$|D(h,k)| \le \frac{1}{12}\|h'\|_\infty \|k'\|_\infty (b-a)^4, \tag{1.3}$$

where $\|h'\|_\infty := \sup_{t\in[a,b]} |h'(t)| < \infty$.

The constant $\frac{1}{12}$ cannot be improved in general in (1.3).

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Čebyšev's inequality (1.3) also holds if $h,\ k : [a,b] \to \mathbb{R}$ are assumed to be absolutely continuous and $h',\ k' \in L_\infty[a,b]$.

In 1970, A. M. Ostrowski [33] proved, amongst others, the following result that is in a sense a combination of the Čebyšev and Grüss results:

$$|D(h,k)| \le \frac{1}{8}(b-a)^3 (M-m)\|k'\|_\infty, \tag{1.4}$$

provided h is Lebesgue integrable on $[a,b]$ and satisfying (1.2) while $k : [a,b] \to \mathbb{R}$ is absolutely continuous and $k' \in L_\infty[a,b]$. Here the constant $\frac{1}{8}$ is also sharp.

In 1973, A. Lupaş [29] (see also [30, p. 210]) obtained the following result as well:

$$|D(h,k)| \le \frac{1}{\pi^2}\|h'\|_2 \|k'\|_2 (b-a)^3, \tag{1.5}$$

provided $h,\ k$ are absolutely continuous and $h',\ k' \in L_2[a,b]$.

Here the constant $\frac{1}{\pi^2}$ is the best possible as well.

In [6], P. Cerone and S. S. Dragomir proved the following inequalities:

$$|D(h,k)| \le (b-a) \times \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|h-\gamma\|_\infty \int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right| dt, \\ \\ \inf\limits_{\gamma\in\mathbb{R}} \|h-\gamma\|_q \left(\int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right|^p dt\right)^{\frac{1}{p}}, \\ \text{where } p>1,\ 1/p+1/q=1. \end{cases} \tag{1.6}$$

For $\gamma = 0$, we get from the first inequality in (1.6)

$$|D(h,k)| \le (b-a)\|h\|_\infty \int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right| dt \tag{1.7}$$

for which the constant 1 cannot be replaced by a smaller constant.

If $m \le h \le M$ for a.e. $x \in [a,b]$, then $\left\|h - \frac{m+M}{2}\right\|_\infty \le \frac{1}{2}(M-m)$ and by the first inequality in (1.6) we can deduce the following result obtained by Cheng and Sun [9]

$$|D(h,k)| \le \frac{1}{2}(b-a)(M-m)\int_a^b \left|k(t) - \frac{1}{b-a}\int_a^b k(s)\,ds\right| dt. \tag{1.8}$$

The constant $\frac{1}{2}$ is best in (1.8) as shown by Cerone and Dragomir in [7].

In [12] we obtained the following refinement of Ostrowski's inequality (1.4)

$$\begin{aligned} |D(h,k)| &\le \frac{1}{2}(b-a)^3 \frac{\left(\frac{1}{b-a}\int_a^b h(t)\,dt - m\right)\left(M - \frac{1}{b-a}\int_a^b h(t)\,dt\right)}{M-m}\|k'\|_\infty \\ &\le \frac{1}{8}(b-a)^3 (M-m)\|k'\|_\infty, \end{aligned} \tag{1.9}$$

provided $m \le h \le M$ a.e. on $[a,b]$ and k is absolutely continuous on $[a,b]$.

In [8], Cerone & Dragomir obtained the following result

$$|D(p,k)| \leq \frac{1}{2}(b-a)\left(\int_a^b (t-a)(b-t)[p'(t)]^2\,dt\right)^{1/2} \times \left(\int_a^b (t-a)(b-t)[k'(t)]^2\,dt\right)^{1/2} \tag{1.10}$$

where h,k are absolutely continuous on $[a,b]$ where $(\ell-a)(b-\ell)[p']^2, (\ell-a)(b-\ell)[k']^2 \in L[a,b]$ and $\ell(t)=t$, $t\in[a,b]$. The constant $\frac{1}{2}$ is the best possible constant in (1.10).

For more recent upper bounds related to the Čebyšev functional see [1]-[9], [11]-[20] and [24]-[32].

We say that a function $v:[a,b]\to\mathbb{R}$ is K-Lipschitzian with $K>0$ if $|v(t)-v(s)|\leq K|t-s|$ for any $t,s\in[a,b]$.

The following result can be stated as well [15]:

Theorem 1. *Assume that v is bounded variation on $[a,b]$.*

If k is of bounded variation on $[a,b]$, then

$$|C(k,v)| \leq \frac{1}{4}(b-a)^2 \bigvee_a^b (v) \bigvee_a^b (k). \tag{1.11}$$

The constant $\frac{1}{4}$ is best possible in (1.11).

If k is monotonic nondecreasing, then

$$|C(k,v)| \leq 2\bigvee_a^b (v) \int_a^b \left(t-\frac{a+b}{2}\right) k(t)\,dt \tag{1.12}$$

$$\leq (b-a)\bigvee_a^b (v) \times \begin{cases} \frac{1}{2}(b-a)\max\{|k(a)|,|k(b)|\}; \\ \frac{1}{(q+1)^{1/q}}(b-a)^{1/q}\|k\|_p \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \|k\|_1. \end{cases}$$

The multiplicative constants 2 and $\frac{1}{2}$ are best possible in (1.12).

We also obtained [15]:

Theorem 2. *Assume that v is K-Lipschitzian on $[a,b]$.*

If k is of bounded variation, then

$$|C(k,v)| \leq \frac{1}{8}(b-a)^3 K \bigvee_a^b (k). \tag{1.13}$$

The constant $\frac{1}{8}$ is best possible.

If k is monotonic nondecreasing, then

$$|C(k,v)| \leq K(b-a)\int_a^b \left(t-\frac{a+b}{2}\right)k(t)\,dt \tag{1.14}$$

$$\leq \frac{1}{2}K(b-a)^2 \times \begin{cases} \frac{1}{2}(b-a)\max\{|k(a)|,|k(b)|\}; \\ \\ \frac{1}{(q+1)^{1/q}}(b-a)^{1/q}\|k\|_p \\ \text{if } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \|k\|_1 . \end{cases}$$

The first inequality is sharp. The constant $\frac{1}{4}$ is best possible.

In this paper we show among others that, if f and g are complex-valued functions of bounded variation on $[a,b]$, then

$$|D_w(f,g)| \leq \frac{1}{4}\left(\int_a^b w(s)\,ds\right)^2 \bigvee_a^b (f) \bigvee_a^b (g).$$

Applications for continuous probability density functions supported on infinite intervals are also given.

2. Main Results

We start with the following result for two functions of bounded variation:

Theorem 3. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f and g are complex-valued functions of bounded variation on $[a,b]$, then*

$$|D_{h'}(f,g)| \leq \frac{1}{4}[h(b)-h(a)]^2 \bigvee_a^b (f) \bigvee_a^b (g). \tag{2.1}$$

Proof. If we use the inequality (1.11) for the functions $f\circ h^{-1}$ and $g\circ h^{-1}$ on the interval $[h(a),h(b)]$, then we get

$$\begin{aligned} &\Bigg|[h(b)-h(a)]\int_{h(a)}^{h(b)} f\circ h^{-1}(u)g\circ h^{-1}(u)du \\ &-\int_{h(a)}^{h(b)} f\circ h^{-1}(u)du\int_{h(a)}^{h(b)} g\circ h^{-1}(u)du\Bigg| \\ &\leq \frac{1}{4}[h(b)-h(a)]^2 \bigvee_{h(a)}^{h(b)}\left(f\circ h^{-1}\right)\bigvee_{h(a)}^{h(b)}\left(g\circ h^{-1}\right) \end{aligned} \tag{2.2}$$

If $h(a)=u_0\leq u_1\leq ...\leq u_n=h(b)$ is a division of $[h(a),h(b)]$, then $a=t_0:=h^{-1}(u_0)\leq t_1:=h^{-1}(u_1)\leq ...\leq t_n:=h^{-1}(u_n)=b$ is a division of $[a,b]$ and

$$\sum_{i=0}^n \left|f\circ h^{-1}(u_{i+1})-f\circ h^{-1}(u_i)\right| = \sum_{i=0}^n |f(t_{i+1})-f(t_i)|.$$

Therefore

$$\sup_{Div[h(a),h(b)]}\sum_{i=0}^{n}\left|f\circ h^{-1}(u_{i+1})-f\circ h^{-1}(u_i)\right|=\sup_{Div[a,b]}\sum_{i=0}^{n}\left|f(t_{i+1})-f(t_i)\right|,$$

which shows that $\bigvee_{h(a)}^{h(b)}\left(f\circ h^{-1}\right)=\bigvee_a^b(f)$.

Observe also that, by the change of variable $t=h^{-1}(u)$, $u\in[h(a),h(b)]$, we have $u=h(t)$ that gives $du=h'(t)\,dt$ and

$$\int_{h(a)}^{h(b)}\left(f\circ h^{-1}\right)(u)\,du=\int_a^b f(t)\,h'(t)\,dt,$$
$$\int_{h(a)}^{h(b)}g\circ h^{-1}(u)du=\int_a^b g(t)\,h'(t)\,dt,$$
$$\int_{h(a)}^{h(b)}f\circ h^{-1}(u)g\circ h^{-1}(u)du=\int_a^b f(t)\,g(t)\,h'(t)\,dt.$$

By making use of (2.2) we derive (2.1). □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Consider the weighted Čebyšev functional

$$D_w(f,g):=\int_a^b w(t)\,dt\int_a^b f(t)\,g(t)\,w(t)\,dt-\int_a^b f(t)\,w(t)\,dt\int_a^b g(t)\,w(t)\,dt,$$

then we have the weighted inequalities:

Corollary 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$. If f and g are complex-valued functions of bounded variation on $[a,b]$, then*

$$|D_w(f,g)|\le\frac{1}{4}\left(\int_a^b w(s)\,ds\right)^2\bigvee_a^b(f)\bigvee_a^b(g). \tag{2.3}$$

In the case when we have one function of bounded variation and the second function monotonic nondecreasing we derive the following result:

Theorem 4. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If f is monotonic nondecreasing on $[a,b]$ and g*

is complex-valued functions of bounded variation on $[a,b]$, *then*

$$|D_{h'}(f,g)| \leq 2\bigvee_a^b (g) \int_a^b \left(h(t) - \frac{h(a)+h(b)}{2}\right) f(t)\, h'(t)\, dt \tag{2.4}$$

$$\leq [h(b)-h(a)] \bigvee_a^b (g) \times \begin{cases} \frac{1}{2}[h(b)-h(a)] \max\{|f(a)|, |f(b)|\}; \\ \\ \frac{1}{(q+1)^{1/q}} [h(b)-h(a)]^{1/q} \left(\int_a^b |f(t)|^p h'(t)\, dt\right)^{1/p} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_a^b |f(t)|\, h'(t)\, dt. \end{cases}$$

Proof. If we use the inequality (1.12) for the functions $f \circ h^{-1}$ and $g \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$\left|[h(b)-h(a)] \int_{h(a)}^{h(b)} f \circ h^{-1}(u) g \circ h^{-1}(u) du - \int_{h(a)}^{h(b)} f \circ h^{-1}(u) du \int_{h(a)}^{h(b)} g \circ h^{-1}(u) du\right| \tag{2.5}$$

$$\leq 2 \bigvee_{h(a)}^{h(b)} \left(g \circ h^{-1}\right) \int_{h(a)}^{h(b)} \left(u - \frac{h(a)+h(b)}{2}\right) \left(f \circ h^{-1}\right)(u)\, du$$

$$\leq [h(b)-h(a)] \bigvee_{h(a)}^{h(b)} \left(g \circ h^{-1}\right) \times \begin{cases} \frac{1}{2}[h(b)-h(a)] \max\left\{\left|f \circ h^{-1}(h(a))\right|, \left|g \circ h^{-1}(h(b))\right|\right\}; \\ \\ \frac{1}{(q+1)^{1/q}} [h(b)-h(a)]^{1/q} \left\|f \circ h^{-1}\right\|_{[h(a),h(b)],p} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left\|f \circ h^{-1}\right\|_{[h(a),h(b)],1}. \end{cases}$$

By the change of variable $t = h^{-1}(u)$, $u \in [h(a), h(b)]$, we have $u = h(t)$ that gives $du = h'(t)\, dt$ and

$$\int_{h(a)}^{h(b)} \left(u - \frac{h(a)+h(b)}{2}\right) \left(f \circ h^{-1}\right)(u)\, du = \int_a^b \left(h(t) - \frac{h(a)+h(b)}{2}\right) f(t)\, h'(t)\, dt,$$

$$\left\|f \circ h^{-1}\right\|_{[h(a),h(b)],p} = \left(\int_{h(a)}^{h(b)} \left|f \circ h^{-1}(u)\right|^p du\right)^{1/p} = \left(\int_a^b |f(t)|^p h'(t)\, dt\right)^{1/p}$$

for $p \geq 1$.

By making use of (2.5) and the calculations performed in the proof of Theorem 3, we derive the desired result (2.4). □

Corollary 2. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$. *If* f *is monotonic nondecreasing on* $[a,b]$ *and* g *is complex-valued functions of bounded variation on* $[a,b]$, *then*

$$\begin{aligned}(2.6)\qquad |D_w(f,g)| &\le \bigvee_a^b(g)\int_a^b\left(\int_a^t w(s)\,ds-\int_t^b w(s)\,ds\right)f(t)\,w(t)\,dt\\ &\le\left(\int_a^b w(s)\,ds\right)\bigvee_a^b(g)\\ &\times\begin{cases}\frac{1}{2}\left(\int_a^b w(s)\,ds\right)\max\{|f(a)|,|f(b)|\};\\ \frac{1}{(q+1)^{1/q}}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b|f(t)|^p w(t)\,dt\right)^{1/p}\\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \int_a^b|f(t)|\,w(t)\,dt.\end{cases}\end{aligned}$$

We say that a function $v:[a,b]\to\mathbb{R}$ is K-Lipschitzian with $K>0$ if $|v(t)-v(s)|\le K|t-s|$ for any $t,\ s\in[a,b]$.

Theorem 5. *Let* $h:[a,b]\to[h(a),h(b)]$ *be a continuous strictly increasing function that is differentiable on* (a,b). *If* f *satisfies the condition*

$$(2.7)\qquad |f(t)-f(s)|\le K|h(t)-h(s)|$$

and g *is complex-valued function of bounded variation on* $[a,b]$, *then*

$$(2.8)\qquad |D_{h'}(f,g)|\le\frac{1}{8}K[h(b)-h(a)]^3\bigvee_a^b(g).$$

If f *is differentiable and*

$$(2.9)\qquad |f'(t)|\le Kh'(t)\ \text{ for all } t\in(a,b),$$

then (2.8) also holds.

Proof. The fact that $f\circ h^{-1}$ is K-Lipschitzian on $[h(a),h(b)]$, namely

$$\left|f\circ h^{-1}(u)-f\circ h^{-1}(z)\right|\le K|u-z|$$

for all $u,\ z\in[h(a),h(b)]$ is equivalent, via $t=h^{-1}(u)$, $s=h^{-1}(z)$, to (2.7) for all $t,\ s\in[a,b]$.

If $k=g\circ h^{-1}$ is of bounded variation, then by (1.13) for $v=f\circ h^{-1}$ on the interval $[h(a),h(b)]$, we get

$$\begin{aligned}(2.10)\qquad &\Big|[h(b)-h(a)]\int_{h(a)}^{h(b)} f\circ h^{-1}(u)g\circ h^{-1}(u)du \\ &-\int_{h(a)}^{h(b)} f\circ h^{-1}(u)du\int_{h(a)}^{h(b)} g\circ h^{-1}(u)du\Big| \\ &\le \frac{1}{8}[h(b)-h(a)]^3 K\bigvee_{h(a)}^{h(b)}\left(g\circ h^{-1}\right).\end{aligned}$$

By utilising the calculations from above, we then deduce the inequality (2.7).

If f is differentiable, then $k=f\circ h^{-1}$ is K-Lipschitzian on $[h(a),h(b)]$ is equivalent to $|k'(u)|\le K$ for all $u\in(h(a),h(b))$. Now

$$\left(f\circ h^{-1}\right)'(u)=\left(f'\circ h^{-1}\right)(u)\left(h^{-1}\right)'(u)=\frac{\left(f'\circ h^{-1}\right)(u)}{\left(h'\circ h^{-1}\right)(u)}$$

and $\left|\frac{\left(f'\circ h^{-1}\right)(u)}{\left(h'\circ h^{-1}\right)(u)}\right|\le K$ for all $u\in(h(a),h(b))$ that is equivalent to (2.9). □

Corollary 3. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$. If f is differentiable and*

$$(2.11)\qquad |f'(t)|\le Kw(t)\ \text{ for all } t\in(a,b),$$

then for g a complex-valued function of bounded variation on $[a,b]$,

$$(2.12)\qquad |D_w(f,g)|\le\frac{1}{8}K\left(\int_a^b w(s)\,ds\right)^3\bigvee_a^b(g).$$

Finally, by the use of (1.14) we can also state the following result as well:

Theorem 6. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is differentiable on (a,b). If $g:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and f is differentiable satisfying the condition (2.11), then*

$$(2.13)\qquad |D_{h'}(f,g)|\le\frac{1}{2}K[h(b)-h(a)]^2 \times\begin{cases}\frac{1}{2}[h(b)-h(a)]\max\{|g(a)|,|g(b)|\};\\[2mm] \frac{1}{(q+1)^{1/q}}[h(b)-h(a)]^{1/q}\left(\int_a^b|g(t)|^p h'(t)\,dt\right)^{1/p}\\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm] \int_a^b|g(t)|h'(t)\,dt.\end{cases}$$

The case of weighted integrals is as follows:

Corollary 4. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$. If f is differentiable and satisfies the condition (2.11), then for $g:[a,b]\to\mathbb{R}$ monotonic* *nondecreasing we have*

$$|D_w(f,g)| \le \frac{1}{2}K\left(\int_a^b w(s)\,ds\right)^2 \tag{2.14}$$

$$\times\begin{cases} \frac{1}{2}\left(\int_a^b w(s)\,ds\right)\max\{|g(a)|,|g(b)|\}; \\ \\ \frac{1}{(q+1)^{1/q}}\left(\int_a^b w(s)\,ds\right)^{1/q}\left(\int_a^b |g(t)|^p w(t)\,dt\right)^{1/p} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_a^b |g(t)|\,w(t)\,dt. \end{cases}$$

3. Some Examples

For $\alpha \in \mathbb{R}$, $\alpha \neq 0$, we consider the weight $w(t) = \exp(\alpha t)$ and the functional

$$D_{\exp(\alpha\cdot)}(f,g) := \frac{1}{\alpha}\left[\exp(\alpha b) - \exp(\alpha a)\right]\int_a^b f(t)\,g(t)\exp(\alpha t)\,dt - \int_a^b f(t)\exp(\alpha t)\,dt\int_a^b g(t)\exp(\alpha t)\,dt,$$

on the real interval $[a,b]$. If f and g are complex-valued functions of bounded variation on $[a,b]$, then

$$\left|D_{\exp(\alpha\cdot)}(f,g)\right| \le \frac{1}{4\alpha^2}\left[\exp(\alpha b) - \exp(\alpha a)\right]^2 \bigvee_a^b(f)\bigvee_a^b(g). \tag{3.1}$$

If f is monotonic nondecreasing on $[a,b]$ and g is complex-valued functions of bounded variation on $[a,b]$, then

$$\begin{aligned} |D_w(f,g)| &\le 2\bigvee_a^b(g)\int_a^b\left(\exp(\alpha t) - \frac{\exp(\alpha a)+\exp(\alpha b)}{2}\right)f(t)\exp(\alpha t)\,dt \\ &\le \frac{1}{\alpha}\left[\exp(\alpha b) - \exp(\alpha a)\right]\bigvee_a^b(g) \\ &\quad\times\begin{cases} \frac{1}{2\alpha}\left[\exp(\alpha b) - \exp(\alpha a)\right]\max\{|f(a)|,|f(b)|\}; \\ \\ \frac{1}{\alpha^{1/q}(q+1)^{1/q}}\left[\exp(\alpha b) - \exp(\alpha a)\right]^{1/q}\left(\int_a^b |f(t)|^p\exp(\alpha t)\,dt\right)^{1/p} \\ \text{if } p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_a^b |f(t)|\exp(\alpha t)\,dt. \end{cases} \end{aligned} \tag{3.2}$$

If f is differentiable and

$$|f'(t)| \le K\exp(\alpha t) \text{ for all } t \in (a,b), \tag{3.3}$$

then for g a complex-valued function of bounded variation on $[a,b]$,

$$|D_w(f,g)| \le \frac{1}{8}K\left[\frac{\exp(\alpha b)-\exp(\alpha a)}{\alpha}\right]^3 \bigvee_a^b (g). \tag{3.4}$$

If f is differentiable and satisfies the condition (3.3), then for $g:[a,b]\to\mathbb{R}$ monotonic nondecreasing we have

$$\begin{aligned} &|D_w(f,g)| \\ &\le \frac{1}{2\alpha^2}K\left[\exp(\alpha b)-\exp(\alpha a)\right]^2 \\ &\begin{cases} \frac{1}{2\alpha}\left[\exp(\alpha b)-\exp(\alpha a)\right]\max\{|g(a)|,|g(b)|\}; \\ \\ \frac{1}{(q+1)^{1/q}}\left(\frac{1}{\alpha}\left[\exp(\alpha b)-\exp(\alpha a)\right]\right)^{1/q}\left(\int_a^b |g(t)|^p \exp(\alpha t)\,dt\right)^{1/p} \\ \text{if } \ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_a^b |g(t)|\exp(\alpha t)\,dt. \end{cases} \end{aligned} \tag{3.5}$$

The interested reader may also consider the weights $w(t)=\frac{1}{t}$, $t\in[a,b]$ or $w(t)=r\ell^{r-1}$, $r>0$, $t\in[a,b]$, see for instance the preprint version of [21].

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R}=(-\infty,\infty)$. Namely, if $I=(-\infty,\infty)$, $w(s)>0$ for $s\in\mathbb{R}$ with $\int_{-\infty}^{\infty} w(s)\,ds=1$, i.e., w is a probability density function on $(-\infty,\infty)$, f and g are *Lebesgue measurable* with wf, wg, $wfg\in L(-\infty,\infty)$, then we can consider the functional

$$D_{w,\mathbb{R}}(f,g) := \int_{-\infty}^{\infty} w(t)f(t)g(t)\,dt - \int_{-\infty}^{\infty} w(t)f(t)\,dt\int_{-\infty}^{\infty} w(t)g(t)\,dt,$$

provided that all integrals are convergent.

It is know that, if f is differentiable on (a,b), then

$$\bigvee_a^b (f) = \int_a^b |f'(t)|\,dt.$$

If f and g are complex-valued functions and differentiable on $(-\infty,\infty)$ and such that $\int_{-\infty}^{\infty}|f'(t)|\,dt<\infty$, $\int_{-\infty}^{\infty}|g'(t)|\,dt<\infty$ then by (2.3),

$$|D_{w,\mathbb{R}}(f,g)| \le \frac{1}{4}\int_{-\infty}^{\infty}|f'(t)|\,dt\int_{-\infty}^{\infty}|g'(t)|\,dt. \tag{3.6}$$

If f is differentiable and

$$|f'(t)| \le Kw(t) \text{ for all } t\in(-\infty,\infty), \tag{3.7}$$

then for g a complex-valued function differentiable on $(-\infty,\infty)$, we derive from (2.8)

$$|D_{w,\mathbb{R}}(f,g)| \le \frac{1}{8}K\int_{-\infty}^{\infty}|g'(t)|\,dt, \tag{3.8}$$

provided that $\int_{-\infty}^{\infty}|g'(t)|\,dt<\infty$.

If f is monotonic nondecreasing on $(-\infty,\infty)$ and g is a complex-valued function differentiable on $(-\infty,\infty)$, then by (2.6),

$$
\begin{aligned}
(3.9)\qquad & |D_{w,\mathbb{R}}(f,g)| \\
& \leq \left(\int_{-\infty}^{\infty}|g'(t)|\,dt\right)\int_{-\infty}^{\infty}\left(\int_{-\infty}^{t}w(s)\,ds-\int_{t}^{\infty}w(s)\,ds\right)f(t)\,w(t)\,dt \\
& \leq \left(\int_{-\infty}^{\infty}|g'(t)|\,dt\right) \\
& \times\begin{cases}\frac{1}{2}\max\{|f(-\infty)|,|f(\infty)|\}; \\ \\ \frac{1}{(q+1)^{1/q}}\left(\int_{-\infty}^{\infty}|f(t)|^{p}w(t)\,dt\right)^{1/p} \\ \text{if }\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_{-\infty}^{\infty}|f(t)|\,w(t)\,dt,\end{cases}
\end{aligned}
$$

provided that $\int_{-\infty}^{\infty}|g'(t)|\,dt<\infty$, $f(-\infty):=\lim_{t\to-\infty}f(t)$, $f(\infty):=\lim_{t\to\infty}f(t)$ are finite and the integrals $\int_{-\infty}^{\infty}|f(t)|^{p}w(t)\,dt$ and $\int_{-\infty}^{\infty}|f(t)|\,w(t)\,dt$ are convergent.

If f is differentiable and satisfies the condition (3.7), then for $g:(-\infty,\infty)\to\mathbb{R}$ monotonic nondecreasing we have

$$
(3.10)\qquad |D_{w,\mathbb{R}}(f,g)|\leq\frac{1}{2}K\times\begin{cases}\frac{1}{2}\max\{|g(-\infty)|,|g(\infty)|\}; \\ \\ \frac{1}{(q+1)^{1/q}}\left(\int_{-\infty}^{\infty}|g(t)|^{p}w(t)\,dt\right)^{1/p} \\ \text{if }\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \int_{-\infty}^{\infty}|g(t)|\,w(t)\,dt,\end{cases}
$$

provided that the quantities in the right side are finite.

In probability theory and statistics, the *beta prime distribution* (also known as *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x>0$ with two parameters α and β, having the probability density function:

$$
w_{\alpha,\beta}(x):=\frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)}
$$

where B is *Beta function,* $B(\alpha,\beta):=\int_{0}^{1}t^{\alpha-1}(1-t)^{\beta-1}$, $\alpha,\beta>0$. Consider the functional

$$
\begin{aligned}
C_{B,\alpha,\beta}(f,g) &:= B(\alpha,\beta)\int_{0}^{\infty}t^{\alpha-1}(1+t)^{-\alpha-\beta}f(t)\,g(t)\,dt \\
&\quad-\int_{0}^{\infty}t^{\alpha-1}(1+t)^{-\alpha-\beta}f(t)\,dt\int_{0}^{\infty}t^{\alpha-1}(1+t)^{-\alpha-\beta}g(t)\,dt
\end{aligned}
$$

where $\alpha,\beta>0$. The interested reader may state similar inequalities for $C_{B,\alpha,\beta}(\cdot,\cdot)$, see [21].

The probability density of the *normal distribution* on $(-\infty, \infty)$ is

$$w_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \ x \in \mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

Consider the functional

$$C_{N,\sigma,\mu}(f,g) := \sqrt{2\pi}\sigma \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t) g(t) dt - \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t) dt \int_{-\infty}^{\infty} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) g(t) dt$$

with the parameters μ and σ as above. One can state similar inequalities for $C_{N,\sigma,\mu}(\cdot,\cdot)$, see [21].

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