p-NORMS INTEGRAL INEQUALITIES FOR THE WEIGHTED ČEBYŠEV FUNCTIONAL WITH APPLICATIONS

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Abstract. Assume that $w : [a,b] \to [0,\infty)$ is integrable with $\int_a^b w(t)\,dt = 1$ and f, g are Lebesgue integrable on $[a,b]$. Consider the Čebyšev functional

$$D_w(f,g) := \int_a^b f(t)\,g(t)\,w(t)\,dt - \int_a^b f(t)\,w(t)\,dt \int_a^b g(t)\,w(t)\,dt.$$

In ths paper we show among other that, if f, g are absolutely continuous with $\|f'\|_{[a,b],p} := \left(\int_a^b |f'(u)|^p\,du\right)^{1/p} < \infty$ and $\|g'\|_{[a,b],q} := \left(\int_a^b |g'(u)|^q\,du\right)^{1/q} < \infty$ for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned}
|D_w(f,g)| &\le \left[D_w\left(\ell, \int_a^\cdot |f'(u)|^p\,du\right)\right]^{1/p} \left[D_w\left(\ell, \int_a^\cdot |g'(u)|^q\,du\right)\right]^{1/q} \\
&\le \frac{1}{2}\|f'\|_{[a,b],p}\|g'\|_{[a,b],q} \int_a^b w(t)\left|t - \int_a^b s w(s)\,ds\right| dt \\
&\le \frac{1}{2}\|f'\|_{[a,b],p}\|g'\|_{[a,b],q} \left[\int_a^b s^2 w(s)\,ds - \left(\int_a^b s w(s)\,ds\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{2}(b-a)\|f'\|_{[a,b],p}\|g'\|_{[a,b],q}.
\end{aligned}$$

Applications for continuous probability density functions supported on infinite intervals and for norms and semi-inner products are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \ge 0$ for μ-a.e. $x \in \Omega$, consider the Lebesgue space $L_w(\Omega,\mu) := \{h : \Omega \to \mathbb{R},\ h$ is μ-measurable and $\int_\Omega w(x)|h(x)|\,d\mu(x) < \infty\}$. Assume $\int_\Omega w(x)\,d\mu(x) = 1$. In order to simplify the notation for the integrals, we do not write the variable, namely, instead of $\int_\Omega w(x)\,d\mu(x)$ we simply write $\int_\Omega w d\mu$.

If h, $k : \Omega \to \mathbb{R}$ are μ-measurable functions and h, k, $hk \in L_w(\Omega,\mu)$, then we may consider the weighted Čebyšev functional in the following form

$$D_w(h,k) := \int_\Omega whk d\mu - \int_\Omega wh d\mu \int_\Omega wk d\mu. \tag{1.1}$$

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The following result is known in the literature as the Grüss inequality, see for instance [7]:

$$|D_w(h,k)| \le \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{1.2}$$

provided

$$-\infty<\gamma\le h\le\Gamma<\infty,\quad -\infty<\delta\le k\le\Delta<\infty \tag{1.3}$$

μ-a.e. on Ω. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [7] Cerone and Dragomir proved among others the following *refinement of Grüss' inequality:*

Theorem 1. *For $h,\ k:\Omega\to\mathbb{R}$, μ-measurable functions and so that $-\infty<\gamma\le h\le\Gamma<\infty$, $-\infty<\delta\le k\le\Delta<\infty$ μ-a.e. on Ω,*

$$\begin{aligned}|D_w(h,k)| &\le \frac{1}{2}(\Delta-\delta)\int_\Omega w\left|h-\int_\Omega whd\mu\right|d\mu \\ &\le \frac{1}{2}(\Delta-\delta)\left[\int_\Omega wh^2d\mu-\left(\int_\Omega whd\mu\right)^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma),\end{aligned} \tag{1.4}$$

provided that $h,\ k,\ hk\in L_w(\Omega,\mu)$. The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

Consider a probability density function w on $[a,b]$, i.e., $w\ge0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt=1$, and the weighted Čebyšev functional for functions defined on a finite interval $[a,b]$,

$$D_w(h,k):=\int_a^b h(t)\,k(t)\,w(t)\,dt-\int_a^b h(t)\,w(t)\,dt\int_a^b k(t)\,w(t)\,dt.$$

From (1.4) we get

$$\begin{aligned}|D_w(h,k)| &\le \frac{1}{2}(\Delta-\delta)\int_a^b w(t)\left|h(t)-\int_a^b h(s)\,w(s)\,ds\right|dt \\ &\le \frac{1}{2}(\Delta-\delta)\left[\int_a^b w(s)\,h^2(s)\,ds-\left(\int_a^b h(s)\,w(s)\,ds\right)^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma),\end{aligned} \tag{1.5}$$

provided that $h,\ k:[a,b]\to\mathbb{R}$ are measurable functions and so that $-\infty<\gamma\le h\le\Gamma<\infty$, $-\infty<\delta\le k\le\Delta<\infty$ a.e. on $[a,b]$, and $h,\ k,\ hk\in L_w[a,b]$.

For more recent upper bounds related to the Čebyšev functional see [1]-[9], [11]-[19] and [22]-[29].

In this paper we show among other that, if $f,\ g$ are absolutely continuous with $\|f'\|_{[a,b],p}:=\left(\int_a^b|f'(u)|^p\,du\right)^{1/p}<\infty$ and $\|g'\|_{[a,b],q}:=\left(\int_a^b|g'(u)|^q\,du\right)^{1/q}<\infty$

for $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned}|D_w(f,g)| &\le \left[D_w\left(\ell, \int_a^{\cdot} |f'(u)|^p\, du\right)\right]^{1/p}\left[D_w\left(\ell, \int_a^{\cdot} |g'(u)|^q\, du\right)\right]^{1/q}\\ &\le \frac{1}{2}\|f'\|_{[a,b],p}\|g'\|_{[a,b],q}\int_a^b w(t)\left|t - \int_a^b sw(s)\,ds\right| dt\\ &\le \frac{1}{2}\|f'\|_{[a,b],p}\|g'\|_{[a,b],q}\left[\int_a^b s^2 w(s)\,ds - \left(\int_a^b sw(s)\,ds\right)^2\right]^{\frac{1}{2}}\\ &\le \frac{1}{2}(b-a)\|f'\|_{[a,b],p}\|g'\|_{[a,b],q}.\end{aligned}$$

Applications for continuous probability density functions supported on infinite intervals and for norms and semi-inner products are also given.

2. Main Results

The first main results is as follows:

Theorem 2. *Assume that $w : [a,b] \to [0,\infty)$ is integrable with $\int_a^b w(t)\,dt = 1$ and $f,\, g$ absolutely continuous with $\|f'\|_{[a,b],\infty} := \operatorname{esssup}_{t\in[a,b]}|f'(t)| < \infty$, then*

$$\begin{aligned}|D_w(f,g)| &\le \|f'\|_{[a,b],\infty} D_w\left(\ell, \int_a^{\cdot}|g'(u)|\,du\right)\\ &\le \frac{1}{2}\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],1}\int_a^b w(t)\left|t - \int_a^b sw(s)\,ds\right| dt\\ &\le \frac{1}{2}\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],1}\left[\int_a^b s^2 w(s)\,ds - \left(\int_a^b sw(s)\,ds\right)^2\right]^{\frac{1}{2}}\\ &\le \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],1},\end{aligned}\tag{2.1}$$

where $\|g'\|_{[a,b],1} := \left(\int_a^b |g'(u)|\,du\right)$ and $\ell(t) = t,\ t \in [a,b]$.

Also,

$$\begin{aligned}&|D_w(f,g)|\\ &\le \|f'\|_{[a,b],\infty} D_w\left(\ell, \int_a^{\cdot}|g'(u)|\,du\right)\\ &\le \frac{1}{2}(b-a)\|f'\|_{[a,b],\infty}\\ &\times w(t)\left|\int_a^t\left(\int_a^s w(u)\,du\right)|g'(s)|\,ds - \int_t^b\left(\int_s^b w(u)\,du\right)|g'(s)|\,ds\right| dt\end{aligned}\tag{2.2}$$

$$\leq \frac{1}{2}(b-a)\|f'\|_{[a,b],\infty}$$
$$\times\left[\int_a^b w(s)\left(\int_a^s |g'(u)|\,du\right)^2 ds - \left(\int_a^b \left(\int_a^s |g'(u)|\,du\right) w(s)\,ds\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}(b-a)\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],1}.$$

Proof. Observe that, by the use of integral's properties,

$$\int_a^b\int_a^b w(t)\,w(s)\,[f(t)-f(s)]\,[g(t)-g(s)]\,dtds$$
$$=\int_a^b\int_a^b w(t)\,w(s)\,(f(t)\,g(t)-f(s)\,g(t)-f(t)\,g(s)+f(s)\,g(s))\,dtds$$
$$=\int_a^b w(s)\,ds\int_a^b f(t)\,g(t)\,dt-\int_a^b w(s)\,f(s)\,ds\int_a^b w(t)\,g(t)\,dt$$
$$-\int_a^b w(t)\,f(t)\,dt\int_a^b w(s)\,g(s)\,ds+\int_a^b w(t)\,dt\int_a^b f(s)\,g(s)\,ds=2D_w(f,g),$$

which give the weighted Korkine's identity for functions with complex values

$$D_w(f,g)=\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,[f(t)-f(s)]\,[g(t)-g(s)]\,dtds.$$

For Korkine's classical identity for real-valued functions, see [27, p. 242].

If we take the modulus and use the integral's properties, we get

$$|D_w(f,g)|\leq\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,|[f(t)-f(s)]\,[g(t)-g(s)]|\,dtds \tag{2.3}$$
$$=\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,|f(t)-f(s)|\,|g(t)-g(s)|\,dtds.$$

Observe that for $s,\ t\in[a,b]$

$$f(t)-f(s)=\int_s^t f'(u)\,du,\ \ g(t)-g(s)=\int_s^t g'(u)\,du,$$

which implies that

$$|f(t)-f(s)|\,|g(t)-g(s)|=\left|\int_s^t f'(u)\,du\right|\left|\int_s^t g'(u)\,du\right|$$
$$\leq\left|\int_s^t |f'(u)|\,du\right|\left|\int_s^t |g'(u)|\,du\right|$$
$$\leq\sup_{t\in(a,b)}|f'(u)|\,|t-s|\left|\int_s^t |g'(u)|\,du\right|$$
$$=\sup_{t\in(a,b)}|f'(u)|\,(t-s)\int_s^t |g'(u)|\,du,$$

for all $s,\ t\in[a,b]$.

By (2.3) we get

$$|D_w(f,g)| \le \sup_{t\in(a,b)} |f'(u)| \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,(t-s)\left(\int_s^t |g'(u)|\,du\right)dtds. \tag{2.4}$$

Since

$$(t-s)\left(\int_s^t |g'(u)|\,du\right) = (t-s)\left(\int_a^t |g'(u)|\,du - \int_a^s |g'(u)|\,du\right),$$

hence by Korkine's identity for real valued functions $f(t)=\ell(t)$ and $\int_a^t |g'(u)|\,du$, we have

$$\begin{aligned}\frac{1}{2}\int_a^b\int_a^b & w(t)\,w(s)\,(t-s)\left(\int_a^t |g'(u)|\,du - \int_a^s |g'(u)|\,du\right)dtds \\ &= D_w\left(\ell, \int_a^{\cdot} |g'(u)|\,du\right).\end{aligned} \tag{2.5}$$

By utilising (2.4) and (2.5), we deduce the first inequality in (2.1).

Observe that

$$0 \le \int_a^t |g'(u)|\,du \le \int_a^b |g'(u)|\,du$$

for all $t\in[a,b]$, then by (1.5) for the functions $h(t)=\ell(t)$ and $k(t)=\int_a^t |g'(u)|\,du$, $t\in[a,b]$, we get

$$\begin{aligned}&\left|D_w\left(\ell, \int_a^t |g'(u)|\,du\right)\right| \\ &\le \frac{1}{2}\left(\int_a^b |g'(u)|\,du\right)\int_a^b w(t)\left|t-\int_a^b s w(s)\,ds\right|dt \\ &\le \frac{1}{2}\left(\int_a^b |g'(u)|\,du\right)\left[\int_a^b s^2 w(s)\,ds - \left(\int_a^b s w(s)\,ds\right)^2\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}(b-a)\left(\int_a^b |g'(u)|\,du\right),\end{aligned} \tag{2.6}$$

which proves the last part of (2.1).

If we use (1.5) for the functions $k(t) = \ell(t)$ and $h(t) = \int_a^t |g'(u)|\,du$, $t \in [a,b]$, we get

$$\left|D_w\left(\int_a^{\cdot} |g'(u)|\,du, \ell\right)\right| \tag{2.7}$$
$$\leq \frac{1}{2}(b-a)\int_a^b w(t)\left|\int_a^t |g'(u)|\,du - \int_a^b\left(\int_a^s |g'(u)|\,du\right)w(s)\,ds\right|dt$$
$$\leq \frac{1}{2}(b-a)$$
$$\times\left[\int_a^b w(s)\left(\int_a^s |g'(u)|\,du\right)^2 ds - \left(\int_a^b\left(\int_a^s |g'(u)|\,du\right)w(s)\,ds\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}(b-a)\left(\int_a^b |g'(u)|\,du\right).$$

Observe that

$$\int_a^b w(t)\left|\int_a^t |g'(u)|\,du - \int_a^b w(s)\left(\int_a^s |g'(u)|\,du\right)ds\right|dt$$
$$= \int_a^b w(t)\left|\int_a^t |g'(u)|\,du - \int_a^b\left(\int_a^s |g'(u)|\,du\right)d\left(\int_a^s w(u)\,du\right)\right|dt$$
$$= \int_a^b w(t)\left|\int_a^t |g'(u)|\,du\right.$$
$$\left.-\left(\left(\int_a^b |g'(s)|\,ds\right)\int_a^b w(u)\,du - \int_a^b |g'(s)|\left(\int_a^s w(u)\,du\right)ds\right)dt\right|$$
$$= \int_a^b w(t)\left|\int_a^t |g'(u)|\,du - \left(\int_a^b |g'(s)|\left(\int_s^b w(u)\,du\right)ds\right)\right|dt$$

$$= \int_a^b w(t)\left|\int_a^t |g'(u)|\,du - \left(\int_a^b |g'(s)|\left(\int_s^b w(u)\,du\right)ds\right)\right|dt$$
$$= \int_a^b w(t)\left|\int_a^t |g'(u)|\,du - \int_a^t |g'(s)|\left(\int_s^b w(u)\,du\right)ds\right.$$
$$\left.-\int_t^b |g'(s)|\left(\int_s^b w(u)\,du\right)dsdt\right|$$
$$= \int_a^b w(t)\left|\int_a^t\left(1 - \int_s^b w(u)\,du\right)|g'(u)|\,du - \int_t^b |g'(s)|\left(\int_s^b w(u)\,du\right)ds\right|dt$$
$$= \int_a^b w(t)\left|\int_a^t\left(\int_a^s w(u)\,du\right)|g'(s)|\,ds - \int_t^b\left(\int_s^b w(u)\,du\right)|g'(s)|\,ds\right|dt$$

and by (2.7) we derive (2.2). □

Theorem 3. *Assume that* $w : [a,b] \to [0,\infty)$ *is integrable with* $\int_a^b w(t)\,dt = 1$ *and* f, g *absolutely continuous with* $\|f'\|_{[a,b],p} := \left(\int_a^b |f'(u)|^p\,du\right)^{1/p} < \infty$ *and* $\|g'\|_{[a,b],q} := \left(\int_a^b |g'(u)|^q\,du\right)^{1/q} < \infty$ *for* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$
\begin{aligned}
(2.8)\qquad |D_w(f,g)| &\le \left[D_w\left(\ell, \int_a^{\cdot} |f'(u)|^p\,du\right)\right]^{1/p} \left[D_w\left(\ell, \int_a^{\cdot} |g'(u)|^q\,du\right)\right]^{1/q} \\
&\le \frac{1}{2}\|f'\|_{[a,b],p}\|g'\|_{[a,b],q} \int_a^b w(t)\left|t - \int_a^b s w(s)\,ds\right| dt \\
&\le \frac{1}{2}\|f'\|_{[a,b],p}\|g'\|_{[a,b],q} \left[\int_a^b s^2 w(s)\,ds - \left(\int_a^b s w(s)\,ds\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{2}(b-a)\|f'\|_{[a,b],p}\|g'\|_{[a,b],q}.
\end{aligned}
$$

In particular, if $\|f'\|_{[a,b],2} := \left(\int_a^b |f'(u)|^2\,du\right)^{1/2} < \infty$ *and* $\|g'\|_{[a,b],2} := \left(\int_a^b |g'(u)|^2\,du\right)^{1/2} < \infty$, *then*

$$
\begin{aligned}
(2.9)\qquad |D_w(f,g)| &\le \left[D_w\left(\ell, \int_a^{\cdot} |f'(u)|^2\,du\right)\right]^{1/2} \left[D_w\left(\ell, \int_a^{\cdot} |g'(u)|^2\,du\right)\right]^{1/2} \\
&\le \frac{1}{2}\|f'\|_{[a,b],2}\|g'\|_{[a,b],2} \int_a^b w(t)\left|t - \int_a^b s w(s)\,ds\right| dt \\
&\le \frac{1}{2}\|f'\|_{[a,b],2}\|g'\|_{[a,b],2} \left[\int_a^b s^2 w(s)\,ds - \left(\int_a^b s w(s)\,ds\right)^2\right]^{\frac{1}{2}} \\
&\le \frac{1}{2}(b-a)\|f'\|_{[a,b],2}\|g'\|_{[a,b],2}.
\end{aligned}
$$

Proof. Using Hölder's inequality for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$
\begin{aligned}
&|f(t) - f(s)|\,|g(t) - g(s)| \\
&= \left|\int_s^t f'(u)\,du\right| \left|\int_s^t g'(u)\,du\right| \\
&\le \left|\int_s^t |f'(u)|\,du\right| \left|\int_s^t |g'(u)|\,du\right| \\
&\le |t-s|^{1/q} \left|\int_s^t |f'(u)|^p\,du\right|^{1/p} |t-s|^{1/p} \left|\int_s^t |g'(u)|^q\,du\right|^{1/q} \\
&= |t-s| \left|\int_s^t |f'(u)|^p\,du\right|^{1/p} \left|\int_s^t |g'(u)|^q\,du\right|^{1/q}
\end{aligned}
$$

for all t, $s \in [a,b]$.

By the weighted Hölder's inequality for double integral, we also have

$$\begin{aligned}
&\int_a^b \int_a^b w(s)\, w(t)\, |f(t) - f(s)|\, |g(t) - g(s)|\, dtds \\
&\leq \int_a^b \int_a^b w(s)\, w(t)\, |t-s| \left| \int_s^t |f'(u)|^p\, du \right|^{1/p} \left| \int_s^t |g'(u)|^q\, du \right|^{1/q} dtds \\
&\leq \left(\int_a^b \int_a^b w(s)\, w(t)\, |t-s| \left(\left| \int_s^t |f'(u)|^p\, du \right|^{1/p} \right)^p dtds \right)^{1/p} \\
&\times \left(\int_a^b \int_a^b w(s)\, w(t)\, |t-s| \left(\left| \int_s^t |g'(u)|^q\, du \right|^{1/q} \right)^q dtds \right)^{1/q} \\
&= \left(\int_a^b \int_a^b w(s)\, w(t)\, |t-s| \left| \int_s^t |f'(u)|^p\, du \right| dtds \right)^{1/p} \\
&\times \left(\int_a^b \int_a^b w(s)\, w(t)\, |t-s| \left| \int_s^t |g'(u)|^q\, du \right| dtds \right)^{1/q}.
\end{aligned} \tag{2.10}$$

Observe that

$$\begin{aligned}
&\int_a^b \int_a^b w(s)\, w(t)\, |t-s| \left| \int_s^t |f'(u)|^p\, du \right| dtds \\
&= \int_a^b \int_a^b w(s)\, w(t)\, (t-s) \left(\int_s^t |f'(u)|^p\, du \right) dtds \\
&= \int_a^b \int_a^b w(s)\, w(t)\, (t-s) \left(\int_a^t |f'(u)|^p\, du - \int_a^s |f'(u)|^p\, du \right) dtds \\
&= 2D_w \left(\ell, \int_a^{\cdot} |f'(u)|^p\, du \right)
\end{aligned}$$

and, similarly

$$\int_a^b \int_a^b w(s)\, w(t)\, |t-s| \left| \int_s^t |g'(u)|^q\, du \right| dtds = 2D_w \left(\ell, \int_a^{\cdot} |g'(u)|^q\, du \right).$$

Therefore, by (2.3)

$$\begin{aligned}
|D(f,g)| &\leq \frac{1}{2} \int_a^b \int_a^b w(s)\, w(t)\, |f(t) - f(s)|\, |g(t) - g(s)|\, dtds \\
&\leq \frac{1}{2} \left[2D_w \left(\ell, \int_a^{\cdot} |f'(u)|^p\, du \right) \right]^{1/p} \left[2D_w \left(\ell, \int_a^{\cdot} |g'(u)|^q\, du \right) \right]^{1/q} \\
&= \left[D_w \left(\ell, \int_a^{\cdot} |f'(u)|^p\, du \right) \right]^{1/p} \left[D_w \left(\ell, \int_a^{\cdot} |g'(u)|^q\, du \right) \right]^{1/q},
\end{aligned}$$

which proves the first inequality in (2.8).

If we use inequality (1.5) for $h = \ell$ and $k = \int_a^{\cdot} |f'(u)|^p \, du$ and observe that $0 \leq \int_a^t |f'(u)|^p \, du \leq \int_a^b |f'(u)|^p \, du$ while $a \leq \ell(t) \leq b$ for $t \in [a, b]$, then

$$\begin{aligned} &D_w\left(\ell, \int_a^{\cdot} |f'(u)|^p \, du\right) \\ &\leq \frac{1}{2}\left(\int_a^b |f'(u)|^p \, du\right) \int_a^b w(t) \left|t - \int_a^b s w(s) \, ds\right| dt \\ &\leq \frac{1}{2}\left(\int_a^b |f'(u)|^p \, du\right) \left[\int_a^b s^2 w(s) \, ds - \left(\int_a^b s w(s) \, ds\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(b-a)\left(\int_a^b |f'(u)|^p \, du\right), \end{aligned} \tag{2.11}$$

and, similarly

$$\begin{aligned} &D_w\left(\ell, \int_a^{\cdot} |g'(u)|^q \, du\right) \\ &\leq \frac{1}{2}\left(\int_a^b |g'(u)|^q \, du\right) \int_a^b w(t) \left|t - \int_a^b s w(s) \, ds\right| dt \\ &\leq \frac{1}{2}\left(\int_a^b |g'(u)|^q \, du\right) \left[\int_a^b s^2 w(s) \, ds - \left(\int_a^b s w(s) \, ds\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(b-a)\left(\int_a^b |g'(u)|^q \, du\right). \end{aligned} \tag{2.12}$$

Therefore, by (2.11) and (2.12) we derive

$$\begin{aligned} &\left[D_w\left(\ell, \int_a^{\cdot} |f'(u)|^p \, du\right)\right]^{1/p} \left[D_w\left(\ell, \int_a^{\cdot} |g'(u)|^q \, du\right)\right]^{1/q} \\ &\leq \left[\frac{1}{2}\left(\int_a^b |f'(u)|^p \, du\right) \int_a^b w(t) \left|t - \int_a^b s w(s) \, ds\right| dt\right]^{1/p} \\ &\times \left[\frac{1}{2}\left(\int_a^b |g'(u)|^q \, du\right) \int_a^b w(t) \left|t - \int_a^b s w(s) \, ds\right| dt\right]^{1/q} \end{aligned}$$

$$\leq \left(\frac{1}{2} \left(\int_a^b |f'(u)|^p \, du \right) \left[\int_a^b s^2 w(s) \, ds - \left(\int_a^b sw(s) \, ds \right)^2 \right]^{\frac{1}{2}} \right)^{1/p}$$
$$\times \left(\frac{1}{2} \left(\int_a^b |g'(u)|^q \, du \right) \left[\int_a^b s^2 w(s) \, ds - \left(\int_a^b sw(s) \, ds \right)^2 \right]^{\frac{1}{2}} \right)^{1/q}$$
$$\leq \left[\frac{1}{4} (b-a) \left(\int_a^b |f'(u)|^p \, du \right) \right]^{1/p} \left[\frac{1}{4} (b-a) \left(\int_a^b |g'(u)|^q \, du \right) \right]^{1/q},$$

which proves the last part of (2.8). □

3. The Case of Uniform Distribution

If we consider the uniform distribution $w_0(t) = 1/(b-a)$ on the interval $[a, b]$, then we get

$$D_{w_0}(h, k) := \frac{1}{b-a} \int_a^b h(t) k(t) \, dt - \frac{1}{b-a} \int_a^b h(t) \, dt \frac{1}{b-a} \int_a^b k(t) \, dt,$$

$$D_{w_0}\left(\ell, \int_a^{\cdot} |g'(u)| \, du \right)$$
$$= \frac{1}{b-a} \int_a^b t \left(\int_a^t |g'(u)| \, du \right) dt - \frac{a+b}{2} \frac{1}{b-a} \int_a^b \left(\int_a^t |g'(u)| \, du \right) dt$$

Integrating by parts, we have

$$\frac{1}{2} \int_a^b (b-t)(t-a) |g'(t)| \, dt$$
$$= \frac{1}{2} \int_a^b (b-t)(t-a) \, d\left(\int_a^t |g'(u)| \, du \right)$$
$$= \frac{1}{2} \left[(b-t)(t-a) \int_a^t |g'(u)| \, du \Big|_a^b + \int_a^b (2t-a-b) \left(\int_a^t |g'(u)| \, du \right) dt \right]$$
$$= \int_a^b \left(t - \frac{a+b}{2} \right) \left(\int_a^t |g'(u)| \, du \right) dt$$
$$= \int_a^b t \left(\int_a^t |g'(u)| \, du \right) dt - \frac{a+b}{2} \int_a^b \left(\int_a^t |g'(u)| \, du \right) dt.$$

Therefore,

$$D_{w_0}\left(\ell, \int_a^{\cdot} |g'(u)| \, du \right) = \frac{1}{2(b-a)} \int_a^b (b-t)(t-a) |g'(t)| \, dt.$$

Also

$$\int_a^b w_0(t) \left| t - \int_a^b s w_0(s) \, ds \right| dt = \frac{1}{b-a} \int_a^b \left| t - \frac{1}{b-a} \int_a^b s ds \right| dt$$
$$= \frac{1}{b-a} \int_a^b \left| t - \frac{a+b}{2} \right| dt = \frac{1}{4}(b-a),$$

therefore, by the first two inequalities in (2.1) we derive the following inequality of interest:

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.1}$$
$$\leq \frac{1}{2(b-a)}\left\|f'\right\|_{[a,b],\infty}\int_a^b (b-t)(t-a)\left|g'(t)\right|dt$$
$$\leq \frac{1}{8}(b-a)\left\|f'\right\|_{[a,b],\infty}\left\|g'\right\|_{[a,b],1}.$$

Observe also that

$$D_{w_0}\left(\ell,\int_a^{\cdot}\left|f'(u)\right|^p du\right)=\frac{1}{2(b-a)}\int_a^b (b-t)(t-a)\left|f'(u)\right|^p dt$$

and

$$D_{w_0}\left(\ell,\int_a^{\cdot}\left|g'(u)\right|^q du\right)=\frac{1}{2(b-a)}\int_a^b (b-t)(t-a)\left|g'(u)\right|^q dt.$$

By utilising the first two inequalities in (2.8) we also get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.2}$$
$$\leq \frac{1}{2(b-a)}\left[\int_a^b (b-t)(t-a)\left|f'(u)\right|^p dt\right]^{1/p}$$
$$\times\left[\int_a^b (b-t)(t-a)\left|g'(u)\right|^q\right]^{1/q}$$
$$\leq \frac{1}{8}(b-a)\left\|f'\right\|_{[a,b],p}\left\|g'\right\|_{[a,b],q},$$

provided that $\left\|f'\right\|_{[a,b],p}<\infty$ and $\left\|g'\right\|_{[a,b],q}<\infty$ for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

4. The Case of Infinite Intervals

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R}=(-\infty,\infty)$. Namely, if $I=(-\infty,\infty)$ and $w(s)>0$ for $s\in\mathbb{R}$ with $\int_{-\infty}^{\infty}w(s)\,ds=1$, namely w is a probability density function on $(-\infty,\infty)$, then for $f,\ g$ *Lebesgue measurable* functions on $(-\infty,\infty)$, we can consider the functional the functional

$$D_{w,\mathbb{R}}(f,g):=\int_{-\infty}^{\infty}w(t)\,f(t)\,g(t)\,dt-\int_{-\infty}^{\infty}w(t)\,f(t)\,dt\int_{-\infty}^{\infty}w(t)\,g(t)\,dt.$$

From (2.1) we then have

$$
\begin{aligned}
(4.1)\quad & |D_{w,\mathbb{R}}(f,g)| \\
&\leq \|f'\|_{(-\infty,\infty),\infty} D_{w,\mathbb{R}}\left(\ell, \int_{-\infty}^{\cdot} |g'(u)|\,du\right) \\
&\leq \frac{1}{2}\|f'\|_{(-\infty,\infty),\infty}\|g'\|_{(-\infty,\infty),1}\int_{-\infty}^{\infty} w(t)\left|t-\int_{-\infty}^{\infty} sw(s)\,ds\right| dt \\
&\leq \frac{1}{2}\|f'\|_{(-\infty,\infty),\infty}\|g'\|_{(-\infty,\infty),1}\left[\int_{-\infty}^{\infty} s^2 w(s)\,ds-\left(\int_{-\infty}^{\infty} sw(s)\,ds\right)^2\right]^{\frac{1}{2}}
\end{aligned}
$$

with the assumptions that f and g are locally absolutely continuous, $\|f'\|_{(-\infty,\infty),\infty} = \operatorname{esssup}_{t\in(-\infty,\infty)}|f'(t)| < \infty$ and $\|g'\|_{(-\infty,\infty),1} := \left(\int_{-\infty}^{\infty}|g'(u)|\,du\right) < \infty$.

Also, if $\|f'\|_{(-\infty,\infty),p} = \left(\int_{-\infty}^{\infty}|f'(u)|^p\,du\right)^{1/p} < \infty$ and $\|g'\|_{(-\infty,\infty),q} = \left(\int_{-\infty}^{\infty}|g'(u)|^q\,du\right)^{1/q} < \infty$ for p, $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$. Then from (2.8) we derive

$$
\begin{aligned}
(4.2)\quad & |D_{w,\mathbb{R}}(f,g)| \\
&\leq \left[D_{w,\mathbb{R}}\left(\ell, \int_{-\infty}^{\cdot} |f'(u)|^p\,du\right)\right]^{1/p}\left[D_{w,\mathbb{R}}\left(\ell, \int_{-\infty}^{\cdot} |g'(u)|^q\,du\right)\right]^{1/q} \\
&\leq \frac{1}{2}\|f'\|_{(-\infty,\infty),p}\|g'\|_{(-\infty,\infty),q}\int_{-\infty}^{\infty} w(t)\left|t-\int_{-\infty}^{\infty} sw(s)\,ds\right| dt \\
&\leq \frac{1}{2}\|f'\|_{(-\infty,\infty),p}\|g'\|_{(-\infty,\infty),q}\left[\int_{-\infty}^{\infty} s^2 w(s)\,ds-\left(\int_{-\infty}^{\infty} sw(s)\,ds\right)^2\right]^{\frac{1}{2}}
\end{aligned}
$$

In particular, if $\|f'\|_{(-\infty,\infty),2} = \left(\int_{-\infty}^{\infty}|f'(u)|^2\,du\right)^{1/2} < \infty$ and $\|g'\|_{(-\infty,\infty),2} = \left(\int_{-\infty}^{\infty}|g'(u)|^2\,du\right)^{1/2} < \infty$, then

$$
\begin{aligned}
(4.3)\quad & |D_{w,\mathbb{R}}(f,g)| \\
&\leq \left[D_{w,\mathbb{R}}\left(\ell, \int_{-\infty}^{\cdot} |f'(u)|^2\,du\right)\right]^{1/2}\left[D_{w,\mathbb{R}}\left(\ell, \int_{-\infty}^{\cdot} |g'(u)|^2\,du\right)\right]^{1/2} \\
&\leq \frac{1}{2}\|f'\|_{(-\infty,\infty),2}\|g'\|_{(-\infty,\infty),2}\int_{-\infty}^{\infty} w(t)\left|t-\int_{-\infty}^{\infty} sw(s)\,ds\right| dt \\
&\leq \frac{1}{2}\|f'\|_{(-\infty,\infty),2}\|g'\|_{(-\infty,\infty),2}\left[\int_{-\infty}^{\infty} s^2 w(s)\,ds-\left(\int_{-\infty}^{\infty} sw(s)\,ds\right)^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

The probability density of the *normal distribution* on $(-\infty,\infty)$ is

$$
w_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x\in\mathbb{R},
$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

The cumulative distribution function is

$$W_{\mu,\sigma^2}(x) = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right),$$

where the *error function* erf is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right) dt.$$

Consider the functional

$$\begin{aligned} &D_{N,\sigma,\mu}(f,g) \\ &:= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t)\, g(t)\, dt \\ &- \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) f(t)\, dt \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right) g(t)\, dt \end{aligned}$$

with the parameters μ and σ as above.

Then from (4.1) we get

$$\begin{aligned} &|D_{N,\sigma,\mu}(f,g)| \\ &\le \|f'\|_{(-\infty,\infty),\infty} D_{N,\sigma,\mu}\left(\ell, \int_{-\infty}^{\cdot} |g'(u)|\, du\right) \\ &\le \frac{1}{2\sqrt{2\pi}\sigma}\|f'\|_{(-\infty,\infty),\infty}\|g'\|_{(-\infty,\infty),1}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)|t-\mu|\, dt \\ &\le \frac{1}{2}\sigma\|f'\|_{(-\infty,\infty),\infty}\|g'\|_{(-\infty,\infty),1} \end{aligned} \tag{4.4}$$

with the assumptions that f and g are locally absolutely continuous, $\|f'\|_{(-\infty,\infty),\infty} = \operatorname{esssup}_{t\in(-\infty,\infty)}|f'(t)| < \infty$ and $\|g'\|_{(-\infty,\infty),1} := \left(\int_{-\infty}^{\infty}|g'(u)|\, du\right) < \infty$.

From (4.2) we derive

$$\begin{aligned} &|D_{N,\sigma,\mu}(f,g)| \\ &\le \left[D_{N,\sigma,\mu}\left(\ell, \int_{-\infty}^{\cdot}|f'(u)|^p du\right)\right]^{1/p}\left[D_{N,\sigma,\mu}\left(\ell, \int_{-\infty}^{\cdot}|g'(u)|^q du\right)\right]^{1/q} \\ &\le \frac{1}{2\sqrt{2\pi}\sigma}\|f'\|_{(-\infty,\infty),p}\|g'\|_{(-\infty,\infty),q}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)|t-\mu|\, dt \\ &\le \frac{1}{2}\sigma\|f'\|_{(-\infty,\infty),p}\|g'\|_{(-\infty,\infty),q} \end{aligned} \tag{4.5}$$

provided that $\|f'\|_{(-\infty,\infty),p} = \left(\int_{-\infty}^{\infty}|f'(u)|^p du\right)^{1/p} < \infty$ and $\|g'\|_{(-\infty,\infty),q} = \left(\int_{-\infty}^{\infty}|g'(u)|^q du\right)^{1/q} < \infty$ for $p,\, q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$.

5. Applications for Norms and Semi-Inner Products

Let X be a real linear space, $x, y \in X$, $x \ne y$ and let $[x,y] := \{(1-\lambda)x + \lambda y,\ \lambda \in [0,1]\}$ be the *segment* generated by x and y. We consider the function $f : [x,y] \to \mathbb{R}$ and the attached function $g(x,y) : [0,1] \to \mathbb{R}$, $g(x,y)(t) := f[(1-t)x + ty]$, $t \in [0,1]$.

It is well known that f is convex on $[x,y]$ iff $g(x,y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_{\pm}(x,y)(s) = \triangledown_{\pm} f[(1-s)x+sy](y-x)$, $s \in [0,1)$,
(ii) $g'_{+}(x,y)(0) = \triangledown_{+} f(x)(y-x)$,
(iii) $g'_{-}(x,y)(1) = \triangledown_{-} f(y)(y-x)$,

where $\triangledown_{\pm} f(x)(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\begin{aligned} \triangledown_{+} f(x)(y) &:= \lim_{h\to 0+} \frac{f(x+hy)-f(x)}{h}, \\ \triangledown_{-} f(x)(y) &:= \lim_{k\to 0-} \frac{f(x+ky)-f(x)}{k}, \quad x,\ y \in X. \end{aligned}$$

We remark also that

$$\triangledown_{+} f[(1-s)x+sy](y-x) = \triangledown_{-} f[(1-s)x+sy](y-x)$$

for almost every $s \in [0,1]$, being the lateral derivatives of a convex function. In integrals we can then write $\triangledown$ instead of $\triangledown_{+}$ or $\triangledown_{-}$.

Now, assume that $(X, \|\cdot\|)$ is a normed linear space. The function $f_0(s) = \frac{1}{2}\|x\|^2$, $x \in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s := \triangledown_{+} f_0(y)(x) = \lim_{t\to 0+} \frac{\|y+tx\|^2-\|y\|^2}{2t}$;
(v) $\langle x,y\rangle_i := \triangledown_{-} f_0(y)(x) = \lim_{s\to 0-} \frac{\|y+sx\|^2-\|y\|^2}{2s}$;

for any x, $y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [21]), assuming that p, $q \in \{s,i\}$ and $p \neq q$:

(a) $\langle x,x\rangle_p = \|x\|^2$ for all $x \in X$;
(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta \langle x,y\rangle_p$ if α, $\beta \geq 0$ and x, $y \in X$;
(aaa) $\left|\langle x,y\rangle_p\right| \leq \|x\|\|y\|$ for all x, $y \in X$;
(av) $\langle \alpha x+y, x\rangle_p = \alpha\langle x,x\rangle_p + \langle y,x\rangle_p$ if x, $y \in X$ and $\alpha \in \mathbb{R}$;
(v) $\langle -x,y\rangle_p = -\langle x,y\rangle_q$ for all x, $y \in X$;
(va) $\langle x+y,z\rangle_p \leq \|x\|\|z\| + \langle y,z\rangle_p$ for all x, y, $z \in X$;
(vaa) The mapping $\langle \cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);
(vaaa) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X\setminus\{0\}$ if and only if $\langle y,x_0\rangle_s = \langle y,x_0\rangle_i$ for all $y \in X$; in general $\langle y,x\rangle_i \leq \langle y,x\rangle_s$ for all x, $y \in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $\langle y,x\rangle_i = \langle y,x\rangle = \langle y,x\rangle_s$ for all x, $y \in X$.

For $m \geq 1$ the function $f_m(x) = \|x\|^m$ is convex on X. Therefore

$$\triangledown_{+(-)} f_m(y)(x) = p\|y\|^{m-2}\langle x,y\rangle_{s(i)} \tag{5.1}$$

which exists for all x, $y \in X$ whenever $m \geq 2$. If $1 \leq m < 2$ the equality (4.4) holds for all $x \in X$ and nonzero $y \in X$.

Observe also that

$$\triangledown_{\pm} f_m[(1-s)x+sy](y-x) = m\|(1-s)x+sy\|^{m-2}\langle y-x,(1-s)x+sy\rangle_{s(i)}$$

which exists for all $x, y \in X$ whenever $m \geq 2$. If $1 \leq m < 2$ the equality (4.4) holds for all x, y such that $(1-s)x + sy \neq 0$ for all $s \in [0,1]$.

Now, assume that $f, g : C \to \mathbb{R}$ are convex on the convev subset C in the linear space X. Assume also that $w : [0,1] \to [0,\infty)$ is integrable and $\int_0^1 w(t)\,dt = 1$. For distinct vectors x, y we consider the functional

$$D_{w,x,y}(f,g) := \int_0^1 f((1-t)x + ty)\, g((1-t)x + ty)\, w(t)\, dt - \int_0^1 f((1-t)x + ty)\, w(t)\, dt \int_0^1 g((1-t)x + ty)\, w(t)\, dt.$$

From (2.1) we then obtain

$$\begin{aligned} &|D_{w,x,y}(f,g)| \\ &\leq \sup_{t\in[0,1]} |\nabla f[(1-s)x + sy](y-x)| \\ &\times D_w\left(\ell, \int_0^{\cdot} |\nabla g[(1-u)x + uy](y-x)|\, du\right) \\ &\leq \frac{1}{2} \sup_{t\in[0,1]} |\nabla f[(1-s)x + sy](y-x)| \int_0^1 |\nabla g[(1-s)x + sy](y-x)|\, ds \\ &\times \int_0^1 w(t) \left| t - \int_a^b s w(s)\, ds \right| dt \\ &\leq \frac{1}{2} \sup_{t\in[0,1]} |\nabla f[(1-s)x + sy](y-x)| \int_0^1 |\nabla g[(1-s)x + sy](y-x)|\, ds \\ &\times \left[\int_0^1 s^2 w(s)\, ds - \left(\int_0^1 s w(s)\, ds \right)^2 \right]^{\frac{1}{2}} \\ &\leq \frac{1}{4} \sup_{t\in[0,1]} |\nabla f[(1-s)x + sy](y-x)| \int_0^1 |\nabla g[(1-s)x + sy](y-x)|\, ds, \end{aligned} \tag{5.2}$$

provided that the sup and the integral in the right side are finite.

From (2.8) we derive

$$\begin{aligned} &|D_{w,x,y}(f,g)| \\ &\leq \left[D_w\left(\ell, \int_0^{\cdot} |\nabla f[(1-u)x + uy](y-x)|^r\, du\right) \right]^{1/r} \\ &\times \left[D_w\left(\ell, \int_0^{\cdot} |\nabla g[(1-u)x + uy](y-x)|^q\, du\right) \right]^{1/q} \\ &\leq \frac{1}{2} \left(\int_0^1 |\nabla f[(1-s)x + sy](y-x)|^r\, ds \right)^{1/r} \\ &\times \left(\int_0^1 |\nabla g[(1-s)x + sy](y-x)|^q\, ds \right)^{1/q} \int_a^b w(t) \left| t - \int_a^b s w(s)\, ds \right| dt \end{aligned} \tag{5.3}$$

$$\leq \frac{1}{2}\left(\int_0^1 \left|\triangledown f\left[(1-s)x+sy\right](y-x)\right|^r ds\right)^{1/r}$$
$$\times\left(\int_0^1 \left|\triangledown g\left[(1-s)x+sy\right](y-x)\right|^q ds\right)^{1/q}$$
$$\times\left[\int_a^b s^2 w(s)\,ds-\left(\int_a^b sw(s)\,ds\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{2}\left(\int_0^1 \left|\triangledown f\left[(1-s)x+sy\right](y-x)\right|^r ds\right)^{1/r}$$
$$\times\left(\int_0^1 \left|\triangledown g\left[(1-s)x+sy\right](y-x)\right|^q ds\right)^{1/q}$$

for $r,\,q>1$ with $\frac{1}{r}+\frac{1}{q}=1$, provided the integrals from the right side are finite.

In the case when $w=w_0\equiv 1$, then from (5.2) we have

$$\left|\int_0^1 f((1-t)x+ty)\,g((1-t)x+ty)\,dt - \int_0^1 f((1-t)x+ty)\,dt\int_0^1 g((1-t)x+ty)\,dt\right| \leq \frac{1}{8}\sup_{t\in[0,1]}\left|\triangledown f\left[(1-s)x+sy\right](y-x)\right|\int_0^1 \left|\triangledown g\left[(1-s)x+sy\right](y-x)\right|ds \tag{5.4}$$

while from (5.3)

$$\left|\int_0^1 f((1-t)x+ty)\,g((1-t)x+ty)\,dt - \int_0^1 f((1-t)x+ty)\,dt\int_0^1 g((1-t)x+ty)\,dt\right| \leq \frac{1}{8}\left(\int_0^1 \left|\triangledown f\left[(1-s)x+sy\right](y-x)\right|^r ds\right)^{1/r} \times\left(\int_0^1 \left|\triangledown g\left[(1-s)x+sy\right](y-x)\right|^q ds\right)^{1/q} \tag{5.5}$$

for $r,\,q>1$ with $\frac{1}{r}+\frac{1}{q}=1$.

Now, if we write (5.4) for the convex functions $f_m(x)=\|x\|^m$, $m\geq 1$ and $g_n(x)=\|x\|^n$, $n\geq 1$, then for

$$D_{\|\cdot\|;m,n}(x,y) := \int_0^1 \|(1-t)x+ty\|^{m+n}dt-\int_0^1 \|(1-t)x+ty\|^m dt\int_0^1 \|(1-t)x+ty\|^n dt$$

we get

$$\begin{aligned}
&\left|D_{m,n}(x,y)\right| \\
&\leq \frac{1}{8}mn\sup_{t\in[0,1]}\left\{\|(1-s)x+sy\|^{m-2}\left|\langle y-x,(1-s)x+sy\rangle_p\right|\right\} \\
&\times\int_0^1\|(1-s)x+sy\|^{n-2}\left|\langle y-x,(1-s)x+sy\rangle_p\right|ds,\ p=s,i,
\end{aligned}$$

which exists for all x, $y\in X$ whenever m, $n\geq 2$. If either $1\leq m<2$ or $1\leq n<2$, the inequality (4.4) holds for all x, y such that $(1-s)x+sy\neq 0$ for all $s\in[0,1]$.

Using the Schwarz inequality for the semi-inner products, we have

$$\left|\langle y-x,(1-s)x+sy\rangle_p\right|\leq\|y-x\|\,\|(1-s)x+sy\|$$

and by (5.6) we derive

$$\begin{aligned}
&\left|D_{\|\cdot\|;m,n}(x,y)\right| \\
&\leq\frac{1}{8}mn\|y-x\|^2\sup_{t\in[0,1]}\left\{\|(1-s)x+sy\|^{m-1}\right\}\int_0^1\|(1-s)x+sy\|^{n-1}ds,
\end{aligned}\tag{5.6}$$

for m, $n\geq 1$.

Now, if we write (5.4) for the convex functions $f_m(x)=\|x\|^m$, $m\geq 1$ and $g_n(x)=\|x\|^n$, $n\geq 1$, then we get

$$\begin{aligned}
&\left|D_{\|\cdot\|;m,n}(x,y)\right| \\
&\leq\frac{1}{8}m^{1/p}n^{1/q}\left(\int_0^1\|(1-s)x+sy\|^{r(m-2)}\left|\langle y-x,(1-s)x+sy\rangle_p\right|^r ds\right)^{1/r} \\
&\times\left(\int_0^1\|(1-s)x+sy\|^{q(n-2)}\left|\langle y-x,(1-s)x+sy\rangle_p\right|^q ds\right)^{1/q}
\end{aligned}\tag{5.7}$$

for r, $q>1$ with $\frac{1}{r}+\frac{1}{q}=1$, which exists for all x, $y\in X$ whenever m, $n\geq 2$. If either $1\leq m<2$ or $1\leq n<2$, the inequality (4.4) holds for all x, y such that $(1-s)x+sy\neq 0$ for all $s\in[0,1]$.

Using the Schwarz inequality for the semi-inner products, we have

$$\begin{aligned}
\left|D_{\|\cdot\|;m,n}(x,y)\right|&\leq\frac{1}{8}m^{1/r}n^{1/q}\|y-x\|^2\left(\int_0^1\|(1-s)x+sy\|^{p(m-1)}ds\right)^{1/r} \\
&\times\left(\int_0^1\|(1-s)x+sy\|^{q(n-1)}ds\right)^{1/q},
\end{aligned}\tag{5.8}$$

which holds for all x, $y\in X$ and r, $q>1$ with $\frac{1}{r}+\frac{1}{q}=1$.

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