INTEGRAL INEQUALITIES FOR THE WEIGHTED ČEBYŠEV FUNCTIONAL OF A MONOTONIC FUNCTION WITH APPLICATIONS

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Abstract. Assume that $w : [a,b] \to [0,\infty)$ is integrable with $\int_a^b w(t)\,dt = 1$ and f, g are Lebesgue integrable on $[a,b]$. Consider the Čebyšev functional

$$D_w(f,g) := \int_a^b f(t)\,g(t)\,w(t)\,dt - \int_a^b f(t)\,w(t)\,dt \int_a^b g(t)\,w(t)\,dt.$$

In this paper we show among others that, if $f : [a,b] \to \mathbb{R}$ is monotonic nondecreasing, $g : [a,b] \to \mathbb{C}$ is absolutely continuous and w a probability density function on $[a,b]$, then

$$\begin{aligned}|D_w(f,g)| &\le D_w\left(f, \int_a^{\cdot} |g'(u)|\,du\right) \\ &\le \frac{1}{2}\|g'\|_{[a,b],1} \int_a^b w(t)\left|f(t) - \int_a^b w(s) f(s)\,ds\right| dt \\ &\le \frac{1}{2}\|g'\|_{[a,b],1}\left(\int_a^b w(t)\left|f(t) - \int_a^b w(s) f(s)\,ds\right|^2 dt\right)^{1/2} \\ &\le \frac{1}{4}[f(b) - f(a)]\,\|g'\|_{[a,b],1},\end{aligned}$$

where $\|g'\|_{[a,b],1} = \int_a^b |g'(u)|\,du$. Applications for continuous probability density functions supported on infinite intervals are also given.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R} \cup \{\infty\}$.

For a μ-measurable function $w : \Omega \to \mathbb{R}$, with $w(x) \ge 0$ for μ-a.e. $x \in \Omega$, consider the Lebesgue space $L_w(\Omega,\mu) := \{h : \Omega \to \mathbb{R},\ h$ is μ-measurable and $\int_\Omega w(x)\,|h(x)|\,d\mu(x) < \infty\}$. Assume $\int_\Omega w(x)\,d\mu(x) = 1$. In order to simplify the notation for the integrals, we do not write the variable, namely, instead of $\int_\Omega w(x)\,d\mu(x)$ we simply write $\int_\Omega w\,d\mu$.

If $h,\ k : \Omega \to \mathbb{R}$ are μ-measurable functions and $h,\ k,\ hk \in L_w(\Omega,\mu)$, then we may consider the weighted Čebyšev functional in the following form

$$D_w(h,k) := \int_\Omega whk\,d\mu - \int_\Omega wh\,d\mu \int_\Omega wk\,d\mu. \tag{1.1}$$

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The following result is known in the literature as the Grüss inequality, see for instance [7]:

$$|D_w(h,k)| \leq \frac{1}{4}(\Gamma-\gamma)(\Delta-\delta), \tag{1.2}$$

provided

$$-\infty<\gamma\leq h\leq\Gamma<\infty,\ \ -\infty<\delta\leq k\leq\Delta<\infty \tag{1.3}$$

μ-a.e. on Ω. The constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [7] Cerone and Dragomir proved among others the following *refinement of Grüss' inequality:*

Theorem 1. *For $h,\ k:\Omega\to\mathbb{R}$, μ-measurable functions and so that $-\infty<\gamma\leq h\leq\Gamma<\infty$, $-\infty<\delta\leq k\leq\Delta<\infty$ μ-a.e. on Ω,*

$$\begin{aligned} |D_w(h,k)| &\leq \frac{1}{2}(\Delta-\delta)\int_\Omega w\left|h-\int_\Omega whd\mu\right|d\mu \\ &\leq \frac{1}{2}(\Delta-\delta)\left[\int_\Omega wh^2d\mu-\left(\int_\Omega whd\mu\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma), \end{aligned} \tag{1.4}$$

provided that $h,\ k,\ hk\in L_w(\Omega,\mu)$. The constants $\frac{1}{2}$ and $\frac{1}{4}$ are best possible.

Consider a probability density function w on $[a,b]$, i.e., $w\geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt=1$, and the weighted Čebyšev functional for functions defined on a finite interval $[a,b]$,

$$D_w(h,k):=\int_a^b h(t)\,k(t)\,w(t)\,dt-\int_a^b h(t)\,w(t)\,dt\int_a^b k(t)\,w(t)\,dt.$$

From (1.4) we get

$$\begin{aligned} |D_w(h,k)| &\leq \frac{1}{2}(\Delta-\delta)\int_a^b w(t)\left|h(t)-\int_a^b h(s)\,w(s)\,ds\right|dt \\ &\leq \frac{1}{2}(\Delta-\delta)\left[\int_a^b w(s)\,h^2(s)\,ds-\left(\int_a^b h(s)\,w(s)\,ds\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma), \end{aligned} \tag{1.5}$$

provided that $h,\ k:[a,b]\to\mathbb{R}$ are measurable functions and so that $-\infty<\gamma\leq h\leq\Gamma<\infty$, $-\infty<\delta\leq k\leq\Delta<\infty$ a.e. on $[a,b]$, and $h,\ k,\ hk\in L_w[a,b]$.

For more recent upper bounds related to the Čebyšev functional see [1]-[9], [11]-[19] and [22]-[29].

2. Main Results

Theorem 2. *Assume that* $f:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing,* $g:[a,b]\to\mathbb{C}$ *is absolutely continuous and* w *a probability density function on* $[a,b]$, *then*

$$\begin{aligned} &|D_w(f,g)| \\ &\leq D_w\left(f,\int_a^{\cdot}|g'(u)|\,du\right) \\ &\leq \frac{1}{2}\|g'\|_{[a,b],1}\int_a^b w(t)\left|f(t)-\int_a^b w(s)f(s)\,ds\right|dt \\ &\leq \frac{1}{2}\|g'\|_{[a,b],1}\left(\int_a^b w(t)\left|f(t)-\int_a^b w(s)f(s)\,ds\right|^2 dt\right)^{1/2} \\ &\leq \frac{1}{4}[f(b)-f(a)]\,\|g'\|_{[a,b],1}, \end{aligned} \tag{2.1}$$

where $\|g'\|_{[a,b],1}=\int_a^b|g'(u)|\,du$.

We also have,

$$\begin{aligned} &|D_w(f,g)| \\ &\leq D_w\left(f,\int_a^{\cdot}|g'(u)|\,du\right) \\ &\leq \frac{1}{2}[f(b)-f(a)] \\ &\times\int_a^b w(t)\left|\int_a^t\left(\int_a^s w(u)\,du\right)|g'(s)|\,ds-\int_t^b\left(\int_s^b w(u)\,du\right)|g'(s)|\,ds\right|dt \\ &\leq \frac{1}{2}[f(b)-f(a)] \\ &\times\left[\int_a^b w(s)\left(\int_a^s|g'(u)|\,du\right)^2 ds-\left(\int_a^b\left(\int_a^s|g'(u)|\,du\right)w(s)\,ds\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4}[f(b)-f(a)]\,\|g'\|_{[a,b],1}. \end{aligned} \tag{2.2}$$

Proof. Observe that for $f,\ g:[a,b]\to\mathbb{C}$ we have weighted Korkine's identity

$$D_w(f,g)=\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,[f(t)-f(s)]\,[g(t)-g(s)]\,dtds.$$

For Korkine's classical identity for real-valued functions, see [27, p. 242].

If we take the modulus and use the integral's properties, we get

$$\begin{aligned} |D_w(f,g)| &\leq \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,|[f(t)-f(s)]\,[g(t)-g(s)]|\,dtds \\ &= \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,|f(t)-f(s)|\,|g(t)-g(s)|\,dtds. \end{aligned} \tag{2.3}$$

Since $g:[a,b]\to\mathbb{C}$ is absolutely continuous, then

$$\left|g(t)-g(s)\right|=\left|\int_s^t g'(u)\,du\right|\le\left|\int_s^t \left|g'(u)\right|du\right|$$

for all $t,\ s\in[a,b]$.

Therefore

$$\begin{aligned}(2.4)\qquad &\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left|f(t)-f(s)\right|\left|g(t)-g(s)\right|dtds\\ &\le\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left|f(t)-f(s)\right|\left|\int_s^t \left|g'(u)\right|du\right|dtds\\ &=\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left|(f(t)-f(s))\int_s^t \left|g'(u)\right|du\right|dtds.\end{aligned}$$

Since f is monotone nondecreasing, then

$$\left|(f(t)-f(s))\int_s^t \left|g'(u)\right|du\right|=(f(t)-f(s))\int_s^t \left|g'(u)\right|du\ge 0$$

for all $t,\ s\in[a,b]$, which implies that

$$\begin{aligned}&\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left|(f(t)-f(s))\int_s^t \left|g'(u)\right|du\right|dtds\\ &=\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,(f(t)-f(s))\int_s^t \left|g'(u)\right|dudtds\\ &=\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,(f(t)-f(s))\left(\int_a^t \left|g'(u)\right|du-\int_a^s \left|g'(u)\right|du\right)dtds\\ &=D_w\left(f,\int_a^{\cdot}\left|g'(u)\right|du\right).\end{aligned}$$

By utilising (2.3) and (2.4) we derive

$$\left|D_w(f,g)\right|\le D_w\left(f,\int_a^{\cdot}\left|g'(u)\right|du\right),$$

and the first inequality in (2.1) is proved.

Now, if we write the inequality (1.5) for $h=f$ and $k=\int_a^{\cdot}\left|g'(u)\right|du$, which satisfy the bounds

$$0\le\int_a^t \left|g'(u)\right|du\le\int_a^b \left|g'(u)\right|du \text{ for all } t\in[a,b],$$

then we get

$$\begin{aligned}&D_w\left(f,\int_a^{\cdot}\left|g'(u)\right|du\right)\\ &\le\frac{1}{2}\int_a^b \left|g'(u)\right|du\int_a^b w(t)\left|f(t)-\int_a^b w(s)\,f(s)\,ds\right|dt.\end{aligned}$$

By Schwarz and Grüss' inequalities we have

$$\int_a^b w(t)\left|f(t)-\int_a^b w(s)f(s)\,ds\right|dt$$
$$\leq\left(\int_a^b w(t)\left|f(t)-\int_a^b w(s)f(s)\,ds\right|^2 dt\right)^{1/2}$$
$$=\left[\int_a^b w(t)f^2(t)\,dt-\left(\int_a^b w(s)f(s)\,ds\right)^2\right]^{1/2}\leq\frac{1}{2}\left[f(b)-f(a)\right],$$

which proves the second part of (2.1).

If we write the inequality (1.5) for $h=\int_a^{\cdot}|g'(u)|\,du$ and $k=f$, then we get

$$\left|D\left(\int_a^{\cdot}|g'(u)|\,du,f\right)\right| \tag{2.5}$$
$$\leq\frac{1}{2}\left[f(b)-f(a)\right]$$
$$\times\int_a^b w(t)\left|\int_a^t|g'(u)|\,du-\int_a^b w(s)\left(\int_a^s|g'(u)|\,du\right)ds\right|dt$$

$$\leq\frac{1}{2}\left[f(b)-f(a)\right]$$
$$\times\left[\int_a^b w(s)\left(\int_a^s|g'(u)|\,du\right)^2 ds-\left(\int_a^b\left(\int_a^s|g'(u)|\,du\right)w(s)\,ds\right)^2\right]^{\frac{1}{2}}$$
$$\leq\frac{1}{4}\left[f(b)-f(a)\right]\left(\int_a^b|g'(u)|\,du\right).$$

Observe that

$$\int_a^b w(t)\left|\int_a^t|g'(u)|\,du-\int_a^b w(s)\left(\int_a^s|g'(u)|\,du\right)ds\right|dt$$
$$=\int_a^b w(t)\left|\int_a^t|g'(u)|\,du-\int_a^b\left(\int_a^s|g'(u)|\,du\right)d\left(\int_a^s w(u)\,du\right)\right|dt$$
$$=\int_a^b w(t)\left|\int_a^t|g'(u)|\,du\right.$$
$$\left.-\left(\left(\int_a^b|g'(s)|\,ds\right)\int_a^b w(u)\,du-\int_a^b|g'(s)|\left(\int_a^s w(u)\,du\right)ds\right)dt\right|$$
$$=\int_a^b w(t)\left|\int_a^t|g'(u)|\,du-\left(\int_a^b|g'(s)|\left(\int_s^b w(u)\,du\right)ds\right)\right|dt$$

$$
\begin{aligned}
&= \int_a^b w(t) \left| \int_a^t |g'(u)|\, du - \left(\int_a^b |g'(s)| \left(\int_s^b w(u)\, du \right) ds \right) \right| dt \\
&= \int_a^b w(t) \left| \int_a^t |g'(u)|\, du - \int_a^t |g'(s)| \left(\int_s^b w(u)\, du \right) ds \right. \\
&\left. - \int_t^b |g'(s)| \left(\int_s^b w(u)\, du \right) ds dt \right| \\
&= \int_a^b w(t) \left| \int_a^t \left(1 - \int_s^b w(u)\, du \right) |g'(u)|\, du - \int_t^b |g'(s)| \left(\int_s^b w(u)\, du \right) ds \right| dt \\
&= \int_a^b w(t) \left| \int_a^t \left(\int_a^s w(u)\, du \right) |g'(s)|\, ds - \int_t^b \left(\int_s^b w(u)\, du \right) |g'(s)|\, ds \right| dt
\end{aligned}
$$

and by (2.5) we get (2.2). □

We also have:

Theorem 3. *Assume that* $f : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing and* $g : [a,b] \to \mathbb{C}$ *is absolutely continuous with* $\|g'\|_{[a,b],p} := \left(\int_a^b |g'(t)|^p \right)^{1/p} < \infty$. *Then*

$$
|D_w(f,g)| \le [D_w(f,\ell)]^{1/q} \left[D_w\left(f, \int_a^{\cdot} |g'(u)|^p\, du \right) \right]^{1/p} \tag{2.6}
$$

for all $p,\ q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$.

In particular, we have

$$
|D_w(f,g)| \le [D_w(f,\ell)]^{1/2} \left[D_w\left(f, \int_a^{\cdot} |g'(u)|^p\, du \right) \right]^{1/2}. \tag{2.7}
$$

Proof. Using Hölder's inequality, we also have for all $s,\ t \in [a,b]$ that

$$
|g(t) - g(s)| = \left| \int_s^t g'(u)\, du \right| \le \left| \int_s^t |g'(u)|\, du \right| \le |t-s|^{1/q} \left| \int_s^t |g'(u)|^p\, du \right|^{1/p}
$$

for $p,\ q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Therefore

$$
\begin{aligned}
&\frac{1}{2} \int_a^b \int_a^b w(s) w(t) |f(t) - f(s)| |g(t) - g(s)|\, dtds \\
&\le \frac{1}{2} \int_a^b \int_a^b w(s) w(t) |f(t) - f(s)| |t-s|^{1/q} \left| \int_s^t |g'(u)|^p\, du \right|^{1/p} dtds.
\end{aligned} \tag{2.8}
$$

On making use of weighted Hölder's inequality for double integral, we have for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\begin{aligned}
&\frac{1}{2}\int_a^b\int_a^b w(s)\,w(t)\,|f(t)-f(s)|\,|t-s|^{1/q}\left|\int_s^t |g'(u)|^p\,du\right|^{1/p} dtds \qquad (2.9)\\
&\le\left[\frac{1}{2}\int_a^b\int_a^b w(s)\,w(t)\,|f(t)-f(s)|\left(|t-s|^{1/q}\right)^q dtds\right]^{1/q}\\
&\times\left[\frac{1}{2}\int_a^b\int_a^b w(s)\,w(t)\,|f(t)-f(s)|\left(\left|\int_s^t |g'(u)|^p\,du\right|^{1/p}\right)^p dtds\right]^{1/p}
\end{aligned}$$

$$\begin{aligned}
&=\left[\frac{1}{2}\int_a^b\int_a^b w(s)\,w(t)\,|f(t)-f(s)|\,|t-s|\,dtds\right]^{1/q}\\
&\times\left[\frac{1}{2}\int_a^b\int_a^b w(s)\,w(t)\,|f(t)-f(s)|\left|\int_s^t |g'(u)|^p\,du\right| dtds\right]^{1/p}\\
&=\left[\frac{1}{2}\int_a^b\int_a^b w(s)\,w(t)\,[f(t)-f(s)]\,(t-s)\,dtds\right]^{1/q}\\
&\times\left[\frac{1}{2}\int_a^b\int_a^b w(s)\,w(t)\,[f(t)-f(s)]\left(\int_a^t |g'(u)|^p\,du-\int_a^s |g'(u)|^p\,du\right) dtds\right]^{1/p}\\
&=[D_w(f,\ell)]^{1/q}\left[D_w\left(f,\int_a^{\cdot} |g'(u)|^p\,du\right)\right]^{1/p}.
\end{aligned}$$

By making use (2.3), (2.8) and (2.9), we derive the desired inequality in (2.6). □

Corollary 1. *With the assumptions of Theorem 3, we have*

$$\begin{aligned}
&|D_w(f,g)| \qquad (2.10)\\
&\le[D_w(f,\ell)]^{1/q}\left[D_w\left(f,\int_a^{\cdot} |g'(u)|^p\,du\right)\right]^{1/p}\\
&\le\frac{1}{2}(b-a)^{1/q}\,\|g'\|_{[a,b],p}\int_a^b w(t)\left|f(t)-\int_a^b w(s)\,f(s)\,ds\right| dt\\
&\le\frac{1}{2}(b-a)^{1/q}\,\|g'\|_{[a,b],p}\left[\int_a^b w(t)\,f^2(t)\,dt-\left(\int_a^b w(s)\,f(s)\,ds\right)^2\right]^{1/2}\\
&\le\frac{1}{4}(b-a)^{1/q}\,[f(b)-f(a)]\,\|g'\|_{[a,b],p}
\end{aligned}$$

for all p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

In particular,

$$
\begin{aligned}
(2.11)\qquad & |D_w(f,g)| \\
&\leq [D_w(f,\ell)]^{1/2}\left[D_w\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/2} \\
&\leq \frac{1}{2}(b-a)^{1/2}\|g'\|_{[a,b],2}\int_a^b w(t)\left|f(t)-\int_a^b w(s)f(s)\,ds\right|dt \\
&\leq \frac{1}{2}(b-a)^{1/2}\|g'\|_{[a,b],2}\left[\int_a^b w(t)f^2(t)\,dt-\left(\int_a^b w(s)f(s)\,ds\right)^2\right]^{1/2} \\
&\leq \frac{1}{4}(b-a)^{1/q}[f(b)-f(a)]\|g'\|_{[a,b],2}
\end{aligned}
$$

Proof. From inequality (2.1) for $g=\ell$, we get

$$
(2.12)\qquad |D_w(f,\ell)| \leq \frac{1}{2}(b-a)\int_a^b w(t)\left|f(t)-\int_a^b w(s)f(s)\,ds\right|dt.
$$

Also, from (2.1) for $\int_a^{\cdot}|g'(u)|^p\,du$ instead of g we get

$$
\begin{aligned}
(2.13)\qquad & |D_w(f,g)| \\
&\leq \frac{1}{2}\left\|\left(\int_a^{\cdot}|g'(u)|^p\,du\right)'\right\|_{[a,b],1}\int_a^b w(t)\left|f(t)-\int_a^b w(s)f(s)\,ds\right|dt \\
&= \frac{1}{2}\left\||g'|^p\right\|_{[a,b],1}\int_a^b w(t)\left|f(t)-\int_a^b w(s)f(s)\,ds\right|dt \\
&= \frac{1}{2}\|g'\|_{[a,b],p}^p\int_a^b w(t)\left|f(t)-\int_a^b w(s)f(s)\,ds\right|dt.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& [D_w(f,\ell)]^{1/q}\left[D_w\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p} \\
&\leq \left[\frac{1}{2}(b-a)\int_a^b w(t)\left|f(t)-\int_a^b w(s)f(s)\,ds\right|dt\right]^{1/q} \\
&\times\left[\frac{1}{2}\|g'\|_{[a,b],p}^p\int_a^b w(t)\left|f(t)-\int_a^b w(s)f(s)\,ds\right|dt\right]^{1/p} \\
&= \frac{1}{2}(b-a)^{1/q}\|g'\|_{[a,b],p}\int_a^b w(t)\left|f(t)-\int_a^b w(s)f(s)\,ds\right|dt,
\end{aligned}
$$

which proves the first part of (2.10).

The second part is obvious. □

3. The Case of Uniform Distribution

If we consider the uniform distribution $w_0(t) = 1/(b-a)$ on the interval $[a,b]$, then we get

$$D_{w_0}(h,k) := \frac{1}{b-a}\int_a^b h(t)\,k(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt \frac{1}{b-a}\int_a^b k(t)\,dt,$$

$$\begin{aligned}
& D_{w_0}\left(f, \int_a^{\cdot} |g'(u)|\,du\right) \\
&= \frac{1}{b-a}\int_a^b f(t)\left(\int_a^t |g'(u)|\,du\right)dt \\
&- \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b \left(\int_a^t |g'(u)|\,du\right)dt \\
&= \frac{1}{b-a}\int_a^b \left(f(t) - \frac{1}{b-a}\int_a^b f(s)\,ds\right)\left(\int_a^t |g'(u)|\,du\right)dt \\
&= \frac{1}{b-a}\int_a^b \left(\int_a^t |g'(u)|\,du\right) d\left(\int_a^t f(s)\,ds - \frac{t-a}{b-a}\int_a^b f(s)\,ds\right)
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
& \int_a^b \left(\int_a^t |g'(u)|\,du\right) d\left(\int_a^t f(s)\,ds - \frac{t-a}{b-a}\int_a^b f(s)\,ds\right) \\
&= \left(\int_a^t |g'(u)|\,du\right)\left(\int_a^t f(s)\,ds - \frac{t-a}{b-a}\int_a^b f(s)\,ds\right)\Bigg|_a^b \\
&- \int_a^b \left(\int_a^t f(s)\,ds - \frac{t-a}{b-a}\int_a^b f(s)\,ds\right)|g'(t)|\,dt
\end{aligned}$$

$$\begin{aligned}
&= -\int_a^b \left(\int_a^t f(s)\,ds - \frac{t-a}{b-a}\int_a^b f(s)\,ds\right)|g'(t)|\,dt \\
&= \int_a^b \left(\frac{t-a}{b-a}\int_a^b f(s)\,ds - \int_a^t f(s)\,ds\right)|g'(t)|\,dt \\
&= \frac{1}{b-a}\int_a^b \left[(t-a)\int_t^b f(s)\,ds - (b-t)\int_a^t f(s)\,ds\right]|g'(t)|\,dt.
\end{aligned}$$

Therefore

$$\begin{aligned}
& D_{w_0}\left(f, \int_a^{\cdot} |g'(u)|\,du\right) \\
&= \frac{1}{(b-a)^2}\int_a^b \left[(t-a)\int_t^b f(s)\,ds - (b-t)\int_a^t f(s)\,ds\right]|g'(t)|\,dt.
\end{aligned} \tag{3.1}$$

By making use of (2.1) we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.2}$$
$$\leq \frac{1}{(b-a)^2}\int_a^b\left[(t-a)\int_t^b f(s)\,ds-(b-t)\int_a^t f(s)\,ds\right]|g'(t)|\,dt$$
$$\leq \frac{1}{2(b-a)}\|g'\|_{[a,b],1}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt$$
$$\leq \frac{1}{2}\|g'\|_{[a,b],1}\left(\frac{1}{b-a}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|^2 dt\right)^{1/2}$$
$$\leq \frac{1}{4}\left[f(b)-f(a)\right]\|g'\|_{[a,b],1},$$

provided that $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and $g:[a,b]\to\mathbb{C}$ is absolutely continuous.

Further, observe that

$$D_{w_0}(f,\ell) \tag{3.3}$$
$$=\frac{1}{b-a}\left(\int_a^b tf(t)\,dt-\frac{a+b}{2}\int_a^b f(t)\,dt\right)=\frac{1}{b-a}\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt$$

and

$$D_{w_0}\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right) \tag{3.4}$$
$$=\frac{1}{(b-a)^2}\int_a^b\left[(t-a)\int_t^b f(s)\,ds-(b-t)\int_a^t f(s)\,ds\right]|g'(t)|^p\,dt.$$

Therefore, by (2.10) we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.5}$$
$$\leq \frac{1}{(b-a)^{1+1/p}}\left[\int_a^b\left(t-\frac{a+b}{2}\right)f(t)\,dt\right]^{1/q}$$
$$\times\left[\int_a^b\left[(t-a)\int_t^b f(s)\,ds-(b-t)\int_a^t f(s)\,ds\right]|g'(t)|^p\,dt\right]^{1/p}$$
$$\leq \frac{1}{2}(b-a)^{1/p}\|g'\|_{[a,b],p}\int_a^b\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|dt$$
$$\leq \frac{1}{2}(b-a)^{1/q}\|g'\|_{[a,b],p}\left[\frac{1}{b-a}\int_a^b f^2(t)\,dt-\left(\int_a^b\frac{1}{b-a}f(s)\,ds\right)^2\right]^{1/2}$$
$$\leq \frac{1}{4}(b-a)^{1/q}\left[f(b)-f(a)\right]\|g'\|_{[a,b],p},$$

for all $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, provided that $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing and $g:[a,b]\to\mathbb{C}$ is absolutely continuous.

4. The Case of Infinite Intervals

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R}=(-\infty,\infty)$. Namely, if $I=(-\infty,\infty)$ and $w(s)>0$ for $s\in\mathbb{R}$ with $\int_{-\infty}^{\infty}w(s)\,ds=1$, namely w is a probability density function on $(-\infty,\infty)$, then for $f,\ g$ *Lebesgue measurable* functions on $(-\infty,\infty)$, we can consider the functional the functional

$$D_{w,\mathbb{R}}(f,g):=\int_{-\infty}^{\infty}w(t)\,f(t)\,g(t)\,dt-\int_{-\infty}^{\infty}w(t)\,f(t)\,dt\int_{-\infty}^{\infty}w(t)\,g(t)\,dt.$$

Assume that $f:(-\infty,\infty)\to\mathbb{R}$ is monotonic nondecreasing with $f(-\infty):=\lim_{t\to-\infty}f(t)$ and $f(\infty):=\lim_{t\to\infty}f(t)$ exists and are finite while $g:(-\infty,\infty)\to\mathbb{C}$ is locally absolutely continuous and w a probability density function on $(-\infty,\infty)$, then from (2.1) by letting $a\to-\infty$, $b\to\infty$, we derive

$$\begin{aligned}
(4.1)\qquad &|D_{w,\mathbb{R}}(f,g)|\\
&\le D_{w,\mathbb{R}}\left(f,\int_{-\infty}^{\cdot}|g'(u)|\,du\right)\\
&\le\frac{1}{2}\|g'\|_{(-\infty,\infty),1}\int_{-\infty}^{\infty}w(t)\left|f(t)-\int_{-\infty}^{\infty}w(s)\,f(s)\,ds\right|dt\\
&\le\frac{1}{2}\|g'\|_{(-\infty,\infty),1}\left(\int_{-\infty}^{\infty}w(t)\left|f(t)-\int_{-\infty}^{\infty}w(s)\,f(s)\,ds\right|^{2}dt\right)^{1/2}\\
&\le\frac{1}{4}\left[f(\infty)-f(-\infty)\right]\|g'\|_{(-\infty,\infty),1},
\end{aligned}$$

where $\|g'\|_{(-\infty,\infty),1}=\int_{-\infty}^{\infty}|g'(u)|\,du$ is assumed to be finite.

We also have by (2.2),

$$\begin{aligned}
(4.2)\quad |D_{w,\mathbb{R}}(f,g)|&\le D_{w,\mathbb{R}}\left(f,\int_{-\infty}^{\cdot}|g'(u)|\,du\right)\\
&\le\frac{1}{2}\left[f(\infty)-f(-\infty)\right]\\
&\times\int_{-\infty}^{\infty}w(t)\left|\int_{-\infty}^{t}\left(\int_{-\infty}^{s}w(u)\,du\right)|g'(s)|\,ds-\int_{t}^{\infty}\left(\int_{s}^{\infty}w(u)\,du\right)|g'(s)|\,ds\right|dt\\
&\le\frac{1}{2}\left[f(\infty)-f(-\infty)\right]\\
&\times\left[\int_{-\infty}^{\infty}w(s)\left(\int_{-\infty}^{s}|g'(u)|\,du\right)^{2}ds-\left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{s}|g'(u)|\,du\right)w(s)\,ds\right)^{2}\right]^{\frac{1}{2}}\\
&\le\frac{1}{4}\left[f(\infty)-f(-\infty)\right]\|g'\|_{(-\infty,\infty),1}.
\end{aligned}$$

From (2.6) we then get

$$(4.3)\qquad |D_{w,\mathbb{R}}(f,g)|\le\left[D_{w,\mathbb{R}}(f,\ell)\right]^{1/q}\left[D_{w,\mathbb{R}}\left(f,\int_{-\infty}^{\cdot}|g'(u)|^{p}\,du\right)\right]^{1/p}$$

for all $p,\, q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

The probability density of the *normal distribution* on $(-\infty,\infty)$ is

$$w_{\mu,\sigma^2}(x):=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x\in\mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

The cumulative distribution function is

$$W_{\mu,\sigma^2}(x)=\frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right),$$

where the *error function* erf is defined by

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x\exp\left(-t^2\right)dt.$$

Consider the functional

$$\begin{aligned}&D_{N,\sigma,\mu}(f,g)\\&:=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)f(t)\,g(t)\,dt\\&-\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)f(t)\,dt\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)g(t)\,dt\end{aligned}$$

with the parameters μ and σ as above.

Then from (4.1) we get

$$\begin{aligned}&|D_{N,\sigma,\mu}(f,g)|\\&\le D_{N,\sigma,\mu}\left(f,\int_{-\infty}^{\cdot}|g'(u)|\,du\right)\\&\le\frac{1}{2\sqrt{2\pi}\sigma}\|g'\|_{(-\infty,\infty),1}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)\\&\times\left|f(t)-\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(s-\mu)^2}{2\sigma^2}\right)f(s)\,ds\right|dt\\&\le\frac{1}{2}\|g'\|_{(-\infty,\infty),1}\left\{\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)\right.\\&\left.\times\left|f(t)-\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left(-\frac{(s-\mu)^2}{2\sigma^2}\right)f(s)\,ds\right|^2dt\right\}^{1/2}\\&\le\frac{1}{4}\left[f(\infty)-f(-\infty)\right]\|g'\|_{(-\infty,\infty),1}.\end{aligned}\tag{4.4}$$

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