SEVERAL INTEGRAL INEQUALITIES FOR THE WEIGHTED ČEBYŠEV FUNCTIONAL WITH APPLICATIONS FOR NORMS AND SEMI-INNER PRODUCTS

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Assume that $w:[a,b]\to[0,\infty)$ is integrable with $\int_a^b w(t)\,dt=1$ and f, g are Lebesgue integrable on $[a,b]$. Consider the Čebyšev functional

$$D_w(f,g):=\int_a^b f(t)\,g(t)\,w(t)\,dt-\int_a^b f(t)\,w(t)\,dt\int_a^b g(t)\,w(t)\,dt.$$

In ths paper we show among other that, if f, g are absolutely continuous with $\|f'\|_{[a,b],p}:=\left(\int_a^b|f'(u)|^p\,du\right)^{1/p}<\infty$ and $\|g'\|_{[a,b],q}:=\left(\int_a^b|g'(u)|^q\,du\right)^{1/q}<\infty$ for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$|D_w(f,g)|\le\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}$$

$$\le\frac{1}{8}\times\begin{cases}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\|f'\|_{[a,b],p}\|g'\|_{[a,b],q}, & \text{if } \frac{1}{w}\in L_\infty[a,b],\\[2ex](b-a)\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty}, & \text{if } \frac{f'}{w^{1/p}},\ \frac{g'}{w^{1/q}}\in L_\infty[a,b].\end{cases}$$

Applications for norms and semi-inner products are also given.

1. Introduction

Consider a probability density function w on $[a,b]$, i.e., $w\ge0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt=1$, and the weighted Čebyšev functional for functions defined on a finite interval $[a,b]$,

$$D_w(h,k):=\int_a^b h(t)\,k(t)\,w(t)\,dt-\int_a^b h(t)\,w(t)\,dt\int_a^b k(t)\,w(t)\,dt.$$

In [7] Cerone and Dragomir proved among others the following *refinement of Grüss' inequality:*

$$|D_w(h,k)|\le\frac{1}{2}(\Delta-\delta)\int_a^b w(t)\left|h(t)-\int_a^b h(s)\,w(s)\,ds\right|dt \tag{1.1}$$

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$$\leq \frac{1}{2}(\Delta-\delta)\left[\int_a^b w(s)h^2(s)\,ds-\left(\int_a^b h(s)w(s)\,ds\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma),$$

provided that $h,\ k:[a,b]\to\mathbb{R}$ are measurable functions and so that $-\infty<\gamma\leq h\leq\Gamma<\infty$, $-\infty<\delta\leq k\leq\Delta<\infty$ a.e. on $[a,b]$, and $h,\ k,\ hk\in L_w[a,b]$.

For more recent upper bounds related to the Čebyšev functional see [1]-[9], [11]-[17] and [20]-[27].

In the recent paper [18] we obtained the following weighted inequalities:

Theorem 1. *Assume that $w:[a,b]\to(0,\infty)$ is a continuous probability density function on $[a,b]$, h is Lebesgue integrable and satisfies the condition $m\leq h(t)\leq M$ for $t\in[a,b]$ and $k:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ with $\frac{k'}{w}$ is essentially bounded, namely $\frac{k'}{w}\in L_\infty[a,b]$, then we have*

$$|D_w(h,k)|\leq\frac{1}{8}(M-m)\left\|\frac{k'}{w}\right\|_{[a,b],\infty}. \tag{1.2}$$

The constant $\frac{1}{8}$ is best possible.

The following bounds in terms of sup-norm of both functions may be provided, [18]:

Theorem 2. *Assume that $w:[a,b]\to(0,\infty)$ is a continuous probability density function on $[a,b]$. If $h,\ k:[a,b]\to\mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{h'}{w}$, $\frac{k'}{w}\in L_\infty[a,b]$, then we have*

$$|D_w(h,k)|\leq\frac{1}{12}\left\|\frac{h'}{w}\right\|_{[a,b],\infty}\left\|\frac{k'}{w}\right\|_{[a,b],\infty}. \tag{1.3}$$

The constant $\frac{1}{12}$ is best possible.

We also have the following bounds in terms of the Euclidian norm, [18]:

Theorem 3. *Assume that $w:[a,b]\to(0,\infty)$ is a continuous probability density function on $[a,b]$. If $h,\ k:[a,b]\to\mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{h'}{w^{1/2}}$, $\frac{k'}{w^{1/2}}\in L_2[a,b]$, then we have*

$$|D_w(h,k)|\leq\frac{1}{\pi^2}\left\|\frac{h'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{k'}{w^{1/2}}\right\|_{[a,b],2}. \tag{1.4}$$

The constant $\frac{1}{\pi^2}$ is best possible.

In ths paper we show among other that, if $f,\ g$ are absolutely continuous with $\|f'\|_{[a,b],p}:=\left(\int_a^b|f'(u)|^p\,du\right)^{1/p}<\infty$ and $\|g'\|_{[a,b],q}:=\left(\int_a^b|g'(u)|^q\,du\right)^{1/q}<\infty$

for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$
\begin{aligned}
|D_w(f,g)| &\le \left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}\\
&\le \frac{1}{8}\times\begin{cases}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\|f'\|_{[a,b],p}\|g'\|_{[a,b],q}\,, & \text{if } \frac{1}{w}\in L_\infty[a,b]\,,\\[2ex] (b-a)\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty}\,, & \text{if } \frac{f'}{w^{1/p}},\ \frac{g'}{w^{1/q}}\in L_\infty[a,b]\,.\end{cases}
\end{aligned}
$$

Applications for norms and semi-inner products are also given.

2. Main Results

The first main results is as follows:

Theorem 4. *Assume that $w:[a,b]\to[0,\infty)$ is integrable with $\int_a^b w(t)\,dt=1$ and f, g absolutely continuous with $\|f'\|_{[a,b],\infty}:=\operatorname{esssup}_{t\in[a,b]}|f'(t)|<\infty$, then*

$$
\begin{aligned}
|D_w(f,g)| &\le \|f'\|_{[a,b],\infty}\,D_w\left(\ell,\int_a^{\cdot}|g'(u)|\,du\right)\\
&\le \frac{1}{8}\|f'\|_{[a,b],\infty}\times\begin{cases}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\|g'\|_{[a,b],1}\,, & \text{if } \frac{1}{w}\in L_\infty[a,b]\,,\\[2ex] (b-a)\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\,, & \text{if } \frac{g'}{w}\in L_\infty[a,b]\,,\end{cases}
\end{aligned}
\tag{2.1}
$$

where $\|g'\|_{[a,b],1}:=\left(\int_a^b|g'(u)|\,du\right)<\infty$ and $\ell(t)=t,\ t\in[a,b]\,.$

We have

$$
\begin{aligned}
|D_w(f,g)| &\le \|f'\|_{[a,b],\infty}\,D_w\left(\ell,\int_a^{\cdot}|g'(u)|\,du\right)\\
&\le \frac{1}{12}\|f'\|_{[a,b],\infty}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\left\|\frac{g'}{w}\right\|_{[a,b],\infty},
\end{aligned}
\tag{2.2}
$$

provided that $\frac{1}{w},\ \frac{g'}{w}\in L_\infty[a,b]\,.$

Also we have the bounds in terms of the Euclidian norm

$$
\begin{aligned}
|D_w(f,g)| &\le \|f'\|_{[a,b],\infty}\,D_w\left(\ell,\int_a^{\cdot}|g'(u)|\,du\right)\\
&\le \frac{1}{\pi^2}\|f'\|_{[a,b],\infty}\left\|\frac{1}{w}\right\|_{[a,b],1}\left\|\frac{g'}{w^{1/2}}\right\|_{[a,b],2},
\end{aligned}
\tag{2.3}
$$

provided that $\frac{1}{w}\in L[a,b]$ and $\frac{g'}{w^{1/2}}\in L_2[a,b]\,.$

Proof. Observe that, by the use of integral's properties,

$$\begin{aligned}
&\int_a^b \int_a^b w(t) w(s) [f(t) - f(s)] [g(t) - g(s)] \, dtds \\
&= \int_a^b \int_a^b w(t) w(s) (f(t) g(t) - f(s) g(t) - f(t) g(s) + f(s) g(s)) \, dtds \\
&= \int_a^b w(s) \, ds \int_a^b f(t) g(t) \, dt - \int_a^b w(s) f(s) \, ds \int_a^b w(t) g(t) \, dt \\
&- \int_a^b w(t) f(t) \, dt \int_a^b w(s) g(s) \, ds + \int_a^b w(t) \, dt \int_a^b f(s) g(s) \, ds = 2D_w(f,g),
\end{aligned}$$

which give the weighted Korkine's identity for functions with complex values

$$D_w(f,g) = \frac{1}{2} \int_a^b \int_a^b w(t) w(s) [f(t) - f(s)] [g(t) - g(s)] \, dtds.$$

For Korkine's classical identity for real-valued functions, see [25, p. 242].

If we take the modulus and use the integral's properties, we get

$$\begin{aligned}
|D_w(f,g)| &\le \frac{1}{2} \int_a^b \int_a^b w(t) w(s) |[f(t) - f(s)] [g(t) - g(s)]| \, dtds \\
&= \frac{1}{2} \int_a^b \int_a^b w(t) w(s) |f(t) - f(s)| |g(t) - g(s)| \, dtds.
\end{aligned} \tag{2.4}$$

Observe that for $s,\, t \in [a,b]$

$$f(t) - f(s) = \int_s^t f'(u) \, du, \;\; g(t) - g(s) = \int_s^t g'(u) \, du,$$

which implies that

$$\begin{aligned}
|f(t) - f(s)| |g(t) - g(s)| &= \left| \int_s^t f'(u) \, du \right| \left| \int_s^t g'(u) \, du \right| \\
&\le \left| \int_s^t |f'(u)| \, du \right| \left| \int_s^t |g'(u)| \, du \right| \\
&\le \sup_{t\in(a,b)} |f'(u)| |t-s| \left| \int_s^t |g'(u)| \, du \right| \\
&= \sup_{t\in(a,b)} |f'(u)| (t-s) \int_s^t |g'(u)| \, du,
\end{aligned}$$

for all $s,\, t \in [a,b]$.

By (2.4) we get

$$|D_w(f,g)| \le \sup_{t\in(a,b)} |f'(u)| \frac{1}{2} \int_a^b \int_a^b w(t) w(s) (t-s) \left(\int_s^t |g'(u)| \, du \right) dtds. \tag{2.5}$$

Since

$$(t-s) \left(\int_s^t |g'(u)| \, du \right) = (t-s) \left(\int_a^t |g'(u)| \, du - \int_a^s |g'(u)| \, du \right),$$

hence by Korkine's identity for real valued functions $f(t) = \ell(t)$ and $\int_a^t |g'(u)|\,du$, we have

$$\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,(t-s)\left(\int_a^t |g'(u)|\,du - \int_a^s |g'(u)|\,du\right)dtds \tag{2.6}$$
$$= D_w\left(\ell, \int_a^{\cdot} |g'(u)|\,du\right).$$

By utilising (2.5) and (2.6), we deduce the first inequality in (2.1).

Now, if we apply the inequality (1.2) for $h = \int_a^{\cdot} |g'(u)|\,du$ and $k = \ell$, and since $0 \le \int_a^{\cdot} |g'(u)|\,du \le \int_a^b |g'(u)|\,du$, then

$$\left|D_w\left(\int_a^{\cdot} |g'(u)|\,du, \ell\right)\right| \le \frac{1}{8}\int_a^b |g'(u)|\,du \left\|\frac{1}{w}\right\|_{[a,b],\infty}, \tag{2.7}$$

which proves the first branch in (2.1).

If we apply the same inequality for $h = \ell$ and $k = \int_a^{\cdot} |g'(u)|\,du$ and since $a \le \ell \le b$, then

$$\left|D_w\left(\int_a^{\cdot} |g'(u)|\,du, \ell\right)\right| \le \frac{1}{8}(b-a)\left\|\frac{g'}{w}\right\|_{[a,b],\infty},$$

which proves the second branch in (2.1).

Now, if we apply the inequality (1.3) for $h = \ell$ and $k = \int_a^{\cdot} |g'(u)|\,du$, then we have

$$\left|D_w\left(\ell, \int_a^{\cdot} |g'(u)|\,du\right)\right| \le \frac{1}{12}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\left\|\frac{g'}{w}\right\|_{[a,b],\infty},$$

which proves (2.2).

If we apply the inequality (1.4) for $h = \ell$ and $k = \int_a^{\cdot} |g'(u)|\,du$, then we have

$$\begin{aligned}\left|D_w\left(\ell, \int_a^{\cdot} |g'(u)|\,du\right)\right| &\le \frac{1}{\pi^2}\left\|\frac{1}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{g'}{w^{1/2}}\right\|_{[a,b],2} \\ &= \frac{1}{\pi^2}\left\|\frac{1}{w}\right\|_{[a,b],1}\left\|\frac{g'}{w^{1/2}}\right\|_{[a,b],2},\end{aligned}$$

which proves (2.3). □

Theorem 5. *Assume that* $w : [a,b] \to [0,\infty)$ *is integrable with* $\int_a^b w(t)\,dt = 1$ *and* f, g *absolutely continuous with* $\|f'\|_{[a,b],p} := \left(\int_a^b |f'(u)|^p\,du\right)^{1/p} < \infty$ *and* $\|g'\|_{[a,b],q} := \left(\int_a^b |g'(u)|^q\,du\right)^{1/q} < \infty$ *for* p, $q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$|D_w(f,g)| \le \left[D_w\left(\ell, \int_a^{\cdot} |f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell, \int_a^{\cdot} |g'(u)|^q\,du\right)\right]^{1/q} \tag{2.8}$$
$$\le \frac{1}{8}\times\begin{cases}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\|f'\|_{[a,b],p}\|g'\|_{[a,b],q}, \\ \\ (b-a)\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty}\end{cases}$$

provided that $\frac{1}{w}$, $\frac{f'}{w^{1/p}}$, $\frac{g'}{w^{1/q}} \in L_\infty[a,b]$.

If $\frac{1}{w},\ \frac{f'}{w^{1/p}},\ \frac{g'}{w^{1/q}} \in L_{\infty}[a,b]$, *then*

$$|D_w(f,g)| \leq \left[D_w\left(\ell, \int_a^{\cdot} |f'(u)|^p du\right)\right]^{1/p} \left[D_w\left(\ell, \int_a^{\cdot} |g'(u)|^q du\right)\right]^{1/q} \leq \frac{1}{12}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty}. \tag{2.9}$$

If $\frac{1}{w} \in L[a,b]$, $\frac{f'}{w^{1/(2p)}} \in L_{2p}[a,b]$, $\frac{g'}{w^{1/(2q)}} \in L_{2q}$, *then*

$$|D_w(f,g)| \leq \left[D_w\left(\ell, \int_a^{\cdot} |f'(u)|^p du\right)\right]^{1/p} \left[D_w\left(\ell, \int_a^{\cdot} |g'(u)|^q du\right)\right]^{1/q} \leq \frac{1}{\pi^2}\left\|\frac{1}{w}\right\|_{[a,b],1}^{1/2}\left\|\frac{f'}{w^{1/(2p)}}\right\|_{[a,b],2p}\left\|\frac{g'}{w^{1/(2q)}}\right\|_{[a,b],2q} \tag{2.10}$$

for $p,\ q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Hölder's inequality for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\begin{aligned}
&|f(t)-f(s)||g(t)-g(s)| \\
&= \left|\int_s^t f'(u)\,du\right|\left|\int_s^t g'(u)\,du\right| \\
&\leq \left|\int_s^t |f'(u)|\,du\right|\left|\int_s^t |g'(u)|\,du\right| \\
&\leq |t-s|^{1/q}\left|\int_s^t |f'(u)|^p du\right|^{1/p} |t-s|^{1/p}\left|\int_s^t |g'(u)|^q du\right|^{1/q} \\
&= |t-s|\left|\int_s^t |f'(u)|^p du\right|^{1/p}\left|\int_s^t |g'(u)|^q du\right|^{1/q}
\end{aligned}$$

for all $t,\ s\in[a,b]$.

By the weighted Hölder's inequality for double integral, we also have

$$\begin{aligned}
&\int_a^b\int_a^b w(s)w(t)|f(t)-f(s)||g(t)-g(s)|\,dtds \\
&\leq \int_a^b\int_a^b w(s)w(t)|t-s|\left|\int_s^t |f'(u)|^p du\right|^{1/p}\left|\int_s^t |g'(u)|^q du\right|^{1/q} dtds \\
&\leq \left(\int_a^b\int_a^b w(s)w(t)|t-s|\left(\left|\int_s^t |f'(u)|^p du\right|^{1/p}\right)^p dtds\right)^{1/p} \\
&\times \left(\int_a^b\int_a^b w(s)w(t)|t-s|\left(\left|\int_s^t |g'(u)|^q du\right|^{1/q}\right)^q dtds\right)^{1/q} \\
&= \left(\int_a^b\int_a^b w(s)w(t)|t-s|\left|\int_s^t |f'(u)|^p du\right| dtds\right)^{1/p} \\
&\times \left(\int_a^b\int_a^b w(s)w(t)|t-s|\left|\int_s^t |g'(u)|^q du\right| dtds\right)^{1/q}.
\end{aligned} \tag{2.11}$$

Observe that

$$\begin{aligned}
&\int_a^b \int_a^b w(s) w(t) |t-s| \left| \int_s^t |f'(u)|^p du \right| dtds \\
&= \int_a^b \int_a^b w(s) w(t) (t-s) \left(\int_s^t |f'(u)|^p du \right) dtds \\
&= \int_a^b \int_a^b w(s) w(t) (t-s) \left(\int_a^t |f'(u)|^p du - \int_a^s |f'(u)|^p du \right) dtds \\
&= 2D_w \left(\ell, \int_a^{\cdot} |f'(u)|^p du \right)
\end{aligned}$$

and, similarly

$$\int_a^b \int_a^b w(s) w(t) |t-s| \left| \int_s^t |g'(u)|^q du \right| dtds = 2D_w \left(\ell, \int_a^{\cdot} |g'(u)|^q du \right).$$

Therefore, by (2.4)

$$\begin{aligned}
|D_w(f,g)| &\le \frac{1}{2} \int_a^b \int_a^b w(s) w(t) |f(t) - f(s)| |g(t) - g(s)| dtds \\
&\le \frac{1}{2} \left[2D_w \left(\ell, \int_a^{\cdot} |f'(u)|^p du \right) \right]^{1/p} \left[2D_w \left(\ell, \int_a^{\cdot} |g'(u)|^q du \right) \right]^{1/q} \\
&= \left[D_w \left(\ell, \int_a^{\cdot} |f'(u)|^p du \right) \right]^{1/p} \left[D_w \left(\ell, \int_a^{\cdot} |g'(u)|^q du \right) \right]^{1/q},
\end{aligned}$$

which proves the first inequality in (2.8).

Now, if we apply the inequality (1.2) for $h = \int_a^{\cdot} |f'(u)|^p du$ and $k = \ell$, and since $0 \le \int_a^{\cdot} |f'(u)|^p du \le \int_a^b |f'(u)|^p du$, then

$$\left| D_w \left(\int_a^{\cdot} |f'(u)|^p du, \ell \right) \right| \le \frac{1}{8} \int_a^b |f'(u)|^p du \left\| \frac{1}{w} \right\|_{[a,b],\infty}. \tag{2.12}$$

Similarly,

$$D_w \left(\int_a^{\cdot} |g'(u)|^q du, \ell \right) \le \frac{1}{8} \int_a^b |g'(u)|^q du \left\| \frac{1}{w} \right\|_{[a,b],\infty}. \tag{2.13}$$

By utilising (2.12) and (2.13) we get

$$\begin{aligned}
&\left[D_w \left(\ell, \int_a^{\cdot} |f'(u)|^p du \right) \right]^{1/p} \left[D_w \left(\ell, \int_a^{\cdot} |g'(u)|^q du \right) \right]^{1/q} \\
&\le \left(\frac{1}{8} \int_a^b |f'(u)|^p du \left\| \frac{1}{w} \right\|_{[a,b],\infty} \right)^{1/p} \left(\frac{1}{8} \int_a^b |g'(u)|^q du \left\| \frac{1}{w} \right\|_{[a,b],\infty} \right)^{1/q} \\
&= \frac{1}{8} \left\| \frac{1}{w} \right\|_{[a,b],\infty} \left(\int_a^b |f'(u)|^p du \right)^{1/p} \left(\int_a^b |g'(u)|^q du \right)^{1/q},
\end{aligned}$$

which proves the first branch in (2.8).

If we apply the same inequality for $h=\ell$ and $k=\int_a^{\cdot}|f'(u)|^p\,du$ and since $a\le\ell\le b$, then

$$\left|D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right|\le\frac{1}{8}(b-a)\left\|\frac{|f'|^p}{w}\right\|_{[a,b],\infty}$$

and, similarly

$$\left|D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right|\le\frac{1}{8}(b-a)\left\|\frac{|g'|^q}{w}\right\|_{[a,b],\infty}.$$

Therefore

$$\begin{aligned}&\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}\\&\le\left(\frac{1}{8}(b-a)\left\|\frac{|f'|^p}{w}\right\|_{[a,b],\infty}\right)^{1/p}\left(\frac{1}{8}(b-a)\left\|\frac{|g'|^q}{w}\right\|_{[a,b],\infty}\right)^{1/q}\\&=\frac{1}{8}(b-a)\left\|\frac{|f'|^p}{w}\right\|_{[a,b],\infty}^{1/p}\left\|\frac{|g'|^q}{w}\right\|_{[a,b],\infty}^{1/q}\\&=\frac{1}{8}(b-a)\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty},\end{aligned}$$

which proves the second branch in (2.8).

Now, if we apply the inequality (1.3) for $h=\ell$ and $k=\int_a^{\cdot}|f'(u)|^p\,du$, then we have

$$\begin{aligned}\left|D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right|&\le\frac{1}{12}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\left\|\frac{|f'|^p}{w}\right\|_{[a,b],\infty}\\&=\frac{1}{12}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}^p\end{aligned}\tag{2.14}$$

and, similarly,

$$\left|D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right|\le\frac{1}{12}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty}^q.\tag{2.15}$$

Therefore

$$\begin{aligned}&\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}\\&\le\left(\frac{1}{12}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}^p\right)^{1/p}\left(\frac{1}{12}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty}^q\right)^{1/q}\\&=\frac{1}{12}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty},\end{aligned}$$

which proves (2.9).

If we use inequality (1.4) for $h=\ell$ and $k=\int_a^{\cdot}|f'(u)|^p du$, then we have

$$\left|D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p du\right)\right| \le \frac{1}{\pi^2}\left\|\frac{1}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{|f'|^p}{w^{1/2}}\right\|_{[a,b],2} = \frac{1}{\pi^2}\left\|\frac{1}{w}\right\|_{[a,b],1}^{1/2}\left\|\frac{f'}{w^{1/(2p)}}\right\|_{[a,b],2p}^{p}$$

and, similarly,

$$\left|D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q du\right)\right| \le \frac{1}{\pi^2}\left\|\frac{1}{w}\right\|_{[a,b],1}^{1/2}\left\|\frac{g'}{w^{1/(2q)}}\right\|_{[a,b],2q}^{q}.$$

Therefore

$$\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q du\right)\right]^{1/q}$$
$$\le \left(\frac{1}{\pi^2}\left\|\frac{1}{w}\right\|_{[a,b],1}^{1/2}\left\|\frac{f'}{w^{1/(2p)}}\right\|_{[a,b],2p}^{p}\right)^{1/p}$$
$$\times\left(\frac{1}{\pi^2}\left\|\frac{1}{w}\right\|_{[a,b],1}^{1/2}\left\|\frac{g'}{w^{1/(2q)}}\right\|_{[a,b],2q}^{q}\right)^{1/q}$$
$$=\frac{1}{\pi^2}\left\|\frac{1}{w}\right\|_{[a,b],1}^{1/2}\left\|\frac{f'}{w^{1/(2p)}}\right\|_{[a,b],2p}\left\|\frac{g'}{w^{1/(2q)}}\right\|_{[a,b],2q},$$

which proves (2.10). □

3. The Case of Uniform Distribution

If we consider the uniform distribution $w_0(t)=1/(b-a)$ on the interval $[a,b]$, then we get

$$D_{w_0}(h,k) := \frac{1}{b-a}\int_a^b h(t)k(t)\,dt - \frac{1}{b-a}\int_a^b h(t)\,dt\frac{1}{b-a}\int_a^b k(t)\,dt,$$

$$D_{w_0}\left(\ell,\int_a^{\cdot}|g'(u)|\,du\right)$$
$$=\frac{1}{b-a}\int_a^b t\left(\int_a^t |g'(u)|\,du\right)dt - \frac{a+b}{2}\frac{1}{b-a}\int_a^b\left(\int_a^t|g'(u)|\,du\right)dt$$

Integrating by parts, we have

$$\begin{aligned}
&\frac{1}{2}\int_a^b (b-t)(t-a)\left|g'(t)\right| dt \\
&= \frac{1}{2}\int_a^b (b-t)(t-a)\, d\left(\int_a^t \left|g'(u)\right| du\right) \\
&= \frac{1}{2}\left[(b-t)(t-a)\int_a^t \left|g'(u)\right| du\Big|_a^b + \int_a^b (2t-a-b)\left(\int_a^t \left|g'(u)\right| du\right) dt\right] \\
&= \int_a^b \left(t-\frac{a+b}{2}\right)\left(\int_a^t \left|g'(u)\right| du\right) dt \\
&= \int_a^b t\left(\int_a^t \left|g'(u)\right| du\right) dt - \frac{a+b}{2}\int_a^b \left(\int_a^t \left|g'(u)\right| du\right) dt.
\end{aligned}$$

Therefore,

$$D_{w_0}\left(\ell, \int_a^{\cdot} \left|g'(u)\right| du\right) = \frac{1}{2(b-a)}\int_a^b (b-t)(t-a)\left|g'(t)\right| dt.$$

Also

$$\begin{aligned}
\int_a^b w_0(t)\left|t-\int_a^b s w_0(s)\, ds\right| dt &= \frac{1}{b-a}\int_a^b \left|t-\frac{1}{b-a}\int_a^b s ds\right| dt \\
&= \frac{1}{b-a}\int_a^b \left|t-\frac{a+b}{2}\right| dt = \frac{1}{4}(b-a),
\end{aligned}$$

therefore, by the first two inequalities in (2.1) we derive the following inequality of interest for f and g absolutely continuous on $[a,b]$:

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\,\frac{1}{b-a}\int_a^b g(t)\,dt\right| \\
&\leq \frac{1}{2(b-a)}\left\|f'\right\|_{[a,b],\infty}\int_a^b (b-t)(t-a)\left|g'(t)\right| dt \\
&\leq \frac{1}{8}(b-a)\left\|f'\right\|_{[a,b],\infty} \times \begin{cases} \left\|g'\right\|_{[a,b],1}, \\ \\ (b-a)\left\|g'\right\|_{[a,b],\infty}, \ g' \in L_\infty[a,b], \end{cases}
\end{aligned} \tag{3.1}$$

and $f' \in L_\infty[a,b]$.

From (2.2) we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt\,\frac{1}{b-a}\int_a^b g(t)\,dt\right| \\
&\leq \frac{1}{2(b-a)}\left\|f'\right\|_{[a,b],\infty}\int_a^b (b-t)(t-a)\left|g'(t)\right| dt \\
&\leq \frac{1}{12}(b-a)^2\left\|f'\right\|_{[a,b],\infty}\left\|g'\right\|_{[a,b],\infty}.
\end{aligned} \tag{3.2}$$

This inequality is better than the second branch of (3.1).

From (2.3) we derive

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.3}$$
$$\leq \frac{1}{2(b-a)}\left\|f'\right\|_{[a,b],\infty}\int_a^b (b-t)(t-a)\left|g'(t)\right|dt$$
$$\leq \frac{1}{\pi^2}(b-a)^{3/2}\left\|f'\right\|_{[a,b],\infty}\left\|g'\right\|_{[a,b],2}.$$

Observe also that

$$D_{w_0}\left(\ell,\int_a^{\cdot}\left|f'(u)\right|^p du\right)=\frac{1}{2(b-a)}\int_a^b (b-t)(t-a)\left|f'(u)\right|^p dt$$

and

$$D_{w_0}\left(\ell,\int_a^{\cdot}\left|g'(u)\right|^q du\right)=\frac{1}{2(b-a)}\int_a^b (b-t)(t-a)\left|g'(u)\right|^q dt.$$

By utilising the first two inequalities in (2.8) we also get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.4}$$
$$\leq \frac{1}{2(b-a)}\left[\int_a^b (b-t)(t-a)\left|f'(u)\right|^p dt\right]^{1/p}$$
$$\times\left[\int_a^b (b-t)(t-a)\left|g'(u)\right|^q\right]^{1/q}$$
$$\leq \frac{1}{8}(b-a)\times\begin{cases}\left\|f'\right\|_{[a,b],p}\left\|g'\right\|_{[a,b],q}, \\ \\ (b-a)\left\|f'\right\|_{[a,b],\infty}\left\|g'\right\|_{[a,b],\infty}\end{cases}$$

provided that $f'\in L_p[a,b]$ and $g'\in L_q[a,b]$ for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, or f', $g'\in L_\infty[a,b]$.

From (2.9) we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.5}$$
$$\leq \frac{1}{2(b-a)}\left[\int_a^b (b-t)(t-a)\left|f'(u)\right|^p dt\right]^{1/p}$$
$$\times\left[\int_a^b (b-t)(t-a)\left|g'(u)\right|^q\right]^{1/q}$$
$$\leq \frac{1}{12}(b-a)^2\left\|f'\right\|_{[a,b],\infty}\left\|g'\right\|_{[a,b],\infty},$$

provided $f',\ g'\in L_\infty[a,b]$.

From (2.10) we also get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.6}$$
$$\leq \frac{1}{2(b-a)}\left[\int_a^b (b-t)(t-a)\left|f'(u)\right|^p dt\right]^{1/p}$$
$$\times\left[\int_a^b (b-t)(t-a)\left|g'(u)\right|^q\right]^{1/q}$$
$$\leq \frac{1}{\pi^2}(b-a)^{3/2}\left\|f'\right\|_{[a,b],2p}\left\|g'\right\|_{[a,b],2q}$$

provided that $f'\in L_{2p}[a,b]$ and $g'\in L_{2q}[a,b]$ for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

4. Applications for Norms and Semi-Inner Products

Let X be a real linear space, $x,y\in X$, $x\neq y$ and let $[x,y]:=\{(1-\lambda)x+\lambda y,\ \lambda\in[0,1]\}$ be the *segment* generated by x and y. We consider the function $f:[x,y]\to\mathbb{R}$ and the attached function $g(x,y):[0,1]\to\mathbb{R}$, $g(x,y)(t):=f[(1-t)x+ty]$, $t\in[0,1]$.

It is well known that f is convex on $[x,y]$ iff $g(x,y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_\pm(x,y)(s)=\triangledown_\pm f[(1-s)x+sy](y-x)$, $s\in[0,1)$,
(ii) $g'_+(x,y)(0)=\triangledown_+ f(x)(y-x)$,
(iii) $g'_-(x,y)(1)=\triangledown_- f(y)(y-x)$,

where $\triangledown_\pm f(x)(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\triangledown_+ f(x)(y) \;:=\; \lim_{h\to 0+}\frac{f(x+hy)-f(x)}{h},$$
$$\triangledown_- f(x)(y) \;:=\; \lim_{k\to 0-}\frac{f(x+ky)-f(x)}{k},\quad x,\ y\in X.$$

We remark also that

$$\triangledown_+ f[(1-s)x+sy](y-x)=\triangledown_- f[(1-s)x+sy](y-x)$$

for almost every $s\in[0,1]$, being the lateral derivatives of a convex function. In integrals we can then write $\triangledown$ instead of $\triangledown_+$ or $\triangledown_-$.

Now, assume that $(X,\|\cdot\|)$ is a normed linear space. The function $f_0(s)=\frac{1}{2}\|x\|^2$, $x\in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s:=\triangledown_+ f_0(y)(x)=\lim_{t\to 0+}\frac{\|y+tx\|^2-\|y\|^2}{2t}$;
(v) $\langle x,y\rangle_i:=\triangledown_- f_0(y)(x)=\lim_{s\to 0-}\frac{\|y+sx\|^2-\|y\|^2}{2s}$;

for any x, $y\in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [19]), assuming that p, $q\in\{s,i\}$ and $p\neq q$:

(a) $\langle x,x\rangle_p=\|x\|^2$ for all $x\in X$;
(aa) $\langle \alpha x,\beta y\rangle_p=\alpha\beta\langle x,y\rangle_p$ if α, $\beta\geq 0$ and x, $y\in X$;

(aaa) $\left|\langle x,y\rangle_p\right| \leq \|x\| \|y\|$ for all $x,\ y\in X$;

(av) $\langle \alpha x+y,x\rangle_p=\alpha\langle x,x\rangle_p+\langle y,x\rangle_p$ if $x,\ y\in X$ and $\alpha\in\mathbb{R}$;

(v) $\langle -x,y\rangle_p=-\langle x,y\rangle_q$ for all $x,\ y\in X$;

(va) $\langle x+y,z\rangle_p\leq \|x\|\|z\|+\langle y,z\rangle_p$ for all $x,\ y,\ z\in X$;

(vaa) The mapping $\langle\cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p=s$ (or $p=i$);

(vaaa) The normed linear space $(X,\|\cdot\|)$ is smooth at the point $x_0\in X\setminus\{0\}$ if and only if $\langle y,x_0\rangle_s=\langle y,x_0\rangle_i$ for all $y\in X$; in general $\langle y,x\rangle_i\leq\langle y,x\rangle_s$ for all x, $y\in X$;

(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle\cdot,\cdot\rangle$, then $\langle y,x\rangle_i=\langle y,x\rangle=\langle y,x\rangle_s$ for all $x,\ y\in X$.

For $m\geq 1$ the function $f_m(x)=\|x\|^m$ is convex on X. Therefore

$$\nabla_{+(-)}f_m(y)(x)=p\|y\|^{m-2}\langle x,y\rangle_{s(i)} \tag{4.1}$$

which exists for all $x,\ y\in X$ whenever $m\geq 2$. If $1\leq m<2$ the equality (4.1) holds for all $x\in X$ and nonzero $y\in X$.

Observe also that

$$\begin{aligned}\nabla_{\pm}f_m&[(1-s)x+sy](y-x)\\&=m\|(1-s)x+sy\|^{m-2}\langle y-x,(1-s)x+sy\rangle_{s(i)}\end{aligned} \tag{4.2}$$

which exists for all $x,\ y\in X$ whenever $m\geq 2$. If $1\leq m<2$ the equality (4.2) holds for all $x,\ y$ such that $(1-s)x+sy\neq 0$ for all $s\in[0,1]$.

Now, assume that $f,\ g:C\to\mathbb{R}$ are convex on the convex subset C in the linear space X. Assume also that $w:[0,1]\to[0,\infty)$ is integrable and $\int_0^1 w(t)\,dt=1$. For distinct vectors $x,\ y$ we consider the functional

$$\begin{aligned}D_{w,x,y}(f,g):=&\int_0^1 f((1-t)x+ty)\,g((1-t)x+ty)\,w(t)\,dt\\&-\int_0^1 f((1-t)x+ty)\,w(t)\,dt\int_0^1 g((1-t)x+ty)\,w(t)\,dt.\end{aligned}$$

From the inequality (2.8) we then have

$$\begin{aligned}&|D_{w,x,y}(f,g)|\\&\leq\frac{1}{8}\times\begin{cases}\sup_{t\in[0,1]}\left(\frac{1}{w(t)}\right)\left(\int_0^1|\nabla f[(1-s)x+sy](y-x)|^p\,ds\right)^{1/p}\\ \times\left(\int_0^1|\nabla g[(1-s)x+sy](y-x)|^q\,ds\right)^{1/q},\\ \\ \sup_{t\in[0,1]}\left|\frac{\nabla f[(1-t)x+ty](y-x)}{w^{1/p}(t)}\right|\sup_{t\in[0,1]}\left|\frac{\nabla g[(1-t)x+ty](y-x)}{w^{1/q}(t)}\right|,\end{cases}\end{aligned} \tag{4.3}$$

where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

From the inequality (2.2) we get

$$
\begin{aligned}
(4.4)\qquad & \left|D_{w,x,y}(f,g)\right| \\
& \leq \frac{1}{12}\sup_{t\in[0,1]}\left(\frac{1}{w(t)}\right) \\
& \times \sup_{s\in[0,1]}\left|\bigtriangledown f[(1-s)x+sy](y-x)\right| \sup_{s\in[0,1]}\left|\frac{\bigtriangledown g[(1-s)x+sy](y-x)}{w(t)}\right|,
\end{aligned}
$$

while from (2.3),

$$
\begin{aligned}
(4.5)\qquad \left|D_{w,x,y}(f,g)\right| &\leq \frac{1}{\pi^2}\int_0^1\frac{dt}{w(t)}\sup_{s\in[0,1]}\left|\bigtriangledown f[(1-s)x+sy](y-x)\right| \\
&\quad\times\left[\int_0^1\frac{\left|\bigtriangledown g[(1-s)x+sy](y-x)\right|^2 ds}{w(s)}\right]^{1/2}.
\end{aligned}
$$

Consider now the convex functions $f_m(x)=\|x\|^m$, $m\geq 1$ and $g_n(x)=\|x\|^n$, $n\geq 1$, then for $w:[0,1]\to[0,\infty)$ is integrable and $\int_0^1 w(t)\,dt=1$, we may introduce the functional

$$
\begin{aligned}
D_{w,\|\cdot\|;m,n}(x,y) &:= \int_0^1\|(1-t)x+ty\|^{m+n}w(t)\,dt \\
&\quad-\int_0^1\|(1-t)x+ty\|^m w(t)\,dt\int_0^1\|(1-t)x+ty\|^n w(t)\,dt.
\end{aligned}
$$

From the first part of (4.3) we get

$$
\begin{aligned}
(4.6)\qquad & \left|D_{w,\|\cdot\|;m,n}(x,y)\right| \\
& \leq \frac{1}{8}m^{1/p}n^{1/q}\sup_{t\in[0,1]}\left(\frac{1}{w(t)}\right) \\
& \times\left(\int_0^1\left|\|(1-s)x+sy\|^{m-2}\langle y-x,(1-s)x+sy\rangle_{s(i)}\right|^p ds\right)^{1/p} \\
& \times\left(\int_0^1\left|\|(1-s)x+sy\|^{n-2}\langle y-x,(1-s)x+sy\rangle_{s(i)}\right|^q ds\right)^{1/q} \\
& \leq \frac{1}{8}m^{1/p}n^{1/q}\|y-x\|^2\sup_{t\in[0,1]}\left(\frac{1}{w(t)}\right) \\
& \times\left(\int_0^1\|(1-s)x+sy\|^{p(m-1)}ds\right)^{1/p}\left(\int_0^1\|(1-s)x+sy\|^{q(n-1)}ds\right)^{1/q},
\end{aligned}
$$

where for the last inequality we used Schwarz inequality *"(aaa)"*.

If we use Hermite-Hadamard inequality for convex functions, we also have for $p(m-1)$, $q(n-1)\geq 1$ that

$$
\int_0^1\|(1-s)x+sy\|^{p(m-1)}ds\leq\frac{\|x\|^{p(m-1)}+\|y\|^{p(m-1)}}{2}
$$

and

$$
\int_0^1\|(1-s)x+sy\|^{q(n-1)}ds\leq\frac{\|x\|^{q(n-1)}+\|y\|^{q(n-1)}}{2},
$$

which give the following simpler upper bound

$$\begin{aligned}
&\left|D_{w,\|\cdot\|;m,n}(x,y)\right| \\
&\leq \frac{1}{8}m^{1/p}n^{1/q}\|y-x\|^2 \sup_{t\in[0,1]}\left(\frac{1}{w(t)}\right) \\
&\times\left(\frac{\|x\|^{p(m-1)}+\|y\|^{p(m-1)}}{2}\right)^{1/p}\left(\frac{\|x\|^{q(n-1)}+\|y\|^{q(n-1)}}{2}\right)^{1/q}
\end{aligned}\tag{4.7}$$

for $p(m-1),\ q(n-1)\geq 1.$

If we use (4.4), then we get

$$\left|D_{w,\|\cdot\|;m,n}(x,y)\right| \leq \frac{1}{12}mn\sup_{t\in[0,1]}\left(\frac{1}{w(t)}\right)\tag{4.8}$$

$$\begin{aligned}
&\times\sup_{s\in[0,1]}\left|\|(1-s)x+sy\|^{m-2}\langle y-x,(1-s)x+sy\rangle_{s(i)}\right| \\
&\times\sup_{s\in[0,1]}\left|\frac{\|(1-s)x+sy\|^{n-2}\langle y-x,(1-s)x+sy\rangle_{s(i)}}{w(t)}\right| \\
&\leq\frac{1}{12}mn\|y-x\|^2\sup_{t\in[0,1]}\left(\frac{1}{w(t)}\right) \\
&\times\sup_{s\in[0,1]}\left|\|(1-s)x+sy\|^{m-1}\right|\sup_{s\in[0,1]}\left|\frac{\|(1-s)x+sy\|^{n-1}}{w(t)}\right|.
\end{aligned}$$

Since $\|(1-s)x+sy\| \leq \max\{\|x\|,\|y\|\}$ for all $s\in[0,1]$, then we can state the following simpler inequality for $m,\ n\geq 1$

$$\begin{aligned}
&\left|D_{w,\|\cdot\|;m,n}(x,y)\right| \\
&\leq\frac{1}{12}mn\|y-x\|^2\sup_{t\in[0,1]}\left(\frac{1}{w^2(t)}\right)[\max\{\|x\|,\|y\|\}]^{m+n-2}.
\end{aligned}\tag{4.9}$$

Similar inequalities may be stated by utilising the inequalities obtained above, however the details are not presented here.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au
URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa