ON INTEGRAL INEQUALITIES FOR THE WEIGHTED ČEBYŠEV FUNCTIONAL OF A MONOTONIC FUNCTION WITH APPLICATIONS FOR NORMS AND SEMI-INNER PRODUCTS

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Abstract. Assume that $w : [a,b] \to [0,\infty)$ is integrable with $\int_a^b w(t)\,dt = 1$ and f, g are Lebesgue integrable on $[a,b]$. Consider the Čebyšev functional

$$D_w(f,g) := \int_a^b f(t)\,g(t)\,w(t)\,dt - \int_a^b f(t)\,w(t)\,dt \int_a^b g(t)\,w(t)\,dt.$$

In this paper we show among other that, if $f : [a,b] \to \mathbb{R}$ is monotonic nondecreasing, $g : [a,b] \to \mathbb{C}$ is absolutely continuous and w a probability density function on $[a,b]$, then

$$\begin{aligned} |D_w(f,g)| &\le D_w\left(f, \int_a^{\cdot} |g'(u)|\,du\right) \\ &\le \begin{cases} \frac{1}{8}\|g'\|_{[a,b],1}\left\|\frac{f'}{w}\right\|_{[a,b],\infty} & \text{if } f \text{ is absolutely continuous} \\ \text{and } \frac{f'}{w} \in L_\infty[a,b], & \\ \frac{1}{8}[f(b)-f(a)]\left\|\frac{g'}{w}\right\|_{[a,b],\infty} & \text{if } \frac{g'}{w} \in L_\infty[a,b], \end{cases} \end{aligned}$$

where $\|g'\|_{[a,b],1} = \int_a^b |g'(u)|\,du$. Applications for norms and semi-inner products are also given.

1. Introduction

Consider a probability density function w on $[a,b]$, i.e., $w \ge 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt = 1$, and the weighted Čebyšev functional for functions defined on a finite interval $[a,b]$,

$$D_w(h,k) := \int_a^b h(t)\,k(t)\,w(t)\,dt - \int_a^b h(t)\,w(t)\,dt \int_a^b k(t)\,w(t)\,dt.$$

In [7] Cerone and Dragomir proved among others the following *refinement of Grüss' inequality:*

$$|D_w(h,k)| \le \frac{1}{2}(\Delta - \delta)\int_a^b w(t)\left|h(t) - \int_a^b h(s)\,w(s)\,ds\right| dt \tag{1.1}$$

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$$\leq \frac{1}{2}(\Delta-\delta)\left[\int_a^b w(s)h^2(s)\,ds-\left(\int_a^b h(s)w(s)\,ds\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma),$$

provided that $h,\ k:[a,b]\to\mathbb{R}$ are measurable functions and so that $-\infty<\gamma\leq h\leq\Gamma<\infty$, $-\infty<\delta\leq k\leq\Delta<\infty$ a.e. on $[a,b]$, and $h,\ k,\ hk\in L_w[a,b]$.

For more recent upper bounds related to the Čebyšev functional see [1]-[9], [11]-[19] and [22]-[29].

In the recent paper [20] we obtained the following weighted inequalities:

Theorem 1. *Assume that $w:[a,b]\to(0,\infty)$ is a continuous probability density function on $[a,b]$, h is Lebesgue integrable and satisfies the condition $m\leq h(t)\leq M$ for $t\in[a,b]$ and $k:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ with $\frac{k'}{w}$ is essentially bounded, namely $\frac{k'}{w}\in L_\infty[a,b]$, then we have*

$$|D_w(h,k)|\leq\frac{1}{8}(M-m)\left\|\frac{k'}{w}\right\|_{[a,b],\infty}. \tag{1.2}$$

The constant $\frac{1}{8}$ is best possible.

The following bounds in terms of sup-norm of both functions may be provided, [20]:

Theorem 2. *Assume that $w:[a,b]\to(0,\infty)$ is a continuous probability density function on $[a,b]$. If $h,\ k:[a,b]\to\mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{h'}{w}$, $\frac{k'}{w}\in L_\infty[a,b]$, then we have*

$$|D_w(h,k)|\leq\frac{1}{12}\left\|\frac{h'}{w}\right\|_{[a,b],\infty}\left\|\frac{k'}{w}\right\|_{[a,b],\infty}. \tag{1.3}$$

The constant $\frac{1}{12}$ is best possible.

We also have the following bounds in terms of the Euclidian norm, [20]:

Theorem 3. *Assume that $w:[a,b]\to(0,\infty)$ is a continuous probability density function on $[a,b]$. If $h,\ k:[a,b]\to\mathbb{R}$ are absolutely continuous on $[a,b]$ and $\frac{h'}{w^{1/2}}$, $\frac{k'}{w^{1/2}}\in L_2[a,b]$, then we have*

$$|D_w(h,k)|\leq\frac{1}{\pi^2}\left\|\frac{h'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{k'}{w^{1/2}}\right\|_{[a,b],2}. \tag{1.4}$$

The constant $\frac{1}{\pi^2}$ is best possible.

Motivated by the above results, we establish in this paper some other upper bounds for the Čebyšev functional in the case if one of the function is monotonic nondecreasing obtaining among others, refinements of the inequalities (1.2)-(1.4). Some applications for norms and semi-inner products are also given.

2. Main Results

We start with the following result:

Theorem 4. *Assume that $f:[a,b]\to\mathbb{R}$ is monotonic nondecreasing, $g:[a,b]\to\mathbb{C}$ is absolutely continuous and w a probability density function on $[a,b]$, then*

$$
\begin{aligned}
(2.1)\quad |D_w(f,g)| &\leq D_w\left(f,\int_a^{\cdot}|g'(u)|\,du\right)\\
&\leq \begin{cases}\frac{1}{8}\|g'\|_{[a,b],1}\left\|\frac{f'}{w}\right\|_{[a,b],\infty} & \text{if } f \text{ is absolutely continuous}\\ \text{and } \frac{f'}{w}\in L_\infty[a,b], & \\ \\ \frac{1}{8}[f(b)-f(a)]\left\|\frac{g'}{w}\right\|_{[a,b],\infty} & \text{if } \frac{g'}{w}\in L_\infty[a,b],\end{cases}
\end{aligned}
$$

where $\|g'\|_{[a,b],1}=\int_a^b|g'(u)|\,du.$

We also have,

$$
|D_w(f,g)|\leq D_w\left(f,\int_a^{\cdot}|g'(u)|\,du\right)\leq\frac{1}{12}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left\|\frac{g'}{w}\right\|_{[a,b],\infty},\tag{2.2}
$$

provided that f is absolutely continuous and $\frac{f'}{w},\ \frac{g'}{w}\in L_\infty[a,b]$.

Moreover, if f is absolutely continuous and $\frac{f'}{w^{1/2}},\ \frac{g'}{w^{1/2}}\in L_2[a,b]$, *then*

$$
|D_w(f,g)|\leq D_w\left(f,\int_a^{\cdot}|g'(u)|\,du\right)\leq\frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{g'}{w^{1/2}}\right\|_{[a,b],2}.\tag{2.3}
$$

Proof. Observe that for $f,\ g:[a,b]\to\mathbb{C}$ we have weighted Korkine's identity

$$
D_w(f,g)=\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,[f(t)-f(s)]\,[g(t)-g(s)]\,dtds.
$$

For Korkine's classical identity for real-valued functions, see [27, p. 242].

If we take the modulus and use the integral's properties, we get

$$
\begin{aligned}
|D_w(f,g)| &\leq\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,|[f(t)-f(s)]\,[g(t)-g(s)]|\,dtds\\
&=\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,|f(t)-f(s)|\,|g(t)-g(s)|\,dtds.
\end{aligned}\tag{2.4}
$$

Since $g:[a,b]\to\mathbb{C}$ is absolutely continuous, then

$$
|g(t)-g(s)|=\left|\int_s^t g'(u)\,du\right|\leq\left|\int_s^t|g'(u)|\,du\right|
$$

for all $t,\ s\in[a,b]$.

Therefore

$$
\begin{aligned}
&\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,|f(t)-f(s)|\,|g(t)-g(s)|\,dtds\\
&\leq\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,|f(t)-f(s)|\left|\int_s^t|g'(u)|\,du\right|dtds\\
&=\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left|(f(t)-f(s))\int_s^t|g'(u)|\,du\right|dtds.
\end{aligned}\tag{2.5}
$$

Since f is monotone nondecreasing, then

$$
\left|(f(t)-f(s))\int_s^t|g'(u)|\,du\right|=(f(t)-f(s))\int_s^t|g'(u)|\,du\geq 0
$$

for all $t,\ s\in[a,b]$, which implies that

$$\begin{aligned}
&\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\left|(f(t)-f(s))\int_s^t |g'(u)|\,du\right| dtds\\
&=\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,(f(t)-f(s))\int_s^t |g'(u)|\,dudtds\\
&=\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,(f(t)-f(s))\left(\int_a^t |g'(u)|\,du-\int_a^s |g'(u)|\,du\right)dtds\\
&=D_w\left(f,\int_a^{\cdot} |g'(u)|\,du\right).
\end{aligned}$$

By utilising (2.4) and (2.5) we derive

$$|D_w(f,g)|\le D_w\left(f,\int_a^{\cdot} |g'(u)|\,du\right),$$

and the first inequality in (2.1) is proved.

Now if we use inequality (1.2) for the choice $h=\int_a^{\cdot}|g'(u)|\,du$ and $k=f$ and observe that $0\le\int_a^{\cdot}|g'(u)|\,du\le\int_a^b|g'(u)|\,du$, then we get

$$D_w\left(f,\int_a^{\cdot} |g'(u)|\,du\right)\le\frac{1}{8}\int_a^b|g'(u)|\,du\left\|\frac{f'}{w}\right\|_{[a,b],\infty},$$

which proves the first branch of (2.1).

By the same inequality for $k=\int_a^{\cdot}|g'(u)|\,du$ and $h=f$ we have

$$D_w\left(f,\int_a^{\cdot} |g'(u)|\,du\right)\le\frac{1}{8}[f(b)-f(a)]\left\|\frac{g'}{w}\right\|_{[a,b],\infty},$$

which proves the second branch of (2.1).

By (1.3) we have

$$D_w\left(f,\int_a^{\cdot} |g'(u)|\,du\right)\le\frac{1}{12}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left\|\frac{g'}{w}\right\|_{[a,b],\infty},$$

which proves (2.2).

If we make use of (**??**), then we get

$$|D_w(h,k)|\le\frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{g'}{w^{1/2}}\right\|_{[a,b],2},$$

which proves (2.3). □

Further, we can state the following result as well:

Theorem 5. *Assume that* $f:[a,b]\to\mathbb{R}$ *is monotonic nondecreasing and* $g:[a,b]\to\mathbb{C}$ *is absolutely continuous with* $\|g'\|_{[a,b],p}:=\left(\int_a^b |g'(t)|^p\right)^{1/p}<\infty$. *Then*

$$
\begin{aligned}
(2.6)\qquad & |D_w(f,g)| \\
&\le [D_w(f,\ell)]^{1/q}\left[D_w\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p} \\
&\le \frac{1}{8}\times\begin{cases}[f(b)-f(a)]\left\|\frac{1}{w}\right\|_{[a,b],\infty}^{1/q}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}^{1/p}, & \text{if } \frac{1}{w},\ \frac{g'}{w}\in L_\infty[a,b]\\[2ex] (b-a)^{1/q}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\|g'\|_{[a,b],p}, & \text{if } f \text{ is absolutely continuous}\\ \text{and } \frac{f'}{w}\in L_\infty[a,b],\ g'\in L_p[a,b]\end{cases}
\end{aligned}
$$

for all $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *where* $\|g'\|_{[a,b],p}=\left(\int_a^b|g'(u)|^p\,du\right)^{1/p}$.

If f *is absolutely continuous and* $\frac{f'}{w},\ \frac{1}{w},\ \frac{g'}{w}\in L_\infty[a,b]$, *then*

$$
\begin{aligned}
(2.7)\qquad |D_w(f,g)| &\le [D_w(f,\ell)]^{1/q}\left[D_w\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p} \\
&\le \frac{1}{12}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left\|\frac{1}{w}\right\|_{[a,b],\infty}^{1/q}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}^{1/p},
\end{aligned}
$$

for all $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Also, if $\frac{f'}{w^{1/2}}\in L_2[a,b]$, $\frac{1}{w}\in L[a,b]$ *and* $\frac{g'}{w^{1/(2p)}}\in L_p[a,b]$, *then*

$$
\begin{aligned}
(2.8)\qquad |D_w(f,g)| &\le [D_w(f,\ell)]^{1/q}\left[D_w\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p} \\
&\le \frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{1}{w}\right\|_{[a,b],1}^{1/q}\left\|\frac{g'}{w^{1/(2p)}}\right\|_{[a,b],p},
\end{aligned}
$$

for all $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Using Hölder's inequality, we also have for all $s,\ t\in[a,b]$ that

$$
|g(t)-g(s)|=\left|\int_s^t g'(u)\,du\right|\le\left|\int_s^t|g'(u)|\,du\right|\le|t-s|^{1/q}\left|\int_s^t|g'(u)|^p\,du\right|^{1/p}
$$

for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Therefore

$$
\begin{aligned}
(2.9)\qquad & \frac{1}{2}\int_a^b\int_a^b w(s)\,w(t)\,|f(t)-f(s)|\,|g(t)-g(s)|\,dtds \\
&\le\frac{1}{2}\int_a^b\int_a^b w(s)\,w(t)\,|f(t)-f(s)|\,|t-s|^{1/q}\left|\int_s^t|g'(u)|^p\,du\right|^{1/p}dtds.
\end{aligned}
$$

On making use of weighted Hölder's inequality for double integral, we have for p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that

$$\begin{aligned}(2.10)\quad &\frac{1}{2}\int_a^b\int_a^b w(s)w(t)|f(t)-f(s)||t-s|^{1/q}\left|\int_s^t |g'(u)|^p du\right|^{1/p} dtds\\ &\leq \left[\frac{1}{2}\int_a^b\int_a^b w(s)w(t)|f(t)-f(s)|\left(|t-s|^{1/q}\right)^q dtds\right]^{1/q}\\ &\times\left[\frac{1}{2}\int_a^b\int_a^b w(s)w(t)|f(t)-f(s)|\left(\left|\int_s^t |g'(u)|^p du\right|^{1/p}\right)^p dtds\right]^{1/p}\end{aligned}$$

$$\begin{aligned}&=\left[\frac{1}{2}\int_a^b\int_a^b w(s)w(t)|f(t)-f(s)||t-s| dtds\right]^{1/q}\\ &\times\left[\frac{1}{2}\int_a^b\int_a^b w(s)w(t)|f(t)-f(s)|\left|\int_s^t |g'(u)|^p du\right| dtds\right]^{1/p}\\ &=\left[\frac{1}{2}\int_a^b\int_a^b w(s)w(t)[f(t)-f(s)](t-s) dtds\right]^{1/q}\\ &\times\left[\frac{1}{2}\int_a^b\int_a^b w(s)w(t)[f(t)-f(s)]\left(\int_a^t |g'(u)|^p du-\int_a^s |g'(u)|^p du\right) dtds\right]^{1/p}\\ &=[D_w(f,\ell)]^{1/q}\left[D_w\left(f,\int_a^{\cdot}|g'(u)|^p du\right)\right]^{1/p}.\end{aligned}$$

By making use (2.4), (2.9) and (2.10), we derive the first inequality in (2.6).

Now, by the inequality (1.2) we have

$$D_w(f,\ell)\leq\frac{1}{8}(f(b)-f(a))\left\|\frac{1}{w}\right\|_{[a,b],\infty}$$

and

$$D_w\left(f,\int_a^{\cdot}|g'(u)|^p du\right)\leq\frac{1}{8}(f(b)-f(a))\left\|\frac{g'}{w}\right\|_{[a,b],\infty}.$$

Therefore,

$$\begin{aligned}&[D_w(f,\ell)]^{1/q}\left[D_w\left(f,\int_a^{\cdot}|g'(u)|^p du\right)\right]^{1/p}\\ &\leq\left[\frac{1}{8}(f(b)-f(a))\left\|\frac{1}{w}\right\|_{[a,b],\infty}\right]^{1/q}\left[\frac{1}{8}(f(b)-f(a))\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\right]^{1/p}\\ &\leq\frac{1}{8}[f(b)-f(a)]\left\|\frac{1}{w}\right\|_{[a,b],\infty}^{1/q}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}^{1/p},\end{aligned}$$

which proves the first branch of inequality (2.6).

From the same inequality, we have

$$D_w(\ell,f)\leq\frac{1}{8}(b-a)\left\|\frac{f'}{w}\right\|_{[a,b],\infty}$$

and

$$D_w\left(\int_a^{\cdot} |g'(u)|^p\,du, f\right) \leq \frac{1}{8}\left(\int_a^b |g'(u)|^p\right)\left\|\frac{f'}{w}\right\|_{[a,b],\infty},$$

which implies that

$$\begin{aligned}
&[D_w(f,\ell)]^{1/q}\left[D_w\left(f, \int_a^{\cdot} |g'(u)|^p\,du\right)\right]^{1/p} \\
&\leq \left[\frac{1}{8}(b-a)\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\right]^{1/q}\left[\frac{1}{8}\left(\int_a^b |g'(u)|^p\right)\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\right] \\
&= \frac{1}{8}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}(b-a)^{1/q}\left(\int_a^b |g'(u)|^p\right)^{1/p}
\end{aligned}$$

and the second branch id also proved.

Further, if we use the inequality (1.3), then we get

$$|D_w(f,\ell)| \leq \frac{1}{12}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left\|\frac{1}{w}\right\|_{[a,b],\infty}$$

and

$$D_w\left(f, \int_a^{\cdot} |g'(u)|^p\,du\right) \leq \frac{1}{12}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left\|\frac{g'}{w}\right\|_{[a,b],\infty},$$

giving that

$$\begin{aligned}
&[D_w(f,\ell)]^{1/q}\left[D_w\left(f, \int_a^{\cdot} |g'(u)|^p\,du\right)\right]^{1/p} \\
&\leq \left(\frac{1}{12}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\right)^{1/q}\left[\frac{1}{12}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\right]^{1/p} \\
&= \frac{1}{12}\left\|\frac{f'}{w}\right\|_{[a,b],\infty}\left\|\frac{1}{w}\right\|_{[a,b],\infty}^{1/q}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}^{1/p}.
\end{aligned}$$

Finally, by (1.4) we have

$$\begin{aligned}
|D_w(f,\ell)| &\leq \frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{1}{w^{1/2}}\right\|_{[a,b],2} \\
&= \frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{1}{w}\right\|_{[a,b],1}
\end{aligned}$$

and

$$D_w\left(f, \int_a^{\cdot} |g'(u)|^p\,du\right) \leq \frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{|g'|^p}{w^{1/2}}\right\|_{[a,b],2}.$$

Therefore,

$$\begin{aligned}
&[D_w(f,\ell)]^{1/q}\left[D_w\left(f,\int_a^{\cdot}|g'(u)|^p\,du\right)\right]^{1/p}\\
\leq &\left(\frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{1}{w}\right\|_{[a,b],1}\right)^{1/q}\left(\frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{|g'|^p}{w^{1/2}}\right\|_{[a,b],2}\right)^{1/p}\\
= &\frac{1}{\pi^2}\left\|\frac{f'}{w^{1/2}}\right\|_{[a,b],2}\left\|\frac{1}{w}\right\|_{[a,b],1}^{1/q}\left\|\frac{g'}{w^{1/(2p)}}\right\|_{[a,b],p},
\end{aligned}$$

which proves the last part of the theorem. □

3. The Case of Uniform Distribution

If we consider the uniform distribution $w_0(t)=1/(b-a)$ on the interval $[a,b]$, then we get

$$D_{w_0}(h,k):=\frac{1}{b-a}\int_a^b h(t)\,k(t)\,dt-\frac{1}{b-a}\int_a^b h(t)\,dt\frac{1}{b-a}\int_a^b k(t)\,dt,$$

$$\begin{aligned}
&D_{w_0}\left(f,\int_a^{\cdot}|g'(u)|\,du\right)\\
&=\frac{1}{b-a}\int_a^b f(t)\left(\int_a^t|g'(u)|\,du\right)dt\\
&-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b\left(\int_a^t|g'(u)|\,du\right)dt\\
&=\frac{1}{b-a}\int_a^b\left(f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right)\left(\int_a^t|g'(u)|\,du\right)dt\\
&=\frac{1}{b-a}\int_a^b\left(\int_a^t|g'(u)|\,du\right)d\left(\int_a^t f(s)\,ds-\frac{t-a}{b-a}\int_a^b f(s)\,ds\right).
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
&\int_a^b\left(\int_a^t|g'(u)|\,du\right)d\left(\int_a^t f(s)\,ds-\frac{t-a}{b-a}\int_a^b f(s)\,ds\right)\\
&=\left(\int_a^t|g'(u)|\,du\right)\left(\int_a^t f(s)\,ds-\frac{t-a}{b-a}\int_a^b f(s)\,ds\right)\Bigg|_a^b\\
&-\int_a^b\left(\int_a^t f(s)\,ds-\frac{t-a}{b-a}\int_a^b f(s)\,ds\right)|g'(t)|\,dt
\end{aligned}$$

$$= -\int_a^b \left(\int_a^t f(s)\,ds - \frac{t-a}{b-a} \int_a^b f(s)\,ds \right) |g'(t)|\,dt$$
$$= \int_a^b \left(\frac{t-a}{b-a} \int_a^b f(s)\,ds - \int_a^t f(s)\,ds \right) |g'(t)|\,dt$$
$$= \frac{1}{b-a} \int_a^b \left[(t-a) \int_t^b f(s)\,ds - (b-t) \int_a^t f(s)\,ds \right] |g'(t)|\,dt.$$

Therefore

$$D_{w_0}\left(f, \int_a^{\cdot} |g'(u)|\,du\right) = \frac{1}{(b-a)^2} \int_a^b \left[(t-a) \int_t^b f(s)\,ds - (b-t) \int_a^t f(s)\,ds \right] |g'(t)|\,dt. \tag{3.1}$$

By making use of (2.1) we get

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a} \int_a^b f(t)\,dt \frac{1}{b-a} \int_a^b g(t)\,dt \right| \\ &\le \frac{1}{(b-a)^2} \int_a^b \left[(t-a) \int_t^b f(s)\,ds - (b-t) \int_a^t f(s)\,ds \right] |g'(t)|\,dt \\ &\le \begin{cases} \frac{1}{8}(b-a)\|g'\|_{[a,b],1}\|f'\|_{[a,b],\infty} & \text{if } f \text{ is absolutely continuous} \\ \text{and } f' \in L_\infty[a,b], & \\ & \\ \frac{1}{8}[f(b)-f(a)](b-a)\|g'\|_{[a,b],\infty} & \text{if } g' \in L_\infty[a,b], \end{cases} \end{aligned} \tag{3.2}$$

provided that $f : [a,b] \to \mathbb{R}$ is monotonic nondecreasing and $g : [a,b] \to \mathbb{C}$ is absolutely continuous.

From (2.2) we also have for $f : [a,b] \to \mathbb{R}$ absolutely continuous and monotonic nondecreasing and $g : [a,b] \to \mathbb{C}$ absolutely continuous

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a} \int_a^b f(t)\,dt \frac{1}{b-a} \int_a^b g(t)\,dt \right| \\ &\le \frac{1}{(b-a)^2} \int_a^b \left[(t-a) \int_t^b f(s)\,ds - (b-t) \int_a^t f(s)\,ds \right] |g'(t)|\,dt \\ &\le \frac{1}{12}(b-a)^2 \|f'\|_{[a,b],\infty} \|g'\|_{[a,b],\infty} \end{aligned} \tag{3.3}$$

provided that $f', g' \in L_\infty[a,b]$.

Moreover, if $f', g' \in L_2[a,b]$, then

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a} \int_a^b f(t)\,dt \frac{1}{b-a} \int_a^b g(t)\,dt \right| \\ &\le \frac{1}{(b-a)^2} \int_a^b \left[(t-a) \int_t^b f(s)\,ds - (b-t) \int_a^t f(s)\,ds \right] |g'(t)|\,dt \\ &\le \frac{1}{\pi^2}(b-a) \|f'\|_{[a,b],2} \|g'\|_{[a,b],2}. \end{aligned} \tag{3.4}$$

We observe that the inequalities (3.2)-(3.4) refine the corresponding inequalities from (1.2)-(1.4).

Further, observe that

$$D_{w_0}(f,\ell) = \frac{1}{b-a}\left(\int_a^b tf(t)\,dt - \frac{a+b}{2}\int_a^b f(t)\,dt\right) = \frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt$$

and

$$D_{w_0}\left(f, \int_a^{\cdot} |g'(u)|^p\,du\right) = \frac{1}{(b-a)^2}\int_a^b \left[(t-a)\int_t^b f(s)\,ds - (b-t)\int_a^t f(s)\,ds\right] |g'(t)|^p\,dt.$$

From (2.6) we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.5}$$
$$\leq \left[\frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt\right]^{1/q}$$
$$\times \left[\frac{1}{(b-a)^2}\int_a^b \left[(t-a)\int_t^b f(s)\,ds - (b-t)\int_a^t f(s)\,ds\right] |g'(t)|^p\,dt\right]^{1/p}$$
$$\leq \frac{1}{8}\times \begin{cases} [f(b)-f(a)]\,(b-a)\,\|g'\|_{[a,b],\infty}^{1/p}, & \text{if } g'\in L_\infty[a,b], \\ (b-a)^{1+1/q}\,\|f'\|_{[a,b],\infty}\,\|g'\|_{[a,b],p}, & \text{if } f \text{ is absolutely continuous} \\ \text{and } f'\in L_\infty[a,b],\ g'\in L_p[a,b]. & \end{cases}$$

From (2.7) we obtain for f monotonic nondecreasing,

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.6}$$
$$\leq \left[\frac{1}{b-a}\int_a^b \left(t-\frac{a+b}{2}\right) f(t)\,dt\right]^{1/q}$$
$$\times \left[\frac{1}{(b-a)^2}\int_a^b \left[(t-a)\int_t^b f(s)\,ds - (b-t)\int_a^t f(s)\,ds\right] |g'(t)|^p\,dt\right]^{1/p}$$
$$\leq \frac{1}{12}(b-a)^2\,\|f'\|_{[a,b],\infty}\,\|g'\|_{[a,b],\infty}^{1/p}.$$

4. Applications for Norms and Semi-Inner Products

Let X be a real linear space, $x, y \in X$, $x \neq y$ and let $[x,y] := \{(1-\lambda)x + \lambda y,\ \lambda \in [0,1]\}$ be the *segment* generated by x and y. We consider the function $f : [x,y] \to \mathbb{R}$ and the attached function $g(x,y) : [0,1] \to \mathbb{R}$, $g(x,y)(t) := f[(1-t)x + ty]$, $t \in [0,1]$.

It is well known that f is convex on $[x,y]$ iff $g(x,y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $g'_{\pm}(x,y)(s) = \triangledown_{\pm} f[(1-s)x + sy](y-x)$, $s \in [0,1)$,
(ii) $g'_{+}(x,y)(0) = \triangledown_{+} f(x)(y-x)$,
(iii) $g'_{-}(x,y)(1) = \triangledown_{-} f(y)(y-x)$,

where $\triangledown_{\pm} f(x)(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\triangledown_{+} f(x)(y) \ :\ = \lim_{h\to 0+} \frac{f(x+hy)-f(x)}{h},$$
$$\triangledown_{-} f(x)(y) \ :\ = \lim_{k\to 0-} \frac{f(x+ky)-f(x)}{k}, \ \ x,\ y \in X.$$

We remark also that

$$\triangledown_{+} f[(1-s)x+sy](y-x) = \triangledown_{-} f[(1-s)x+sy](y-x)$$

for almost every $s \in [0,1]$, being the lateral derivatives of a convex function. In integrals we can then write $\triangledown$ instead of $\triangledown_{+}$ or $\triangledown_{-}$.

Now, assume that $(X, \|\cdot\|)$ is a normed linear space. The function $f_0(s) = \frac{1}{2}\|x\|^2$, $x \in X$ is convex and thus the following limits exist

(iv) $\langle x,y\rangle_s := \triangledown_{+} f_0(y)(x) = \lim_{t\to 0+} \frac{\|y+tx\|^2-\|y\|^2}{2t}$;
(v) $\langle x,y\rangle_i := \triangledown_{-} f_0(y)(x) = \lim_{s\to 0-} \frac{\|y+sx\|^2-\|y\|^2}{2s}$;

for any x, $y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [21]), assuming that p, $q \in \{s,i\}$ and $p \neq q$:

(a) $\langle x,x\rangle_p = \|x\|^2$ for all $x \in X$;
(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta \langle x,y\rangle_p$ if α, $\beta \geq 0$ and x, $y \in X$;
(aaa) $\left|\langle x,y\rangle_p\right| \leq \|x\| \|y\|$ for all x, $y \in X$;
(av) $\langle \alpha x + y, x\rangle_p = \alpha \langle x,x\rangle_p + \langle y,x\rangle_p$ if x, $y \in X$ and $\alpha \in \mathbb{R}$;
(v) $\langle -x,y\rangle_p = -\langle x,y\rangle_q$ for all x, $y \in X$;
(va) $\langle x+y,z\rangle_p \leq \|x\| \|z\| + \langle y,z\rangle_p$ for all x, y, $z \in X$;
(vaa) The mapping $\langle \cdot,\cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);
(vaaa) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X \backslash \{0\}$ if and only if $\langle y,x_0\rangle_s = \langle y,x_0\rangle_i$ for all $y \in X$; in general $\langle y,x\rangle_i \leq \langle y,x\rangle_s$ for all x, $y \in X$;
(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot,\cdot\rangle$, then $\langle y,x\rangle_i = \langle y,x\rangle = \langle y,x\rangle_s$ for all x, $y \in X$.

For $m \geq 1$ the function $f_m(x) = \|x\|^m$ is convex on X. Therefore

$$\triangledown_{+(-)} f_m(y)(x) = p\|y\|^{m-2} \langle x,y\rangle_{s(i)} \tag{4.1}$$

which exists for all x, $y \in X$ whenever $m \geq 2$. If $1 \leq m < 2$ the equality (4.1) holds for all $x \in X$ and nonzero $y \in X$.

Observe also that

$$\begin{aligned} &\triangledown_{\pm} f_m[(1-s)x+sy](y-x) \\ &= m\|(1-s)x+sy\|^{m-2} \langle y-x,(1-s)x+sy\rangle_{s(i)} \end{aligned} \tag{4.2}$$

which exists for all $x, y \in X$ whenever $m \geq 2$. If $1 \leq m < 2$ the equality (4.2) holds for all x, y such that $(1-s)x + sy \neq 0$ for all $s \in [0,1]$.

Assume that $f : [0,1] \to \mathbb{R}$ is *monotonic nondecreasing and absolutely continuous* and $g : C \to \mathbb{R}$ is *convex* on the convex subset C of the linear space X. Consider the functional

$$D_{x,y}(f,g) := \int_0^1 f(t)\, g((1-t)x+ty)\, dt - \int_0^1 f(t)\, dt \int_0^1 g((1-t)x+ty)\, dt,$$

where x, y are distinct vectors in C.

From the inequality (3.2) we get

$$\begin{aligned} &|D_{x,y}(f,g)| \\ &\leq \int_0^1 \left[t\int_t^1 f(s)\, ds - (1-t)\int_0^t f(s)\, ds \right] |\nabla g[(1-t)x+ty](y-x)|\, dt \\ &\leq \frac{1}{8} \times \begin{cases} \|f'\|_{[0,1],\infty} \int_0^1 |\nabla g[(1-t)x+ty](y-x)|\, dt \\ \text{if } f \text{ is absolutely continuous and } f' \in L_\infty[0,1], \\ \\ [f(1)-f(0)] \sup_{t\in[0,1]} |\nabla g[(1-t)x+ty](y-x)|, \end{cases} \end{aligned} \tag{4.3}$$

from (3.3) we get

$$\begin{aligned} &|D_{x,y}(f,g)| \\ &\leq \int_0^1 \left[t\int_t^1 f(s)\, ds - (1-t)\int_0^t f(s)\, ds \right] |\nabla g[(1-t)x+ty](y-x)|\, dt \\ &\leq \frac{1}{12} \|f'\|_{[0,1],\infty} \sup_{t\in[0,1]} |\nabla g[(1-t)x+ty](y-x)| \end{aligned} \tag{4.4}$$

if $f' \in L_\infty[0,1]$, while from (3.4),

$$\begin{aligned} &|D_{x,y}(f,g)| \\ &\leq \int_0^1 \left[t\int_t^1 f(s)\, ds - (1-t)\int_0^t f(s)\, ds \right] |\nabla g[(1-t)x+ty](y-x)|\, dt \\ &\leq \frac{1}{\pi^2} \|f'\|_{[0,1],2} \left(\int_0^1 |\nabla g[(1-t)x+ty](y-x)|^2\, dt \right)^{1/2}. \end{aligned} \tag{4.5}$$

if $f' \in L_2[0,1]$.

If we take $g(x) := \|x\|^m$, $m \geq 1$, then by (4.3)

$$\left|\int_0^1 f(t)\|(1-t)x+ty\|^m dt - \int_0^1 f(t)\,dt \int_0^1 \|(1-t)x+ty\|^m dt\right| \tag{4.6}$$
$$\leq m\int_0^1 \left[t\int_t^1 f(s)\,ds - (1-t)\int_0^t f(s)\,ds\right]$$
$$\times \|(1-t)x+ty\|^{m-2}\left|\langle y-x,(1-t)x+ty\rangle_{s(i)}\right| dt$$
$$\leq \frac{1}{8}m \times \begin{cases} \|f'\|_{[a,b],\infty}\int_0^1 \|(1-t)x+ty\|^{m-2} \\ \times\left|\langle y-x,(1-t)x+ty\rangle_{s(i)}\right| dt \\ \text{if } f \text{ is absolutely continuous and } f' \in L_\infty[a,b], \\ \\ [f(1)-f(0)] \\ \times \sup_{t\in[0,1]}\|(1-t)x+ty\|^{m-2}\left|\langle y-x,(1-t)x+ty\rangle_{s(i)}\right|, \end{cases}$$

if $f' \in L_\infty[0,1]$.

By using Schwarz's inequality "(aaa)" we have

$$\int_0^1 \|(1-t)x+ty\|^{m-2}\left|\langle y-x,(1-t)x+ty\rangle_{s(i)}\right| dt$$
$$\leq \|y-x\|\int_0^1 \|(1-t)x+ty\|^{m-1} dt$$

and

$$\|(1-t)x+ty\|^{m-2}\left|\langle y-x,(1-t)x+ty\rangle_{s(i)}\right| \leq \|y-x\|\,\|(1-t)x+ty\|^{m-1}$$

for all $t \in [0,1]$.

Since $\|(1-t)x+ty\| \leq \max\{\|x\|,\|y\|\}$ for all $t \in [0,1]$, then by (4.6) we derive the simpler inequality

$$\left|\int_0^1 f(t)\|(1-t)x+ty\|^m dt - \int_0^1 f(t)\,dt\int_0^1 \|(1-t)x+ty\|^m dt,\right| \tag{4.7}$$
$$\leq \frac{1}{8}m[f(1)-f(0)]\|y-x\|\max\left\{\|x\|^{m-1},\|y\|^{m-1}\right\},$$

for all $x, y \in X$, $m \geq 1$ and f is monotonic nondecreasing on $[0,1]$.

If $m \geq 2$, then by Hermite-Hadamard inequality for convex functions we have

$$\int_0^1 \|(1-t)x+ty\|^{m-1} dt \leq \frac{1}{2}\left(\|x\|^{m-1}+\|y\|^{m-1}\right)$$

and by (4.6) we get

$$\left|\int_0^1 f(t)\|(1-t)x+ty\|^m dt - \int_0^1 f(t)\,dt\int_0^1 \|(1-t)x+ty\|^m dt\right| \tag{4.8}$$
$$\leq \frac{1}{16}m\|f'\|_{[a,b],\infty}\|y-x\|\left(\|x\|^{m-1}+\|y\|^{m-1}\right)$$

for all $x, y \in X$, $m \geq 2$ and f is monotonic nondecreasing on $[0,1]$.

Now, if we take $f(t) = t$, then

$$\begin{aligned} t\int_t^1 f(s)\,ds - (1-t)\int_0^t f(s)\,ds &= t\left(\frac{1}{2} - \frac{1}{2}t^2\right) - (1-t)\,\frac{1}{2}t^2 \\ &= \frac{1}{2}t - \frac{1}{2}t^2 = \frac{1}{2}t(1-t) \end{aligned}$$

and by the first inequality in (4.6) we get

$$\begin{aligned} &\left|\int_0^1 t\,\|(1-t)x+ty\|^m\,dt - \frac{1}{2}\int_0^1 \|(1-t)x+ty\|^m\,dt\right| \\ &\le \frac{1}{2}m\int_0^1 t(1-t)\,\|(1-t)x+ty\|^{m-2}\left|\langle y-x,(1-t)x+ty\rangle_{s(i)}\right|dt \\ &\le \frac{1}{2}m\,\|y-x\|\int_0^1 t(1-t)\,\|(1-t)x+ty\|^{m-1}\,dt \end{aligned} \tag{4.9}$$

for all $x,\ y \in X$, $m \ge 1$.

From the inequality (4.4) we get

$$\begin{aligned} &\left|\int_0^1 f(t)\,\|(1-t)x+ty\|^m\,dt - \int_0^1 f(t)\,dt\int_0^1 \|(1-t)x+ty\|^m\,dt\right| \\ &\le m\int_0^1 \left[t\int_t^1 f(s)\,ds - (1-t)\int_0^t f(s)\,ds\right] \\ &\times \|(1-t)x+ty\|^{m-2}\left|\langle y-x,(1-t)x+ty\rangle_{s(i)}\right|dt \\ &\le \frac{1}{12}m\,\|f'\|_{[0,1],\infty}\sup_{t\in[0,1]}\left\{\|(1-t)x+ty\|^{m-2}\left|\langle y-x,(1-t)x+ty\rangle_{s(i)}\right|\right\}, \end{aligned} \tag{4.10}$$

while from (4.5),

$$\begin{aligned} &\left|\int_0^1 f(t)\,\|(1-t)x+ty\|^m\,dt - \int_0^1 f(t)\,dt\int_0^1 \|(1-t)x+ty\|^m\,dt\right| \\ &\le m\int_0^1 \left[t\int_t^1 f(s)\,ds - (1-t)\int_0^t f(s)\,ds\right] \\ &\times \|(1-t)x+ty\|^{m-2}\left|\langle y-x,(1-t)x+ty\rangle_{s(i)}\right|dt \\ &\le \frac{1}{\pi^2}m\,\|f'\|_{[0,1],2} \\ &\times \left(\int_0^1 \|(1-t)x+ty\|^{2(m-2)}\left|\langle y-x,(1-t)x+ty\rangle_{s(i)}\right|^2 dt\right)^{1/2}, \end{aligned} \tag{4.11}$$

where f is monotonic nondecreasing and absolutely continuous on $[0,1]$.

Similar simpler upper bounds can be derived, however we omit the details.

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