ON SOME RECENT INTEGRAL INEQUALITIES FOR THE WEIGHTED ČEBYŠEV FUNCTIONAL

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Abstract. Assume that $w:[a,b]\rightarrow[0,\infty)$ is integrable with $\int_a^b w(t)\,dt=1$ and f, g are Lebesgue integrable on $[a,b]$. Consider the Čebyšev functional

$$D_w(f,g):=\int_a^b f(t)\,g(t)\,w(t)\,dt-\int_a^b f(t)\,w(t)\,dt\int_a^b g(t)\,w(t)\,dt.$$

In ths paper we show among other that, if f, g are absolutely continuous and $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\begin{aligned}&|D_w(f,g)|\\ &\leq\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}\\ &\leq\frac{1}{2}\frac{\int_a^b(t-a)\,w(t)\,dt\int_a^b(b-t)\,w(t)\,dt}{b-a}\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty}\\ &\leq\frac{1}{8}(b-a)\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty},\end{aligned}$$

provided that $\frac{f'}{w^{1/p}},\ \frac{g'}{w^{1/q}}\in L_\infty[a,b]$. Applications for uniform distribution are also given.

1. Introduction

Consider a probability density function w on $[a,b]$, i.e., $w\geq 0$ a.e. on $[a,b]$ with $\int_a^b w(t)\,dt=1$, and the weighted Čebyšev functional for functions defined on a finite interval $[a,b]$,

$$D_w(h,k):=\int_a^b h(t)\,k(t)\,w(t)\,dt-\int_a^b h(t)\,w(t)\,dt\int_a^b k(t)\,w(t)\,dt.$$

In [7] Cerone and Dragomir proved among others the following *refinement of Grüss' inequality:*

$$|D_w(h,k)|\leq\frac{1}{2}(\Delta-\delta)\int_a^b w(t)\left|h(t)-\int_a^b h(s)\,w(s)\,ds\right|dt \tag{1.1}$$

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$$\leq \frac{1}{2}(\Delta-\delta)\left[\int_a^b w(s)h^2(s)\,ds-\left(\int_a^b h(s)w(s)\,ds\right)^2\right]^{\frac{1}{2}}$$
$$\leq \frac{1}{4}(\Delta-\delta)(\Gamma-\gamma),$$

provided that $h,\ k:[a,b]\to\mathbb{R}$ are measurable functions and so that $-\infty<\gamma\leq h\leq\Gamma<\infty$, $-\infty<\delta\leq k\leq\Delta<\infty$ a.e. on $[a,b]$, and $h,\ k,\ hk\in L_w[a,b]$.

For more recent upper bounds related to the Čebyšev functional see [1]-[9], [11]-[17] and [21]-[28].

In the recent paper [19] we obtained the following weighted inequalities:

Theorem 1. *Assume that $w:[a,b]\to(0,\infty)$ is a continuous probability density on $[a,b]$, h is Lebesgue integrable and satisfies the condition $m\leq h(t)\leq M$ for $t\in[a,b]$ and $k:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$ with $\frac{k'}{w}$ is essentially bounded, namely $\frac{k'}{w}\in L_\infty[a,b]$, then we have*

$$|D_w(h,k)| \leq \frac{1}{2}\left\|\frac{k'}{w}\right\|_{[a,b],\infty}\frac{\left(\int_a^b h(t)w(t)\,dt-m\right)\left(M-\int_a^b h(t)w(t)\,dt\right)}{M-m} \leq \frac{1}{8}(M-m)\left\|\frac{k'}{w}\right\|_{[a,b],\infty}. \tag{1.2}$$

The constant $\frac{1}{8}$ is best possible.

Also, we have [19]:

Theorem 2. *Assume that $w:[a,b]\to(0,\infty)$ is a continuous probability density on $[a,b]$. If h and k are absolutely continuous and*

$$\frac{\int_\cdot^b w(s)\,ds\int_a^\cdot w(s)\,ds}{w}[h']^2,\ \frac{\int_\cdot^b w(s)\,ds\int_a^\cdot w(s)\,ds}{w}[k']^2\in L[a,b], \tag{1.3}$$

then

$$|D_w(h,k)|\leq\frac{1}{2}\left(\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}[h'(t)]^2\,dt\right)^{1/2}\times\left(\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}[k'(t)]^2\,dt\right)^{1/2}. \tag{1.4}$$

The constant $\frac{1}{2}$ is best possible.

Moreover, we have [19]:

Theorem 3. *Assume that $w:[a,b]\to(0,\infty)$ is a continuous probability density on $[a,b]$. If h and k are absolutely continuous and k is monotonic nondecreasing, then*

$$|D_w(h,k)|\leq\frac{1}{2}\left\|\frac{h'}{w}\right\|_{[a,b],\infty}\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)k'(t)\,dt, \tag{1.5}$$

provided that $\frac{h'}{w}\in L_\infty[a,b]$. The constant $\frac{1}{2}$ is best possible.

Finally, we also obtained [19]:

Theorem 4. *Assume that $w:[a,b]\to(0,\infty)$ is a continuous probability density on $[a,b]$. If h and k are continuous on $[a,b]$ and differentiable on (a,b) with $k'(t)\neq 0$ for each $t\in(a,b)$, then*

$$|D_w(h,k)|\le\frac{1}{2}\left\|\frac{h'}{k'}\right\|_{[a,b],\infty}\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}\left[k'(t)\right]^2dt, \tag{1.6}$$

provided that the norm and the integral in the right term are finite. The constant $\frac{1}{2}$ is best possible.

In ths paper we show among other that, if f, g are absolutely continuous and p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then

$$\begin{aligned}&|D_w(f,g)|\\&\le\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}\\&\le\frac{1}{2}\frac{\int_a^b(t-a)w(t)\,dt\int_a^b(b-t)w(t)\,dt}{b-a}\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty}\\&\le\frac{1}{8}(b-a)\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty},\end{aligned}$$

provided that $\frac{f'}{w^{1/p}}$, $\frac{g'}{w^{1/q}}\in L_\infty[a,b]$. Applications for uniform distribution are also given.

2. Main Results

We start with the following result for absolutely continuous functions, in which one function has the derivative essentially bounded:

Theorem 5. *Assume that $w:[a,b]\to(0,\infty)$ is a continuous probability density on $[a,b]$ and f, g are absolutely continuous on $[a,b]$ and such that $f'\in L_\infty[a,b]$, namely $\|f'\|_{[a,b],\infty}:=\operatorname{esssup}_{t\in[a,b]}|f'(t)|<\infty$. Then*

$$\begin{aligned}|D_w(f,g)|&\le\|f'\|_{[a,b],\infty}D_w\left(\int_a^{\cdot}|g'(u)|\,du,\ell\right)\\&\le\frac{1}{2}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\|f'\|_{[a,b],\infty}\\&\quad\times\frac{\int_a^b\left(\int_a^t|g'(u)|\,du\right)w(t)\,dt\int_a^b\left(\int_t^b|g'(u)|\,du\right)w(t)\,dt}{\int_a^b|g'(u)|\,du}\\&\le\frac{1}{8}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\|f'\|_{[a,b],\infty}\int_a^b|g'(u)|\,du\end{aligned} \tag{2.1}$$

and

$$|D_w(f,g)| \le \|f'\|_{[a,b],\infty} D_w\left(\int_a^{\cdot} |g'(u)|\,du, \ell\right) \le \frac{1}{2}\|f'\|_{[a,b],\infty}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\frac{\int_a^b (t-a)\,w(t)\,dt \int_a^b (b-t)\,w(t)\,dt}{b-a} \le \frac{1}{8}(b-a)\|f'\|_{[a,b],\infty}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}. \tag{2.2}$$

If

$$\frac{\int_{\cdot}^b w(s)\,ds \int_a^{\cdot} w(s)\,ds}{w}[g']^2,\ \ \frac{\int_{\cdot}^b w(s)\,ds \int_a^{\cdot} w(s)\,ds}{w} \in L[a,b], \tag{2.3}$$

then

$$|D_w(f,g)| \le \|f'\|_{[a,b],\infty} D_w\left(\int_a^{\cdot} |g'(u)|\,du, \ell\right) \le \frac{1}{2}\|f'\|_{[a,b],\infty}\left(\int_a^b \frac{\int_t^b w(s)\,ds \int_a^t w(s)\,ds}{w(t)}[g'(t)]^2\,dt\right)^{1/2} \times \left(\int_a^b \frac{\int_t^b w(s)\,ds \int_a^t w(s)\,ds}{w(t)}dt\right)^{1/2}. \tag{2.4}$$

We also have

$$|D_w(f,g)| \le \|f'\|_{[a,b],\infty} D_w\left(\int_a^{\cdot} |g'(u)|\,du, \ell\right) \le \frac{1}{2}\|f'\|_{[a,b],\infty}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)dt, \tag{2.5}$$

for $\frac{g'}{w} \in L_\infty[a,b]$ *and*

$$|D_w(f,g)| \le \|f'\|_{[a,b],\infty} D_w\left(\int_a^{\cdot} |g'(u)|\,du, \ell\right) \le \frac{1}{2}\|f'\|_{[a,b],\infty}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)|g'(t)|\,dt \tag{2.6}$$

if $\frac{1}{w} \in L_\infty[a,b]$.

Moreover, if $g'(u) \neq 0,\ u \in [a,b]$, *then*

$$|D_w(f,g)| \le \|f'\|_{[a,b],\infty} D_w\left(\int_a^{\cdot} |g'(u)|\,du, \ell\right) \le \frac{1}{2}\|f'\|_{[a,b],\infty}\left\|\frac{1}{g'}\right\|_{[a,b],\infty}\int_a^b \frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}[g'(t)]^2\,dt \tag{2.7}$$

if $\frac{1}{g'} \in L_\infty[a,b]$.

Also, we have

$$|D_w(f,g)| \le \|f'\|_{[a,b],\infty} D_w\left(\int_a^{\cdot} |g'(u)|\,du, \ell\right) \le \frac{1}{2}\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\infty}\int_a^b \frac{\int_t^b w(s)\,ds \int_a^t w(s)\,ds}{w(t)}dt, \tag{2.8}$$

provided $g' \in L_\infty[a,b]$.

Proof. We use weighted Korkine's identity for functions with complex values

$$D_w(f,g) = \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,[f(t)-f(s)]\,[g(t)-g(s)]\,dtds.$$

For Korkine's classical identity for real-valued functions, see [26, p. 242].

If we take the modulus and use the integral's properties, we get

$$|D_w(f,g)| \le \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,|[f(t)-f(s)]\,[g(t)-g(s)]|\,dtds = \frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,|f(t)-f(s)|\,|g(t)-g(s)|\,dtds. \tag{2.9}$$

Observe that for $s,\ t \in [a,b]$

$$f(t)-f(s) = \int_s^t f'(u)\,du,\ \ g(t)-g(s) = \int_s^t g'(u)\,du,$$

which implies that

$$\begin{aligned}|f(t)-f(s)|\,|g(t)-g(s)| &= \left|\int_s^t f'(u)\,du\right|\left|\int_s^t g'(u)\,du\right| \\ &\le \left|\int_s^t |f'(u)|\,du\right|\left|\int_s^t |g'(u)|\,du\right| \\ &\le \|f'\|_{[a,b],\infty}|t-s|\left|\int_s^t |g'(u)|\,du\right| \\ &= \|f'\|_{[a,b],\infty}(t-s)\int_s^t |g'(u)|\,du,\end{aligned}$$

for all $s,\ t \in [a,b]$.

By (2.9) we get

$$|D_w(f,g)| \le \|f'\|_{[a,b],\infty}\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,(t-s)\left(\int_s^t |g'(u)|\,du\right)dtds. \tag{2.10}$$

Since

$$(t-s)\left(\int_s^t |g'(u)|\,du\right) = (t-s)\left(\int_a^t |g'(u)|\,du - \int_a^s |g'(u)|\,du\right),$$

hence by Korkine's identity for real valued functions $f(t)=\ell(t)$ and $\int_a^t |g'(u)|\,du$, we have

$$\frac{1}{2}\int_a^b\int_a^b w(t)\,w(s)\,(t-s)\left(\int_a^t |g'(u)|\,du-\int_a^s |g'(u)|\,du\right)dtds \tag{2.11}$$
$$=D_w\left(\ell,\int_a^{\cdot} |g'(u)|\,du\right).$$

By utilising (2.10) and (2.11), we deduce the first inequality in (2.1).

If we use inequality (1.2) for $h=\int_a^{\cdot}|g'(u)|\,du$ and $k=\ell$ and observe that $0\le\int_a^{\cdot}|g'(u)|\,du\le\int_a^b|g'(u)|\,du$ then we get

$$D_w\left(\int_a^{\cdot}|g'(u)|\,du,\ell\right)$$
$$\le\frac{1}{2}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\frac{\left(\int_a^b\left(\int_a^t|g'(u)|\,du\right)w(t)\,dt\right)\left(\int_a^b\left(\int_t^b|g'(u)|\,du\right)w(t)\,dt\right)}{\int_a^b|g'(u)|\,du}$$
$$\le\frac{1}{8}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\int_a^b|g'(u)|\,du,$$

which proves (2.1).

If we use inequality (1.2) for $h=\ell$ and $k=\int_a^{\cdot}|g'(u)|\,du$ and observe that $a\le\ell\le b$, then we get

$$D_w\left(\ell,\int_a^{\cdot}|g'(u)|\,du\right)\le\frac{1}{2}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\frac{\int_a^b(t-a)\,w(t)\,dt\int_a^b(b-t)\,w(t)\,dt}{b-a}$$
$$\le\frac{1}{8}(b-a)\left\|\frac{g'}{w}\right\|_{[a,b],\infty},$$

which proves (2.2).

The inequality (2.4) follows by (1.4) for $h=\int_a^{\cdot}|g'(u)|\,du$ and $k=\ell$.

If we use (1.5) for $h=\int_a^{\cdot}|g'(u)|\,du$ and $k=\ell$, then we get

$$D_w\left(\int_a^{\cdot}|g'(u)|\,du,\ell\right)\le\frac{1}{2}\left\|\frac{g'}{w}\right\|_{[a,b],\infty}\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)dt,$$

which proves (2.5).

If we take in (1.5) $h=\ell$ and $k=\int_a^{\cdot}|g'(u)|\,du$, then we get

$$\left|D_w\left(\ell,\int_a^{\cdot}|g'(u)|\,du\right)\right|$$
$$\le\frac{1}{2}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)|g'(t)|\,dt,$$

which proves (2.6).

Moreover, if $g'(u)\neq 0$, $u\in[a,b]$, then by (1.6) for $k=\int_a^{\cdot}|g'(u)|\,du$ and $h=\ell$,

$$D_w\left(\ell,\int_a^{\cdot}|g'(u)|\,du\right)\le\frac{1}{2}\left\|\frac{1}{g'}\right\|_{[a,b],\infty}\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}\left[g'(t)\right]^2dt,$$

which proves (2.7).

Finally, if $h=\int_a^{\cdot}|g'(u)|\,du$ and $k=\ell$ in (1.6) then we get

$$D_w\left(\int_a^{\cdot}|g'(u)|\,du,\ell\right)\le\frac{1}{2}\|g'\|_{[a,b],\infty}\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}dt,$$

which proves (2.8). □

Further, we have the following bounds in terms of the p-norms:

Theorem 6. *Assume that $w:[a,b]\to(0,\infty)$ is a continuous probability density on $[a,b]$ and f, g are absolutely continuous on $[a,b]$ while p, $q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. Suppose also that $\|f'\|_p:=\left(\int_a^b|f'(u)|^p\,du\right)^{1/p}<\infty$ and $\|g'\|_q:=\left(\int_a^b|g'(u)|^q\,du\right)^{1/q}<\infty$. Then*

$$\begin{aligned}
&|D_w(f,g)|\\
&\le\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}\\
&\le\frac{1}{2}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\\
&\times\frac{\left(\int_a^b\left(\int_a^t|f'(u)|^p\,du\right)w(t)\,dt\right)^{1/p}\left(\int_a^b\left(\int_t^b|f'(u)|^p\,du\right)w(t)\,dt\right)^{1/p}}{\left(\int_a^b|f'(u)|^p\,du\right)^{1/p}}\\
&\times\frac{\left(\int_a^b\left(\int_a^t|g'(u)|^q\,du\right)w(t)\,dt\right)^{1/q}\left(\int_a^b\left(\int_t^b|g'(u)|^q\,du\right)w(t)\,dt\right)^{1/q}}{\left(\int_a^b|g'(u)|^q\,du\right)^{1/q}}\\
&\le\frac{1}{8}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\left(\int_a^b|f'(u)|^p\,du\right)^{1/p}\left(\int_a^b|g'(u)|^q\,du\right)^{1/q}
\end{aligned}\tag{2.12}$$

and

$$\begin{aligned}
&|D_w(f,g)|\\
&\le\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}\\
&\le\frac{1}{2}\frac{\int_a^b(t-a)w(t)\,dt\int_a^b(b-t)w(t)\,dt}{b-a}\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty}\\
&\le\frac{1}{8}(b-a)\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty},
\end{aligned}\tag{2.13}$$

provided that $\frac{f'}{w^{1/p}},\ \frac{g'}{w^{1/q}}\in L_\infty[a,b]$.

We have

$$|D_w(f,g)| \le \left[D_w\left(\ell, \int_a^{\cdot} |f'(u)|^p du\right)\right]^{1/p} \left[D_w\left(\ell, \int_a^{\cdot} |g'(u)|^q du\right)\right]^{1/q} \tag{2.14}$$
$$\le \frac{1}{2}\left(\int_a^b \frac{\int_t^b w(s)\,ds \int_a^t w(s)\,ds}{w(t)} dt\right)^{1/2}$$
$$\times \left(\int_a^b \frac{\int_t^b w(s)\,ds \int_a^t w(s)\,ds}{w(t)} [f'(t)]^{2p}\, dt\right)^{1/(2p)}$$
$$\times \left(\int_a^b \frac{\int_t^b w(s)\,ds \int_a^t w(s)\,ds}{w(t)} [g'(t)]^{2q}\, dt\right)^{1/(2q)},$$

provided that the integrals in the right side are convergent.

Also, if $\frac{1}{w} \in L_\infty[a,b]$,

$$|D_w(f,g)| \le \left[D_w\left(\ell, \int_a^{\cdot} |f'(u)|^p du\right)\right]^{1/p} \left[D_w\left(\ell, \int_a^{\cdot} |g'(u)|^q du\right)\right]^{1/q} \tag{2.15}$$
$$\le \frac{1}{2}\left\|\frac{1}{w}\right\|_{[a,b],\infty} \left(\int_a^b \left(\int_t^b w(s)\,ds \int_a^t w(s)\,ds\right) |f'(t)|^p\, dt\right)^{1/p}$$
$$\times \int_a^b \left(\left(\int_t^b w(s)\,ds \int_a^t w(s)\,ds\right) |g'(t)|^q\, dt\right)^{1/q}$$

and

$$|D_w(f,g)| \tag{2.16}$$
$$\le \left[D_w\left(\ell, \int_a^{\cdot} |f'(u)|^p du\right)\right]^{1/p} \left[D_w\left(\ell, \int_a^{\cdot} |g'(u)|^q du\right)\right]^{1/q}$$
$$\le \frac{1}{2}\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty} \left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty} \int_a^b \left(\int_t^b w(s)\,ds \int_a^t w(s)\,ds\right) dt,$$

provided $\frac{f'}{w^{1/p}}, \frac{g'}{w^{1/q}} \in L_\infty[a,b]$.

Finally, we have

$$|D_w(f,g)| \le \left[D_w\left(\ell, \int_a^{\cdot} |f'(u)|^p du\right)\right]^{1/p} \left[D_w\left(\ell, \int_a^{\cdot} |g'(u)|^q du\right)\right]^{1/q} \tag{2.17}$$
$$\le \frac{1}{2}\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],\infty} \int_a^b \frac{\int_t^b w(s)\,ds \int_a^t w(s)\,ds}{w(t)} dt$$

if $f',\, g' \in L_\infty [a,b]$ *and*

$$|D_w (f,g)| \le \left[D_w\left(\ell, \int_a^{\cdot} |f'(u)|^p du\right)\right]^{1/p} \left[D_w\left(\ell, \int_a^{\cdot} |g'(u)|^q du\right)\right]^{1/q} \le \frac{1}{2}\left\|\frac{1}{f'}\right\|_{[a,b],\infty}\left\|\frac{1}{g'}\right\|_{[a,b],\infty}\left(\int_a^b \frac{\int_t^b w(s)\,ds \int_a^t w(s)\,ds}{w(t)} |f'(t)|^{2p} dt\right)^{1/p} \times\left(\int_a^b \frac{\int_t^b w(s)\,ds \int_a^t w(s)\,ds}{w(t)} |g'(t)|^{2q} dt\right)^{1/q} \tag{2.18}$$

provided $\frac{1}{f'},\, \frac{1}{g'} \in L_\infty [a,b]$.

Proof. Using Hölder's inequality for $p,\, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} &|f(t) - f(s)|\,|g(t) - g(s)| \\ &= \left|\int_s^t f'(u)\,du\right| \left|\int_s^t g'(u)\,du\right| \\ &\le \left|\int_s^t |f'(u)|\,du\right| \left|\int_s^t |g'(u)|\,du\right| \\ &\le |t-s|^{1/q}\left|\int_s^t |f'(u)|^p du\right|^{1/p} |t-s|^{1/p}\left|\int_s^t |g'(u)|^q du\right|^{1/q} \\ &= |t-s|\left|\int_s^t |f'(u)|^p du\right|^{1/p} \left|\int_s^t |g'(u)|^q du\right|^{1/q} \end{aligned}$$

for all $t,\, s \in [a,b]$.

By the weighted Hölder's inequality for double integral, we also have

$$\begin{aligned} &\int_a^b\int_a^b w(s)\,w(t)\,|f(t)-f(s)|\,|g(t)-g(s)|\,dtds \\ &\le \int_a^b\int_a^b w(s)\,w(t)\,|t-s|\left|\int_s^t |f'(u)|^p du\right|^{1/p}\left|\int_s^t |g'(u)|^q du\right|^{1/q} dtds \\ &\le \left(\int_a^b\int_a^b w(s)\,w(t)\,|t-s|\left(\left|\int_s^t |f'(u)|^p du\right|^{1/p}\right)^p dtds\right)^{1/p} \\ &\times \left(\int_a^b\int_a^b w(s)\,w(t)\,|t-s|\left(\left|\int_s^t |g'(u)|^q du\right|^{1/q}\right)^q dtds\right)^{1/q} \\ &= \left(\int_a^b\int_a^b w(s)\,w(t)\,|t-s|\left|\int_s^t |f'(u)|^p du\right| dtds\right)^{1/p} \\ &\times \left(\int_a^b\int_a^b w(s)\,w(t)\,|t-s|\left|\int_s^t |g'(u)|^q du\right| dtds\right)^{1/q}. \end{aligned} \tag{2.19}$$

Observe that

$$\begin{aligned}
&\int_a^b \int_a^b w(s)\, w(t)\, |t-s| \left| \int_s^t |f'(u)|^p\, du \right| dtds \\
&= \int_a^b \int_a^b w(s)\, w(t)\, (t-s) \left(\int_s^t |f'(u)|^p\, du \right) dtds \\
&= \int_a^b \int_a^b w(s)\, w(t)\, (t-s) \left(\int_a^t |f'(u)|^p\, du - \int_a^s |f'(u)|^p\, du \right) dtds \\
&= 2D_w \left(\ell, \int_a^{\cdot} |f'(u)|^p\, du \right)
\end{aligned}$$

and, similarly

$$\int_a^b \int_a^b w(s)\, w(t)\, |t-s| \left| \int_s^t |g'(u)|^q\, du \right| dtds = 2D_w \left(\ell, \int_a^{\cdot} |g'(u)|^q\, du \right).$$

Therefore, by (2.9)

$$\begin{aligned}
|D_w(f,g)| &\leq \frac{1}{2} \int_a^b \int_a^b w(s)\, w(t)\, |f(t)-f(s)|\, |g(t)-g(s)|\, dtds \\
&\leq \frac{1}{2} \left[2D_w \left(\ell, \int_a^{\cdot} |f'(u)|^p\, du \right) \right]^{1/p} \left[2D_w \left(\ell, \int_a^{\cdot} |g'(u)|^q\, du \right) \right]^{1/q} \\
&= \left[D_w \left(\ell, \int_a^{\cdot} |f'(u)|^p\, du \right) \right]^{1/p} \left[D_w \left(\ell, \int_a^{\cdot} |g'(u)|^q\, du \right) \right]^{1/q},
\end{aligned}$$

which proves the first inequality in (2.12).

If we use inequality (1.2) for $h = \int_a^{\cdot} |f'(u)|^p\, du$ and $k = \ell$ and observe that $0 \leq \int_a^{\cdot} |f'(u)|^p\, du \leq \int_a^b |f'(u)|^p\, du$ then we get

$$\begin{aligned}
&D_w \left(\int_a^{\cdot} |f'(u)|^p\, du, \ell \right) \\
&\leq \frac{1}{2} \left\| \frac{1}{w} \right\|_{[a,b],\infty} \frac{\int_a^b \left(\int_a^t |f'(u)|^p\, du \right) w(t)\, dt \left(\int_a^b \left(\int_t^b |f'(u)|^p\, du \right) w(t)\, dt \right)}{\int_a^b |f'(u)|^p\, du} \\
&\leq \frac{1}{8} \left\| \frac{1}{w} \right\|_{[a,b],\infty} \int_a^b |f'(u)|^p\, du
\end{aligned}$$

and, similarly

$$\begin{aligned}
&D_w \left(\int_a^{\cdot} |g'(u)|^q\, du, \ell \right) \\
&\leq \frac{1}{2} \left\| \frac{1}{w} \right\|_{[a,b],\infty} \frac{\int_a^b \left(\int_a^t |g'(u)|^q\, du \right) w(t)\, dt \left(\int_a^b \left(\int_t^b |g'(u)|^q\, du \right) w(t)\, dt \right)}{\int_a^b |g'(u)|^q\, du} \\
&\leq \frac{1}{8} \left\| \frac{1}{w} \right\|_{[a,b],\infty} \int_a^b |g'(u)|^q\, du.
\end{aligned}$$

Therefore

$$\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}$$
$$\leq\frac{1}{2}\left\|\frac{1}{w}\right\|_{[a,b],\infty}$$
$$\times\frac{\left(\int_a^b\left(\int_a^{\cdot}|f'(u)|^p\,du\right)w(t)\,dt\right)^{1/p}\left(\int_a^b\left(\int_t^b|f'(u)|^p\,du\right)w(t)\,dt\right)^{1/p}}{\left(\int_a^b|f'(u)|^p\,du\right)^{1/p}}$$
$$\times\frac{\left(\int_a^b\left(\int_a^{\cdot}|g'(u)|^q\,du\right)w(t)\,dt\right)^{1/q}\left(\int_a^b\left(\int_t^b|g'(u)|^q\,du\right)w(t)\,dt\right)^{1/q}}{\left(\int_a^b|g'(u)|^q\,du\right)^{1/q}}$$
$$\leq\frac{1}{8}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\left(\int_a^b|f'(u)|^p\,du\right)^{1/p}\left(\int_a^b|g'(u)|^q\,du\right)^{1/q},$$

which proves (2.12).

If we use inequality (1.2) for $h=\ell$ and $k=\int_a^{\cdot}|f'(u)|^p\,du$, then we get

$$D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)$$
$$\leq\frac{1}{2}\left\|\frac{|f'|^p}{w}\right\|_{[a,b],\infty}\frac{\int_a^b(t-a)\,w(t)\,dt\int_a^b(b-t)\,w(t)\,dt}{b-a}$$
$$\leq\frac{1}{8}(b-a)\left\|\frac{|f'|^p}{w}\right\|_{[a,b],\infty}.$$

Similarly,

$$D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)$$
$$\leq\frac{1}{2}\left\|\frac{|g'|^p}{w}\right\|_{[a,b],\infty}\frac{\int_a^b(t-a)\,w(t)\,dt\int_a^b(b-t)\,w(t)\,dt}{b-a}$$
$$\leq\frac{1}{8}(b-a)\left\|\frac{|g'|^q}{w}\right\|_{[a,b],\infty}.$$

Therefore

$$\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}$$
$$\leq\frac{1}{2}\frac{\int_a^b(t-a)\,w(t)\,dt\int_a^b(b-t)\,w(t)\,dt}{b-a}\left\|\frac{|f'|^p}{w}\right\|_{[a,b],\infty}^{1/p}\left\|\frac{|g'|^q}{w}\right\|_{[a,b],\infty}^{1/q}$$
$$\leq\frac{1}{8}(b-a)\left\|\frac{|f'|^p}{w}\right\|_{[a,b],\infty}^{1/p}\left\|\frac{|g'|^q}{w}\right\|_{[a,b],\infty}^{1/q},$$

which proves (2.13).

If we use inequality (1.4) we get

$$D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\le\frac{1}{2}\left(\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}\left[f'(t)\right]^{2p}dt\right)^{1/2}\times\left(\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}dt\right)^{1/2}$$

and

$$D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\le\frac{1}{2}\left(\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}\left[g'(t)\right]^{2q}dt\right)^{1/2}\times\left(\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}dt\right)^{1/2}.$$

These imply that

$$\begin{aligned}&\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}\\&\le\frac{1}{2}\left(\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}dt\right)^{1/2}\\&\times\left(\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}\left[f'(t)\right]^{2p}dt\right)^{1/(2p)}\\&\times\left(\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}\left[g'(t)\right]^{2q}dt\right)^{1/(2q)},\end{aligned}$$

which proves (2.14).

From (1.5) we derive

$$D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\le\frac{1}{2}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)|f'(t)|^p\,dt$$

and

$$D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\le\frac{1}{2}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)|g'(t)|^q\,dt,$$

which imply

$$\begin{aligned}&\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}\\&\le\frac{1}{2}\left\|\frac{1}{w}\right\|_{[a,b],\infty}\left(\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)|f'(t)|^p\,dt\right)^{1/p}\\&\times\int_a^b\left(\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)|g'(t)|^q\,dt\right)^{1/q}\end{aligned}$$

and the inequality (2.15) is proved.

By (1.5) we also have

$$D_w\left(\int_a^{\cdot}|f'(u)|^p\,du,\ell\right)\le\frac{1}{2}\left\|\frac{|f'|^p}{w}\right\|_{[a,b],\infty}\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)dt,$$

and

$$D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\le\frac{1}{2}\left\|\frac{|g'|^q}{w}\right\|_{[a,b],\infty}\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)dt,$$

which gives that

$$\begin{aligned}&\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}\\&\le\frac{1}{2}\left\|\frac{|f'|^p}{w}\right\|_{[a,b],\infty}^{1/p}\left\|\frac{|g'|^q}{w}\right\|_{[a,b],\infty}^{1/q}\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)dt\\&=\frac{1}{2}\left\|\frac{f'}{w^{1/p}}\right\|_{[a,b],\infty}\left\|\frac{g'}{w^{1/q}}\right\|_{[a,b],\infty}\int_a^b\left(\int_t^b w(s)\,ds\int_a^t w(s)\,ds\right)dt,\end{aligned}$$

which proves (2.16).

From (1.6) we derive

$$D_w\left(\int_a^{\cdot}|f'(u)|^p\,du,\ell\right)\le\frac{1}{2}\left\||f'|^p\right\|_{[a,b],\infty}\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}dt$$

and

$$D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\le\frac{1}{2}\left\||g'|^q\right\|_{[a,b],\infty}\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}dt,$$

which gives that

$$\begin{aligned}&\left[D_w\left(\int_a^{\cdot}|f'(u)|^p\,du,\ell\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}\\&\le\frac{1}{2}\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}dt\left(\left\||f'|^p\right\|_{[a,b],\infty}\right)^{1/p}\left(\left\||g'|^q\right\|_{[a,b],\infty}\right)^{1/q}\\&=\frac{1}{2}\left\|f'\right\|_{[a,b],\infty}\left\|g'\right\|_{[a,b],\infty}\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}dt\end{aligned}$$

that proves (2.17).

From (1.6) we also obtain,

$$D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\le\frac{1}{2}\left\|\frac{1}{|f'|^p}\right\|_{[a,b],\infty}\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}\left|f'(t)\right|^{2p}dt$$

and

$$D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\le\frac{1}{2}\left\|\frac{1}{|g'|^q}\right\|_{[a,b],\infty}\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}\left|g'(t)\right|^{2q}dt.$$

Therefore,

$$\left[D_w\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)\right]^{1/p}\left[D_w\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)\right]^{1/q}$$
$$\leq\frac{1}{2}\left\|\frac{1}{f'}\right\|_{[a,b],\infty}\left\|\frac{1}{g'}\right\|_{[a,b],\infty}\left(\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}|f'(t)|^{2p}\,dt\right)^{1/p}$$
$$\times\left(\int_a^b\frac{\int_t^b w(s)\,ds\int_a^t w(s)\,ds}{w(t)}|g'(t)|^{2q}\,dt\right)^{1/q}$$

and the inequality (2.18) is thus proved. □

3. The Case of Uniform Distribution

If we consider the uniform distribution $w_0(t)=1/(b-a)$ on the interval $[a,b]$, then we get

$$D_{w_0}(h,k):=\frac{1}{b-a}\int_a^b h(t)\,k(t)\,dt-\frac{1}{b-a}\int_a^b h(t)\,dt\frac{1}{b-a}\int_a^b k(t)\,dt,$$

$$D_{w_0}\left(\ell,\int_a^{\cdot}|g'(u)|\,du\right)$$
$$=\frac{1}{b-a}\int_a^b t\left(\int_a^t|g'(u)|\,du\right)dt-\frac{a+b}{2}\frac{1}{b-a}\int_a^b\left(\int_a^t|g'(u)|\,du\right)dt$$

Integrating by parts, we have

$$\frac{1}{2}\int_a^b(b-t)(t-a)|g'(t)|\,dt$$
$$=\frac{1}{2}\int_a^b(b-t)(t-a)\,d\left(\int_a^t|g'(u)|\,du\right)$$
$$=\frac{1}{2}\left[(b-t)(t-a)\int_a^t|g'(u)|\,du\Big|_a^b+\int_a^b(2t-a-b)\left(\int_a^t|g'(u)|\,du\right)dt\right]$$
$$=\int_a^b\left(t-\frac{a+b}{2}\right)\left(\int_a^t|g'(u)|\,du\right)dt$$
$$=\int_a^b t\left(\int_a^t|g'(u)|\,du\right)dt-\frac{a+b}{2}\int_a^b\left(\int_a^t|g'(u)|\,du\right)dt.$$

Therefore,

$$D_{w_0}\left(\ell,\int_a^{\cdot}|g'(u)|\,du\right)=\frac{1}{2(b-a)}\int_a^b(b-t)(t-a)|g'(t)|\,dt.$$

From (2.1) we derive

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.1}$$
$$\leq \frac{1}{2(b-a)}\|f'\|_{[a,b],\infty}\int_a^b (b-t)(t-a)\,|g'(t)|\,dt$$
$$\leq \frac{1}{2(b-a)}\|f'\|_{[a,b],\infty}\frac{\int_a^b\left(\int_a^t |g'(u)|\,du\right)dt\int_a^b\left(\int_t^b |g'(u)|\,du\right)dt}{\int_a^b |g'(u)|\,du}$$
$$\leq \frac{1}{8}(b-a)\|f'\|_{[a,b],\infty}\|g'\|_{[a,b],1},$$

provided that f and g are absolutely continuous with $f'\in L_\infty[a,b]$.

From (2.4) we obtain

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.2}$$
$$\leq \frac{1}{2(b-a)}\|f'\|_{[a,b],\infty}\int_a^b (b-t)(t-a)\,|g'(t)|\,dt$$
$$\leq \frac{\sqrt{6}}{12}(b-a)^{1/2}\|f'\|_{[a,b],\infty}\left(\int_a^b (b-t)(t-a)\,[g'(t)]^2\,dt\right)^{1/2},$$

provided that f and g are absolutely continuous with $f'\in L_\infty[a,b]$ and $(b-\cdot)(\cdot-a)[g']^2\in L[a,b]$.

From (2.7) we get

$$\left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \tag{3.3}$$
$$\frac{1}{2(b-a)}\|f'\|_{[a,b],\infty}\int_a^b (b-t)(t-a)\,|g'(t)|\,dt$$
$$\leq \frac{1}{2(b-a)}\|f'\|_{[a,b],\infty}\left\|\frac{1}{g'}\right\|_{[a,b],\infty}\int_a^b (b-t)(t-a)\,[g'(t)]^2\,dt$$

provided that $\frac{1}{g'}\in L_\infty[a,b]$ and $(b-\cdot)(\cdot-a)[g']^2\in L[a,b]$.

Observe also that

$$D_{w_0}\left(\ell,\int_a^{\cdot}|f'(u)|^p\,du\right)=\frac{1}{2(b-a)}\int_a^b (b-t)(t-a)\,|f'(u)|^p\,dt$$

and

$$D_{w_0}\left(\ell,\int_a^{\cdot}|g'(u)|^q\,du\right)=\frac{1}{2(b-a)}\int_a^b (b-t)(t-a)\,|g'(u)|^q\,dt.$$

From (2.15) we get

$$\begin{aligned}
(3.4)\qquad & \left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \\
& \leq \frac{1}{2(b-a)}\left[\int_a^b (b-t)(t-a)\left|f'(u)\right|^p dt\right]^{1/p} \\
& \times\left[\int_a^b (b-t)(t-a)\left|g'(u)\right|^q dt\right]^{1/q}
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{1}{2(b-a)}\frac{\left(\int_a^b\left(\int_a^t\left|f'(u)\right|^p du\right)wdt\right)^{1/p}\left(\int_a^b\left(\int_t^b\left|f'(u)\right|^p du\right)dt\right)^{1/p}}{\left(\int_a^b\left|f'(u)\right|^p du\right)^{1/p}} \\
& \times\frac{\left(\int_a^b\left(\int_a^t\left|g'(u)\right|^q du\right)dt\right)^{1/q}\left(\int_a^b\left(\int_t^b\left|g'(u)\right|^q du\right)dt\right)^{1/q}}{\left(\int_a^b\left|g'(u)\right|^q du\right)^{1/q}} \\
& \leq \frac{1}{8}(b-a)\left(\int_a^b\left|f'(u)\right|^p du\right)^{1/p}\left(\int_a^b\left|g'(u)\right|^q du\right)^{1/q}
\end{aligned}$$

for $f'\in L_p[a,b]$, $g'\in L_q[a,b]$ with p, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

From (2.17) we get

$$\begin{aligned}
(3.5)\qquad & \left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \\
& \leq \frac{1}{2(b-a)}\left[\int_a^b (b-t)(t-a)\left|f'(u)\right|^p dt\right]^{1/p} \\
& \times\left[\int_a^b (b-t)(t-a)\left|g'(u)\right|^q dt\right]^{1/q} \\
& \leq \frac{1}{8}(b-a)^2\left\|f'\right\|_{[a,b],\infty}\left\|g'\right\|_{[a,b],\infty}
\end{aligned}$$

for $f'\in L_\infty[a,b]$, $g'\in L_\infty[a,b]$ with p, $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$.

From (2.18) we also derive

$$\begin{aligned}
(3.6)\quad & \left|\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\frac{1}{b-a}\int_a^b g(t)\,dt\right| \\
&\le \frac{1}{2(b-a)} \\
&\times\left[\int_a^b (b-t)(t-a)\left|f'(u)\right|^p dt\right]^{1/p}\left[\int_a^b (b-t)(t-a)\left|g'(u)\right|^q dt\right]^{1/q} \\
&\le \frac{1}{2(b-a)}\left\|\frac{1}{f'}\right\|_{[a,b],\infty}\left\|\frac{1}{g'}\right\|_{[a,b],\infty} \\
&\times\left(\int_a^b (b-t)(t-a)\left|f'(t)\right|^{2p} dt\right)^{1/p}\left(\int_a^b (b-t)(t-a)\left|g'(t)\right|^{2q} dt\right)^{1/q}
\end{aligned}$$

if $\frac{1}{f'},\ \frac{1}{g'}\in L_\infty[a,b]$.

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