ON AN INTEGRAL INEQUALITY OF AGARWAL AND PANG RELATED TO WIRTINGER'S RESULT

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Abstract. In this paper we show among others that, if $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$ with $\int_a^b w(s)\,ds = 1$ and $f \in C^1([a, b], \mathbb{R})$ satisfies the condition $f(a) = f(b) = 0$, then

$$\int_a^b |f(x)|^\lambda w(x)\,dx \le \frac{1}{2} B((\lambda+1)/2, (\lambda+1)/2) \int_a^b \frac{|f'(x)|^\lambda}{w(x)^{\lambda-1}} dx \le \frac{1}{2^\lambda} \int_a^b \frac{|f'(x)|^\lambda}{w(x)^{\lambda-1}} dx,$$

for $\lambda \ge 1$, where B is Beta function. Applications for trapezoid and Grüss' type inequalities are also given.

1. Introduction

It is well known that, see for instance [6], or [10], if $u \in C^1([a, b], \mathbb{R})$ satisfies $u(a) = u(b) = 0$, then

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2} \int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t) = K \sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K \in \mathbb{R}$.

If $u \in C^1([a, b], \mathbb{R})$ satisfies the condition $u(a) = 0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2} \int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t) = L \sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L \in \mathbb{R}$.

For some related Wirtinger type integral inequalities see [2], [3], [6] and [9]-[13].

Agarwal and Pang [1] proved the following Opial-Wirtinger's type inequality:

Theorem 1. *Let $\lambda \ge 1$ be a given real number and let p a nonnegative and continuous function on $[0, 1]$. Further, let x be an aboslutely continuous function on $[0, 1]$ with $x(0) = x(1) = 0$. Then*

$$\int_0^1 p(t)|x(t)|^\lambda\,dt \le \frac{1}{2} \int_0^1 [t(1-t)]^{(\lambda-1)/2} p(t)\,dt \int_0^1 |x'(t)|^\lambda\,dt. \tag{1.3}$$

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Assume that g is absolutely continuous on $[a,b]$ with the property that $g(a) = g(b) = 0$. Assume also that $q : [a,b] \to [0,\infty)$ is continuous. By taking

$$x(t) = f((1-t)a + tb), \ p(t) = q((1-t)a + tb), \ t \in [0,1]$$

and observing that

$$x'(t) = (b-a) f'((1-t)a + tb), \ t \in (0,1)$$

we get from (1.3) that

$$\begin{aligned}\int_0^1 q((1-t)a+tb) |f((1-t)a+tb)|^\lambda dt \\ \leq \frac{1}{2}(b-a)^\lambda \int_0^1 [t(1-t)]^{(\lambda-1)/2} q((1-t)a+tb) dt \\ \times \int_0^1 |f'((1-t)a+tb)|^\lambda dt,\end{aligned} \tag{1.4}$$

and by the change of variable $x = (1-t)a + tb$, then we have $dx = (b-a)dt$, $t = \frac{x-a}{b-a}$ which gives by (1.4) that

$$\begin{aligned}&\frac{1}{b-a}\int_a^b q(x)|f(x)|^\lambda dx \\ &\leq \frac{1}{2}(b-a)^\lambda \frac{1}{(b-a)^2}\int_a^b q(x)\left[\frac{(x-a)(b-x)}{(b-a)^2}\right]^{(\lambda-1)/2} dx \int_a^b |f'(x)|^\lambda dx,\end{aligned}$$

which is equivalent to

$$\int_a^b q(x)|f(x)|^\lambda dx \leq \frac{1}{2}\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2} q(x)dx \int_a^b |f'(x)|^\lambda dx, \tag{1.5}$$

provided that g is absolutely continuous on $[a,b]$ with the property that $g(a) = g(b) = 0$ and that $q : [a,b] \to [0,\infty)$ is continuous.

Since, by the elementary inequality

$$(x-a)(b-x) \leq \frac{1}{4}(b-a)^2,$$

hence

$$\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2} q(x)dx \leq \frac{1}{2^\lambda}(b-a)^{\lambda-1}\int_a^b q(x)dx,$$

and we have the chain of inequalities

$$\begin{aligned}\int_a^b q(x)|f(x)|^\lambda dx &\leq \frac{1}{2}\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2} q(x)dx \int_a^b |f'(x)|^\lambda dx \\ &\leq \frac{1}{2^\lambda}(b-a)^{\lambda-1}\int_a^b q(x)dx \int_a^b |f'(x)|^\lambda dx,\end{aligned} \tag{1.6}$$

for all $\lambda \geq 1$.

We consider the *Beta function*

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt$$

for complex inputs x, y such that $\operatorname{Re} x > 0$, $\operatorname{Re} y > 0$.

Observe that

$$\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2}\,dx = (b-a)^\lambda \int_0^1 [t(1-t)]^{(\lambda-1)/2}\,dt$$
$$= (b-a)^\lambda B((\lambda+1)/2, (\lambda+1)/2),$$

then by (1.6) we get the unweighted inequality

$$\int_a^b |f(x)|^\lambda\,dx \le \frac{1}{2}(b-a)^\lambda B((\lambda+1)/2,(\lambda+1)/2)\int_a^b |f'(x)|^\lambda\,dx \le \frac{1}{2^\lambda}(b-a)^\lambda \int_a^b |f'(x)|^\lambda\,dx, \tag{1.7}$$

for all $\lambda \ge 1$, provided that g is absolutely continuous on $[a,b]$ with the property that $g(a) = g(b) = 0$ and that $q : [a,b] \to [0,\infty)$ is continuous.

In this paper we show among others that, if $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds = 1$ and $f \in C^1([a,b],\mathbb{R})$ satisfies the condition $f(a) = f(b) = 0$, then

$$\int_a^b |f(x)|^\lambda w(x)\,dx \le \frac{1}{2}B((\lambda+1)/2,(\lambda+1)/2)\int_a^b \frac{|f'(x)|^\lambda}{w(x)^{\lambda-1}}dx$$
$$\le \frac{1}{2^\lambda}\int_a^b \frac{|f'(x)|^\lambda}{w(x)^{\lambda-1}}dx,$$

for $\lambda \ge 1$, where B is Beta function. Applications for trapezoid and Grüss' type inequalities are also given.

2. Some Simple Inequalities

We start to the following trapezoid type inequality:

Proposition 1. *Let $g \in C^1([a,b],\mathbb{R})$. Then*

$$\left|\frac{1}{b-a}\int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2}\right| \tag{2.1}$$
$$\le \frac{1}{2^{1+1/\lambda}}(b-a)^{1-1/\lambda}[B((\lambda+1)/2,(\lambda+1)/2)]^{1/\lambda}$$
$$\times \left(\int_a^b |g'(t) - g'(a+b-t)|^\lambda\,dt\right)^{1/\lambda}$$
$$\le \frac{1}{4}(b-a)^{1-1/\lambda}\left(\int_a^b |g'(t) - g'(a+b-t)|^\lambda\,dt\right)^{1/\lambda}$$

for $\lambda \ge 1$.

Proof. If $g \in C^1([a,b],\mathbb{R})$, then by taking

$$f(t) := \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2},\ t \in [a,b]$$

we have $f(a) = f(b) = 0$ and by (1.7) we have

$$\begin{aligned}
&\int_a^b \left| \frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2} \right|^\lambda dt \\
&\leq \frac{1}{2} (b-a)^\lambda B((\lambda+1)/2, (\lambda+1)/2) \\
&\times \int_a^b \left| \frac{g'(t) - g'(a+b-t)}{2} \right|^\lambda dt \\
&\leq \frac{1}{2^\lambda} (b-a)^\lambda \int_a^b \left| \frac{g'(t) - g'(a+b-t)}{2} \right|^\lambda dt,
\end{aligned}$$

namely

$$\begin{aligned}
&\int_a^b \left| \frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2} \right|^\lambda dt \\
&\leq \frac{1}{2^{\lambda+1}} (b-a)^\lambda B((\lambda+1)/2, (\lambda+1)/2) \\
&\times \int_a^b |g'(t) - g'(a+b-t)|^\lambda dt \\
&\leq \frac{1}{4^\lambda} (b-a)^\lambda \int_a^b |g'(t) - g'(a+b-t)|^\lambda dt.
\end{aligned} \tag{2.2}$$

By Jensen's inequality for the power λ we have

$$\begin{aligned}
&\frac{1}{b-a} \int_a^b \left| \frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2} \right|^\lambda dt \\
&\geq \left| \frac{1}{b-a} \int_a^b \left(\frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2} \right) \right|^\lambda dt \\
&= \left| \frac{1}{b-a} \int_a^b g(t)\, dt - \frac{g(a) + g(b)}{2} \right|^\lambda .
\end{aligned}$$

Therefore by (2.2) we get

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b g(t)\, dt - \frac{g(a) + g(b)}{2} \right|^\lambda \\
&\leq \frac{1}{b-a} \int_a^b \left| \frac{g(t) + g(a+b-t)}{2} - \frac{g(a) + g(b)}{2} \right|^\lambda dt \\
&\leq \frac{1}{2^{\lambda+1}} (b-a)^{\lambda-1} B((\lambda+1)/2, (\lambda+1)/2) \\
&\times \int_a^b |g'(t) - g'(a+b-t)|^\lambda dt \\
&\leq \frac{1}{4^\lambda} (b-a)^{\lambda-1} \int_a^b |g'(t) - g'(a+b-t)|^\lambda dt
\end{aligned}$$

and by taking the power $1/\lambda$ we obtain (2.1). □

Remark 1. *For $\lambda = 1$ we get*

$$\left| \frac{1}{b-a} \int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2} \right| \leq \frac{1}{4} \int_a^b \left| g'(t) - g'(a+b-t) \right| dt \tag{2.3}$$

while for $\lambda = 2$ we obtain

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2} \right| \\ &\leq \frac{\pi^{1/2}}{8} (b-a)^{1/2} \left(\int_a^b \left| g'(t) - g'(a+b-t) \right|^2 dt \right)^{1/2}. \end{aligned} \tag{2.4}$$

Proposition 2. *Let $g \in C^1([a,b],\mathbb{R})$. Then*

$$\begin{aligned} &\left| \frac{1}{b-a} \int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2} \right| \\ &\leq \frac{1}{2^{1/\lambda}} (b-a)^{1-1/\lambda} \left[B\left((\lambda+1)/2, (\lambda+1)/2 \right) \right]^{1/\lambda} \\ &\times \left(\int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^\lambda dt \right)^{1/\lambda} \\ &\leq \frac{1}{2} (b-a)^{1-1/\lambda} \left(\int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^\lambda dt \right)^{1/\lambda}, \end{aligned} \tag{2.5}$$

for $\lambda \geq 1$.

Proof. If $g \in C^1([a,b],\mathbb{C})$, then by taking

$$f(t) := g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a}, \ t \in [a,b]$$

we have $f(a) = f(b) = 0$ and by (1.7) we have

$$\begin{aligned} &\int_a^b \left| g(t) - \frac{g(a)(b-t) + g(b)(t-a)}{b-a} \right|^\lambda dt \\ &\leq \frac{1}{2} (b-a)^\lambda B\left((\lambda+1)/2, (\lambda+1)/2 \right) \\ &\times \int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^\lambda dt \\ &\leq \frac{1}{2^\lambda} (b-a)^\lambda \int_a^b \left| g'(t) - \frac{g(b)-g(a)}{b-a} \right|^\lambda dt. \end{aligned} \tag{2.6}$$

By Jensen's inequality for the power λ we have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \left|g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right|^{\lambda} dt \\
&\geq \left|\frac{1}{b-a}\int_a^b \left(g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right) dt\right|^{\lambda} \\
&= \left|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right|^{\lambda}.
\end{aligned}$$

By (2.6) we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right|^{\lambda} \\
&\leq \frac{1}{b-a}\int_a^b \left|g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right|^{\lambda} dt \\
&\leq \frac{1}{2}(b-a)^{\lambda-1} B((\lambda+1)/2,(\lambda+1)/2) \\
&\times \int_a^b \left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^{\lambda} dt \\
&\leq \frac{1}{2^{\lambda}}(b-a)^{\lambda-1}\int_a^b \left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^{\lambda} dt,
\end{aligned}$$

and by taking the power $1/\lambda$ we obtain (2.5). □

Remark 2. *For* $\lambda = 1$ *we get*

$$\left|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right| \leq \frac{1}{2}\int_a^b \left|g'(t)-\frac{g(b)-g(a)}{b-a}\right| dt, \tag{2.7}$$

while for $\lambda = 2$

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right| \\
&\leq \frac{\pi^{1/2}}{4}(b-a)^{1/2}\left(\int_a^b \left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^2 dt\right)^{1/2}.
\end{aligned} \tag{2.8}$$

Proposition 3. *Let* $g \in C^1([a,b],\mathbb{R})$*. Then*

$$\begin{aligned}
&\left|\frac{b+a}{2}\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt\right| \\
&\leq \frac{1}{2^{1/\lambda}}(b-a)^{\lambda-1/\lambda}\left[B\left((\lambda+1)/2,(\lambda+1)/2\right)\right]^{1/\lambda} \\
&\times\left(\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^\lambda dt\right)^{1/\lambda} \\
&\leq \frac{1}{2}(b-a)^{2-1/\lambda}\left(\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^\lambda dt\right)^{1/\lambda}
\end{aligned} \tag{2.9}$$

for $\lambda \geq 1$.

Proof. Assume that $g:[a,b]\to\mathbb{C}$ is continuous, then by taking

$$f(t) := \int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds,\ \ t\in[a,b]$$

we have $f(a)=f(b)=0$, and by (1.7) we have

$$\begin{aligned}
&\int_a^b\left|\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^\lambda dt \\
&\leq \frac{1}{2}(b-a)^\lambda B\left((\lambda+1)/2,(\lambda+1)/2\right) \\
&\times\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^\lambda dt \\
&\leq \frac{1}{2^\lambda}(b-a)^\lambda\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^\lambda dt.
\end{aligned} \tag{2.10}$$

Observe that, integrating by parts, we have

$$\begin{aligned}
&\int_a^b\left(\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right)dt \\
&= \int_a^b\left(\int_a^t g(s)\,ds\right)dt - \frac{b-a}{2}\int_a^b g(s)\,ds \\
&= b\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt - \frac{b-a}{2}\int_a^b g(s)\,ds \\
&= \frac{b+a}{2}\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt.
\end{aligned}$$

By Jensen's inequality we also have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \left|\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^{\lambda} dt \\
&\geq \left|\frac{1}{b-a}\int_a^b \left(\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right)\right|^{\lambda} \\
&= \left|\frac{1}{b-a}\left(\frac{b+a}{2}\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt\right)\right|^{\lambda},
\end{aligned}$$

which gives that

$$\begin{aligned}
&\left|\left(\frac{b+a}{2}\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt\right)\right|^{\lambda} \\
&\leq (b-a)^{\lambda-1}\int_a^b \left|\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^{\lambda} dt.
\end{aligned}$$

By (2.10) we get

$$\begin{aligned}
&\left|\left(\frac{b+a}{2}\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt\right)\right|^{\lambda} \\
&\leq (b-a)^{\lambda-1}\int_a^b \left|\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^{\lambda} dt \\
&\leq \frac{1}{2}(b-a)^{2\lambda-1} B((\lambda+1)/2,(\lambda+1)/2) \\
&\times \int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^{\lambda} dt \\
&\leq \frac{1}{2^{\lambda}}(b-a)^{2\lambda-1}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^{\lambda} dt
\end{aligned} \tag{2.11}$$

and by taking the power $1/\lambda$ we obtain (2.9). □

Remark 3. *For $\lambda = 1$ we get*

$$\left|\frac{b+a}{2}\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt\right| \leq \frac{1}{2}\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right| dt \tag{2.12}$$

and for $\lambda = 2$,

$$\left|\frac{b+a}{2}\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt\right| \tag{2.13}$$
$$\leq \frac{\pi^{1/2}}{4}(b-a)^{1/2}\left(\int_a^b \left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{1/2}$$
$$= \frac{\pi^{1/2}}{4}(b-a)\left(\frac{1}{b-a}\int_a^b g(t)^2\,dt - \left(\frac{1}{b-a}\int_a^b g(s)\,ds\right)^2\right)^{1/2}.$$

3. Weighted Inequalities

We have the following main result:

Theorem 2. *Let* $h : [a,b] \to [h(a), h(b)]$ *be a continuous strictly increasing function that is of class* C^1 *on* (a,b). *If* $f \in C^1([a,b],\mathbb{R})$ *is a function with the properties* $f(a) = f(b) = 0$, *then*

$$\int_a^b |f(x)|^\lambda h'(x)\,dx \leq \frac{1}{2}B((\lambda+1)/2, (\lambda+1)/2)[h(b) - h(a)]^\lambda \tag{3.1}$$
$$\times \int_a^b \frac{|f'(x)|^\lambda}{[h'(x)]^{\lambda-1}}dx$$
$$\leq \frac{1}{2^\lambda}[h(b) - h(a)]^\lambda \int_a^b \frac{|f'(x)|^\lambda}{[h'(x)]^{\lambda-1}}dx.$$

Proof. We write the inequality (1.7) for the function $f \circ h^{-1}$ on the interval $[h(a), h(b)]$ to get

$$\int_{h(a)}^{h(b)} \left|\left(f\circ h^{-1}\right)(z)\right|^\lambda dz \leq \frac{1}{2}[h(b) - h(a)]^\lambda B((\lambda+1)/2, (\lambda+1)/2) \tag{3.2}$$
$$\times \int_{h(a)}^{h(b)} \left|\left(f\circ h^{-1}\right)'(z)\right|^\lambda dz$$
$$\leq \frac{1}{2^\lambda}(b-a)^\lambda \int_{h(a)}^{h(b)} \left|\left(f\circ h^{-1}\right)'(z)\right|^\lambda dz,$$

since $\left(f\circ h^{-1}\right)(h(a)) = f(a) = 0$ and $\left(f\circ h^{-1}\right)(h(b)) = f(b) = 0$.

If $f : [c,d] \to \mathbb{R}$ is absolutely continuous on $[c,d]$, then $f\circ h^{-1} : [h(c), h(d)] \to \mathbb{R}$ is absolutely continuous on $[h(c), h(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(f\circ h^{-1}\right)'(z) = \left(f'\circ h^{-1}\right)(z)\left(h^{-1}\right)'(z) = \frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)} \tag{3.3}$$

for almost every (a.e.) $z \in [h(c), h(d)]$.

Using the inequality (3.2) we then get

$$\int_{h(a)}^{h(b)} \left|\left(f\circ h^{-1}\right)(z)\right|^{\lambda} dz \leq \frac{1}{2}\left[h(b)-h(a)\right]^{\lambda} B\left((\lambda+1)/2,(\lambda+1)/2\right) \times \int_{h(a)}^{h(b)} \left|\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right|^{\lambda} dz \leq \frac{1}{2^{\lambda}}\left[h(b)-h(a)\right]^{\lambda} \int_{h(a)}^{h(b)} \left|\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right|^{\lambda} dz. \tag{3.4}$$

Observe also that, by the change of variable $x=h^{-1}(z)$, $z\in[h(a),h(b)]$, we have $z=h(x)$ that gives $dz=h'(x)\,dx$ and

$$\int_{h(a)}^{h(b)} \left|\left(f\circ h^{-1}\right)(z)\right|^{\lambda} dz=\int_a^b |f(x)|^{\lambda} h'(x)\,dx.$$

We also have

$$\int_{h(a)}^{h(b)} \left|\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right|^{\lambda} dz=\int_a^b \left|\frac{f'(x)}{h'(x)}\right|^{\lambda} h'(x)\,dx=\int_a^b \frac{|f'(x)|^{\lambda}}{[h'(x)]^{\lambda-1}}dx.$$

Therefore, by (3.4) we derive the desired result (3.1). □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Corollary 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and $f\in C^1([a,b],\mathbb{R})$ satisfies the conditiona $f(a)=f(b)=0$, then*

$$\int_a^b |f(x)|^{\lambda} w(x)\,dx \leq \frac{1}{2}B\left((\lambda+1)/2,(\lambda+1)/2\right)\left(\int_a^b w(s)\,ds\right)^{\lambda} \times \int_a^b \frac{|f'(x)|^{\lambda}}{w(x)^{\lambda-1}}dx \leq \frac{1}{2^{\lambda}}\left(\int_a^b w(s)\,ds\right)^{\lambda}\int_a^b \frac{|f'(x)|^{\lambda}}{w(x)^{\lambda-1}}dx. \tag{3.5}$$

If $\int_a^b w(s)\,ds=1$, then we have the simpler inequalities

$$\int_a^b |f(x)|^{\lambda} w(x)\,dx \leq \frac{1}{2}B\left((\lambda+1)/2,(\lambda+1)/2\right)\int_a^b \frac{|f'(x)|^{\lambda}}{w(x)^{\lambda-1}}dx \leq \frac{1}{2^{\lambda}}\int_a^b \frac{|f'(x)|^{\lambda}}{w(x)^{\lambda-1}}dx. \tag{3.6}$$

Some examples are as follows:

a). If we take $w : [a,b] \subset (0,\infty) \to (0,\infty)$, $w(x) = \frac{1}{x}$, then we get from (3.5) that

$$\begin{aligned}
&\int_a^b \frac{|f(x)|^\lambda}{x} dx \\
&\le \frac{1}{2} B\left((\lambda+1)/2, (\lambda+1)/2\right) \left[\ln\left(\frac{b}{a}\right)\right]^\lambda \int_a^b x^{\lambda-1} |f'(x)|^\lambda dx \\
&\le \frac{1}{2^\lambda} \left[\ln\left(\frac{b}{a}\right)\right]^\lambda \int_a^b x^{\lambda-1} |f'(x)|^\lambda dx.
\end{aligned} \tag{3.7}$$

b). If we take $w : [a,b] \to (0,\infty)$, $w(x) = \exp(\alpha x)$, with $\alpha \in \mathbb{R}, \alpha \neq 0$, then we get from (3.5) that

$$\begin{aligned}
&\int_a^b |f(x)|^\lambda \exp(\alpha x)\, dx \\
&\le \frac{1}{2} B\left((\lambda+1)/2, (\lambda+1)/2\right) \left(\frac{\exp(\alpha b) - \exp(\alpha a)}{\alpha}\right)^\lambda \\
&\times \int_a^b |f'(x)|^\lambda \exp\left[-\alpha(\lambda-1)x\right] dx \\
&\le \frac{1}{2^\lambda} \left(\frac{\exp(\alpha b) - \exp(\alpha a)}{\alpha}\right)^\lambda \int_a^b |f'(x)|^\lambda \exp\left[-\alpha(\lambda-1)x\right] dx.
\end{aligned} \tag{3.8}$$

4. Some Weighted Inequalities of Trapezoid Type

We have:

Theorem 3. *Assume that* $w : [a,b] \to (0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\, ds = 1$ *and* $g \in C^1([a,b],\mathbb{R})$, *then*

$$\begin{aligned}
&\left| \int_a^b \frac{w(x) + w(a+b-x)}{2} g(x)\, dx - \frac{g(a)+g(b)}{2} \right| \\
&\le \frac{1}{2^{1+1/\lambda}} \left[B\left((\lambda+1)/2, (\lambda+1)/2\right)\right]^{1/\lambda} \\
&\times \left(\int_a^b |g'(x) - g'(a+b-x)|^\lambda \frac{1}{w(x)^{\lambda-1}} dx \right)^{1/\lambda} \\
&\le \frac{1}{4} \left(\int_a^b |g'(x) - g'(a+b-x)|^\lambda \frac{1}{w(x)^{\lambda-1}} dx \right)^{1/\lambda}.
\end{aligned} \tag{4.1}$$

In particular, if w is symmetrical, i.e. $w(a+b-t)=w(t)$ for any $t\in[a,b]$, then we have

$$\left|\int_a^b w(t)\,g(t)\,dt-\frac{g(a)+g(b)}{2}\right| \tag{4.2}$$
$$\leq\frac{1}{2^{1+1/\lambda}}\left[B\left((\lambda+1)/2,(\lambda+1)/2\right)\right]^{1/\lambda}$$
$$\times\left(\int_a^b\left|g'(x)-g'(a+b-x)\right|^{\lambda}\frac{1}{w(x)^{\lambda-1}}dx\right)^{1/\lambda}$$
$$\leq\frac{1}{4}\left(\int_a^b\left|g'(x)-g'(a+b-x)\right|^{\lambda}\frac{1}{w(x)^{\lambda-1}}dx\right)^{1/\lambda}.$$

Proof. Consider the function

$$f(x):=\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2},\ x\in[a,b],$$

we have $f(a)=f(b)=0$ and by (3.6) we have

$$\int_a^b\left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^{\lambda}w(x)\,dx \tag{4.3}$$
$$\leq\frac{1}{2^{\lambda+1}}B\left((\lambda+1)/2,(\lambda+1)/2\right)$$
$$\times\int_a^b\left|g'(x)-g'(a+b-x)\right|^{\lambda}\frac{1}{w(x)^{\lambda-1}}dx$$
$$\leq\frac{1}{4^{\lambda}}\int_a^b\left|g'(x)-g'(a+b-x)\right|^{\lambda}\frac{1}{w(x)^{\lambda-1}}dx.$$

By Jensen's integral inequality we have

$$\int_a^b\left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^{\lambda}w(x)\,dx$$
$$\geq\left|\int_a^b\left(\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right)w(x)\,dx\right|^{\lambda},$$

namely

$$\int_a^b\left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^{\lambda}w(x)\,dx \tag{4.4}$$
$$\geq\left|\int_a^b\frac{g(x)+g(a+b-x)}{2}w(x)\,dx-\frac{g(a)+g(b)}{2}\int_a^b w(x)\,dx\right|^{\lambda}.$$

Observe that, by the change of variable $s=a+b-x$, $x\in[a,b]$ we have that

$$\int_a^b g(a+b-x)\,w(x)\,dx=\int_a^b g(s)\,w(a+b-s)\,ds$$

and then

$$\int_a^b\frac{g(x)+g(a+b-x)}{2}w(x)\,dx=\int_a^b\frac{w(x)+w(a+b-x)}{2}g(x)\,dx.$$

Therefore (4.4) can be written as

$$\begin{aligned}&\int_a^b \left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^\lambda w(x)\,dx\\ &\geq \left|\int_a^b \frac{w(x)+w(a+b-x)}{2}g(x)\,dx-\frac{g(a)+g(b)}{2}\int_a^b w(x)\,dx\right|^\lambda.\end{aligned}$$

By making use of (4.3) we derive

$$\begin{aligned}&\left|\int_a^b \frac{w(x)+w(a+b-x)}{2}g(x)\,dx-\frac{g(a)+g(b)}{2}\int_a^b w(x)\,dx\right|^\lambda\\ &\leq \int_a^b \left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^\lambda w(x)\,dx\\ &\leq \frac{1}{2^{\lambda+1}}B((\lambda+1)/2,(\lambda+1)/2)\\ &\times \int_a^b |g'(x)-g'(a+b-x)|^\lambda \frac{1}{w(x)^{\lambda-1}}dx\\ &\leq \frac{1}{4^\lambda}\int_a^b |g'(x)-g'(a+b-x)|^\lambda \frac{1}{w(x)^{\lambda-1}}dx.\end{aligned}$$

By taking the power $1/\lambda$ in this inequality, we get the desired result (4.1). □

In 1906, Fejér [8], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 4 (Fejér's Inequality)**.** *Consider the integral* $\int_a^b h(x)\,w(x)\,dx$*, where* h *is a convex function in the interval* (a,b) *and* w *is a positive function in the same interval such that*

$$w(x)=w(a+b-x)\,,\ \textit{for any } x\in[a,b]$$

i.e., $y=w(x)$ *is a symmetric curve with respect to the straight line which contains the point* $\left(\frac{1}{2}(a+b),0\right)$ *and is normal to the* x*-axis. Under those conditions the following inequalities are valid:*

$$h\left(\frac{a+b}{2}\right)\leq\frac{1}{\int_a^b w(x)\,dx}\int_a^b h(x)\,w(x)\,dx\leq\frac{h(a)+h(b)}{2}. \tag{4.5}$$

If h *is concave on* (a,b)*, then the inequalities reverse in (4.5).*

Remark 4. *If* $g:[a,b]\to\mathbb{R}$ *is differentiable convex and* $g'_-(b)$ *and* $g'_+(a)$ *are finite and* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds=1$ *and symmetrical,* *then by (4.2) we get the following reverse of the second inequality in (4.5)*

$$
\begin{aligned}
0 &\leq \frac{g(a)+g(b)}{2} - \int_a^b w(t)\, g(t)\, dt \\
&\leq \frac{1}{2^{1+1/\lambda}} \left[g'_-(b) - g'_+(a) \right] \left[B\left((\lambda+1)/2, (\lambda+1)/2 \right) \right]^{1/\lambda} \\
&\quad \times \left(\int_a^b \frac{1}{w(x)^{\lambda-1}} dx \right)^{1/\lambda} \\
&\leq \frac{1}{4} \left[g'_-(b) - g'_+(a) \right] \left(\int_a^b \frac{1}{w(x)^{\lambda-1}} dx \right)^{1/\lambda},
\end{aligned} \tag{4.6}
$$

provided $\int_a^b \frac{1}{w(x)^{\lambda-1}} dx < \infty$.

Another trapezoid type weighted inequality is as follows:

Theorem 5. *Assume that* $w : [a,b] \to (0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\, ds = 1$ *and* $g \in C^1([a,b],\mathbb{R})$*, then*

$$
\begin{aligned}
&\left| \frac{g(a)\left[b - E(w;[a,b]) \right] + g(b)\left[E(w;[a,b]) - a \right]}{b-a} - \int_a^b g(t)\, w(t)\, dt \right| \\
&\leq \frac{1}{2^{1/\lambda}} \left[B\left((\lambda+1)/2, (\lambda+1)/2 \right) \right]^{1/\lambda} \\
&\quad \times \left(\int_a^b \left| g'(x) - \frac{g(b)-g(a)}{b-a} \right|^{\lambda} \frac{1}{w(x)^{\lambda-1}} dx \right)^{1/\lambda} \\
&\leq \frac{1}{2} \left(\int_a^b \left| g'(x) - \frac{g(b)-g(a)}{b-a} \right|^{\lambda} \frac{1}{w(x)^{\lambda-1}} dx \right)^{1/\lambda},
\end{aligned} \tag{4.7}
$$

where

$$
E(w;[a,b]) := \int_a^b t w(t)\, dt.
$$

Proof. If $g \in C^1([a,b],\mathbb{R})$, then by taking

$$
f(x) := g(x) - \frac{g(a)(b-x) + g(b)(x-a)}{b-a}, \; x \in [a,b]
$$

we have $f(a) = f(b) = 0$ and by (3.6) we have

$$
\begin{aligned}
&\int_a^b \left| g(x) - \frac{g(a)(b-x) + g(b)(x-a)}{b-a} \right|^{\lambda} w(x)\, dx \\
&\leq \frac{1}{2} B\left((\lambda+1)/2, (\lambda+1)/2 \right) \\
&\quad \times \int_a^b \left| g'(x) - \frac{g(b)-g(a)}{b-a} \right|^{\lambda} \frac{1}{w(x)^{\lambda-1}} dx \\
&\leq \frac{1}{2^{\lambda}} \int_a^b \left| g'(x) - \frac{g(b)-g(a)}{b-a} \right|^{\lambda} \frac{1}{w(x)^{\lambda-1}} dx.
\end{aligned} \tag{4.8}
$$

By Jensen's integral inequality we have

$$\begin{aligned}
&\int_a^b \left| g(x) - \frac{g(a)(b-x)+g(b)(x-a)}{b-a} \right|^{\lambda} w(x)\,dx \\
&\geq \left| \int_a^b \left(g(x) - \frac{g(a)(b-x)+g(b)(x-a)}{b-a} \right) w(x)\,dx \right|^{\lambda} \\
&= \left| \frac{g(a)[b-E(w;[a,b])] + g(b)[E(w;[a,b])-a]}{b-a} \right. \\
&\quad \left. - \int_a^b g(t)\,w(t)\,dt \right|^{\lambda},
\end{aligned}$$

then by (4.8) we get

$$\begin{aligned}
&\left| \frac{g(a)[b-E(w;[a,b])] + g(b)[E(w;[a,b])-a]}{b-a} \right. \\
&\quad \left. - \int_a^b g(t)\,w(t)\,dt \right|^{\lambda} \\
&\leq \int_a^b \left| g(x) - \frac{g(a)(b-x)+g(b)(x-a)}{b-a} \right|^{\lambda} w(x)\,dx \\
&\leq \frac{1}{2} B((\lambda+1)/2, (\lambda+1)/2) \\
&\quad \times \int_a^b \left| g'(x) - \frac{g(b)-g(a)}{b-a} \right|^{\lambda} \frac{1}{w(x)^{\lambda-1}}\,dx \\
&\leq \frac{1}{2^{\lambda}} \int_a^b \left| g'(x) - \frac{g(b)-g(a)}{b-a} \right|^{\lambda} \frac{1}{w(x)^{\lambda-1}}\,dx.
\end{aligned}$$

By taking the power $1/\lambda$ in this inequality, we get the desired result (4.7). □

The case of convex function is as follows:

Corollary 2. *If $g : [a,b] \to \mathbb{R}$ is continuously differentiable convex and $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds = 1$, then*

$$\begin{aligned}
0 &\leq \frac{g(a)[b-E(w;[a,b])] + g(b)[E(w;[a,b])-a]}{b-a} \\
&\quad - \int_a^b g(t)\,w(t)\,dt \\
&\leq \frac{1}{2^{1/\lambda}} [B((\lambda+1)/2, (\lambda+1)/2)]^{1/\lambda} \\
&\quad \times \left(\int_a^b \left| g'(x) - \frac{g(b)-g(a)}{b-a} \right|^{\lambda} \frac{1}{w(x)^{\lambda-1}}\,dx \right)^{1/\lambda} \\
&\leq \frac{1}{2} \left(\int_a^b \left| g'(x) - \frac{g(b)-g(a)}{b-a} \right|^{\lambda} \frac{1}{w(x)^{\lambda-1}}\,dx \right)^{1/\lambda}.
\end{aligned} \tag{4.9}$$

The positivity follows by the fact that, for a convex function g on $[a,b]$ we have

$$\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\geq g(t)$$

for any $t\in[a,b]$. The rest is obvious by Theorem 5.

Remark 5. *With the assumption of Corollary 2 we have the simpler inequality*

$$\begin{aligned} 0 &\leq \frac{g(a)[b-E(w;[a,b])]+g(b)[E(w;[a,b])-a]}{b-a} \\ &\quad -\int_a^b g(t)\,w(t)\,dt \\ &\leq \frac{1}{2^{1/\lambda}}[B((\lambda+1)/2,(\lambda+1)/2)]^{1/\lambda} \\ &\quad \times \max_{x\in(a,b)}\left|g'(x)-\frac{g(b)-g(a)}{b-a}\right|\left(\int_a^b \frac{1}{w(x)^{\lambda-1}}dx\right)^{1/\lambda} \\ &\leq \frac{1}{2}\max_{x\in(a,b)}\left|g'(x)-\frac{g(b)-g(a)}{b-a}\right|\left(\int_a^b \frac{1}{w(x)^{\lambda-1}}dx\right)^{1/\lambda}, \end{aligned} \tag{4.10}$$

provided $\int_a^b \frac{1}{w(x)^{\lambda-1}}dx<\infty$.

5. Some Inequalities for the Weighted Čebyšev Functional

Consider now the *weighted Čebyšev functional*

$$C_w(f,g):=\int_a^b w(t)f(t)g(t)\,dt-\int_a^b w(t)f(t)\,dt\int_a^b w(t)g(t)\,dt \tag{5.1}$$

where $f,\ g,\ w:[a,b]\to\mathbb{R}$ and $w(t)\geq 0$ for a.e. $t\in[a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt=1$.

Theorem 6. *Assume that* $w:[a,b]\to[0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds=1$, *then for* $\lambda>1$

$$\begin{aligned} |C_w(f,g)| &\leq \frac{1}{2^{1/\lambda}}[B((\lambda+1)/2,(\lambda+1)/2)]^{1/\lambda}\left(\int_a^b \frac{|g'(x)|^{\frac{\lambda}{\lambda-1}}}{w(x)^{\frac{1}{\lambda-1}}}dx\right)^{\frac{\lambda-1}{\lambda}} \\ &\quad \times\left(\int_a^b\left|\int_a^b f(s)w(s)\,ds-f(x)\right|^{\lambda}w(x)\,dx\right)^{1/\lambda} \\ &\leq \frac{1}{2}\left(\int_a^b \frac{|g'(x)|^{\frac{\lambda}{\lambda-1}}}{w(x)^{\frac{1}{\lambda-1}}}dx\right)^{\frac{\lambda-1}{\lambda}} \\ &\quad \times\left(\int_a^b\left|\int_a^b f(s)w(s)\,ds-f(x)\right|^{\lambda}w(x)\,dx\right)^{1/\lambda}, \end{aligned} \tag{5.2}$$

provided that the involved integrals are finite.

Proof. Integrating by parts, we have

$$\int_a^b \left(\int_a^x f(t) w(t) dt - \int_a^x w(s) ds \int_a^b f(s) w(s) ds \right) g'(x) dx$$
$$= \left[\left(\int_a^x f(t) w(t) dt - \int_a^x w(s) ds \int_a^b f(s) w(s) ds \right) g(x) \right|_a^b$$
$$- \int_a^b g(x) \left(f(x) w(x) - w(x) \int_a^b f(s) w(s) ds \right) dx$$
$$= - \int_a^b f(x) g(x) w(x) dx + \int_a^b f(s) w(s) ds \int_a^b g(x) w(x) dx,$$

which gives that

$$C_w(f,g) = \int_a^b \left(\int_a^x w(s) ds \int_a^b f(s) w(s) ds - \int_a^x f(t) w(t) dt \right) g'(x) dx. \tag{5.3}$$

If we take

$$h(x) := \int_a^x w(s) ds \int_a^b f(s) w(s) ds - \int_a^x f(t) w(t) dt, \;\; x \in [a,b]$$

we observe that $h(a) = h(b) = 0$ and $h \in C^1([a,b], \mathbb{C})$.

If we use (3.6) for $f = h$ defined above, then we get

$$\int_a^b \left| \int_a^x w(s) ds \int_a^b f(s) w(s) ds - \int_a^x f(t) w(t) dt \right|^\lambda w(x) dx$$
$$\leq \frac{1}{2} B((\lambda+1)/2, (\lambda+1)/2)$$
$$\times \int_a^b \frac{\left| w(x) \int_a^b f(s) w(s) ds - f(x) w(x) \right|^\lambda}{w(x)^{\lambda-1}} dx$$
$$\leq \frac{1}{2^\lambda} \int_a^b \frac{\left| w(x) \int_a^b f(s) w(s) ds - f(x) w(x) \right|^\lambda}{w(x)^{\lambda-1}} dx,$$

namely

$$\int_a^b \left| \int_a^x w(s) ds \int_a^b f(s) w(s) ds - \int_a^x f(t) w(t) dt \right|^\lambda w(x) dx$$
$$\leq \frac{1}{2} B((\lambda+1)/2, (\lambda+1)/2)$$
$$\times \int_a^b \left| \int_a^b f(s) w(s) ds - f(x) \right|^\lambda w(x) dx$$
$$\leq \frac{1}{2^\lambda} \int_a^b \left| \int_a^b f(s) w(s) ds - f(x) \right|^\lambda w(x) dx.$$

This implies that

$$\left(\int_a^b \left|\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right|^{\lambda} w(x)\,dx\right)^{1/\lambda} \tag{5.4}$$
$$\leq \frac{1}{2^{1/\lambda}}\left[B\left((\lambda+1)/2,(\lambda+1)/2\right)\right]^{1/\lambda}$$
$$\times \left(\int_a^b \left|\int_a^b f(s)\,w(s)\,ds - f(x)\right|^{\lambda} w(x)\,dx\right)^{1/\lambda}$$
$$\leq \frac{1}{2}\left(\int_a^b \left|\int_a^b f(s)\,w(s)\,ds - f(x)\right|^{\lambda} w(x)\,dx\right)^{1/\lambda}.$$

If we use Hölder's weighted inequality for $p=\lambda>1$ and $q=\frac{\lambda}{\lambda-1}$, then we get

$$|C_w(f,g)|$$
$$= \left|\int_a^b \left(\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right) g'(x)\,dx\right|$$
$$= \left|\int_a^b \left(\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right) \frac{g'(x)}{w(x)} w(x)\,dx\right|$$
$$\leq \left|\int_a^b \left|\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right|^{\lambda} w(x)\,dx\right|^{1/\lambda}$$
$$\times \left(\int_a^b \left|\frac{g'(x)}{w(x)}\right|^{\frac{\lambda}{\lambda-1}} w(x)\,dx\right)^{\frac{\lambda-1}{\lambda}}$$
$$= \left|\int_a^b \left|\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right|^{\lambda} w(x)\,dx\right|^{1/\lambda}$$
$$\times \left(\int_a^b \frac{|g'(x)|^{\frac{\lambda}{\lambda-1}}}{w(x)^{\frac{1}{\lambda-1}}}\,dx\right)^{\frac{\lambda-1}{\lambda}}$$

and by (5.4) we derive the desired result (5.2). □

Remark 6. *For $\lambda = 2$ we derive*

$$|C_w(f,g)| \tag{5.5}$$
$$\leq \frac{\pi^{1/2}}{4}\left(\int_a^b \frac{|g'(x)|^2}{w(x)}\,dx\right)^{1/2}\left(\int_a^b \left|\int_a^b f(s)\,w(s)\,ds - f(x)\right|^2 w(x)\,dx\right)^{1/2}$$
$$\leq \frac{1}{2}\left(\int_a^b \frac{|g'(x)|^2}{w(x)}\,dx\right)^{1/2}\left(\int_a^b \left|\int_a^b f(s)\,w(s)\,ds - f(x)\right|^2 w(x)\,dx\right)^{1/2},$$

provided that the involved integrals are finite.

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