WEIGHTED INTEGRAL INEQUALITIES OF BEESACK TYPE RELATED TO WIRTINGER'S RESULT

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Abstract. In this paper we show among others that, if $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds = 1$ and $f \in C^1([a,b],\mathbb{R})$ satisfies the conditions $f(a) = f(b) = 0$, then

$$\int_a^b |f(x)|^{2k} w(x)\,dx \le \frac{1}{2k-1}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\int_a^b \frac{|f'(x)|^{2k}}{[w(x)]^{2k-1}}dx$$

for $k \ge 1$ and provided that the involved integrals are finite. Applications for trapezoid and Grüss' type inequalities are also given.

1. Introduction

It is well known that, see for instance [6], or [10], if $u \in C^1([a,b],\mathbb{R})$ satisfies $u(a) = u(b) = 0$, then

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t) = K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K \in \mathbb{R}$.

If $u \in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a) = 0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t) = L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L \in \mathbb{R}$.

In the recent paper [7] we obtained the following weighted version of Wirtinger inequalities (1.1) and (1.2):

Theorem 1. *Assume that $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ and $f \in C^1([a,b],\mathbb{C})$ is a function with complex values and $f(a) = f(b) = 0$, then*

$$\int_a^b |f(t)|^2 w(t)\,dt \le \frac{1}{\pi^2}\left(\int_a^b w(s)\,ds\right)^2\int_a^b \frac{|f'(t)|^2}{w(t)}dt. \tag{1.3}$$

The equality holds in (1.3) iff

$$f(t) = K\sin\left[\frac{\pi\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\right],\ K \in \mathbb{C}.$$

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If $f(a)=0$, then

$$\int_a^b |f(t)|^2 w(t)\,dt \le \frac{4}{\pi^2}\left(\int_a^b w(s)\,ds\right)^2 \int_a^b \frac{|f'(t)|^2}{w(t)}dt \tag{1.4}$$

with equality iff

$$f(t) = K\sin\left[\frac{\pi\int_a^t w(s)\,ds}{2\int_a^b w(s)\,ds}\right],\ K\in\mathbb{C}.$$

For some related Wirtinger type integral inequalities see [2], [3], [6] and [9]-[13].

In [4, p. 59], P. R. Beesack proved the following result:

Theorem 2. *If $u\in C^1[0,\pi]$ with $u(0)=u(\pi)=0$, then*

$$\int_0^\pi u^{2k}(x)\,dx \le \frac{1}{2k-1}\left[k\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\int_0^\pi [u'(x)]^{2k}\,dx \tag{1.5}$$

for $k\ge 1$.

If we change the variable $s=\frac{x}{\pi}$, $x=\pi s$, $s\in[0,1]$ then

$$\int_0^\pi u^{2k}(x)\,dx = \pi\int_0^1 u^{2k}(\pi s)\,ds,\quad \int_0^\pi [u'(x)]^{2k}\,dx = \pi\int_0^1 [u'(\pi s)]^{2k}\,ds,$$

and by (1.5) we get

$$\int_0^1 u^{2k}(\pi s)\,ds \le \frac{1}{2k-1}\left[k\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\int_0^1 [u'(\pi s)]^{2k}\,ds. \tag{1.6}$$

Consider the function $v(s)=u(\pi s)$, $s\in[0,1]$. Then $v(0)=v(1)=0$,

$$v'(s)=\pi u'(\pi s),\ s\in(0,1)$$

and by (1.6) we derive

$$\begin{aligned}\int_0^1 v^{2k}(s)\,ds &\le \frac{1}{2k-1}\left[k\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\int_0^1\left[\frac{v'(s)}{\pi}\right]^{2k}ds\\ &= \frac{1}{2k-1}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\int_0^1 [v'(s)]^{2k}\,ds,\end{aligned}$$

therefore we can write Theorem 2 as follows:

Theorem 3. *If $v\in C^1[0,1]$ with $v(0)=v(1)=0$, then*

$$\int_0^1 v^{2k}(t)\,dt \le \frac{1}{2k-1}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\int_0^1 [v'(t)]^{2k}\,dt \tag{1.7}$$

for $k\ge 1$.

If we use (1.7) for $v(t)=g((1-t)a+tb)$, $t\in[0,1]$, we get

$$\begin{aligned}\int_0^1 g^{2k}((1-t)a+tb)\,dt &\le \frac{1}{2k-1}(b-a)^{2k}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\\ &\quad\times\int_0^1 [g'((1-t)a+tb)]^{2k}\,dt\end{aligned}$$

and by the change of variable $x=(1-t)a+tb$, we get

$$\int_a^b g^{2k}(x)\,dx \le \frac{1}{2k-1}(b-a)^{2k}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\int_a^b [g'(x)]^{2k}\,dx \tag{1.8}$$

for $k \geq 1$, provided that g is absolutely continuous on $[a,b]$ with the property that $g(a) = g(b) = 0$. We observe that for $k = 1$ we recapture inequality (1.1).

In this paper we show among others that, if $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds = 1$ and $f \in C^1([a,b],\mathbb{R})$ satisfies the conditions $f(a) = f(b) = 0$, then

$$\int_a^b |f(x)|^{2k} w(x)\,dx \leq \frac{1}{2k-1}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k} \int_a^b \frac{|f'(x)|^{2k}}{[w(x)]^{2k-1}}dx$$

for $k \geq 1$ and provided that the involved integrals are finite. Applications for trapezoid and Grüss' type inequalities are also given.

2. Some Simple Inequalities

We start to the following trapezoid type inequality:

Proposition 1. *Let $g \in C^1([a,b],\mathbb{R})$. Then*

$$\left|\frac{1}{b-a}\int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2}\right| \leq \frac{k}{2(2k-1)^{\frac{1}{2k}}\pi}\sin\left(\frac{\pi}{2k}\right)(b-a) \times \left(\frac{1}{b-a}\int_a^b [g'(t) - g'(a+b-t)]^{2k}\,dt\right)^{\frac{1}{2k}}, \tag{2.1}$$

for $k \geq 1$.

Proof. If $g \in C^1([a,b],\mathbb{R})$, then by taking

$$f(t) := \frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2},\ t \in [a,b]$$

we have $f(a) = f(b) = 0$ and by (1.8) we obtain

$$\int_a^b \left(\frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2}\right)^{2k} dt \leq \frac{1}{(2k-1)2^{2k}}(b-a)^{2k}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k} \times \int_a^b (g'(t) - g'(a+b-t))^{2k}\,dt. \tag{2.2}$$

By Jensen's inequality for the power $2k$ we have

$$\frac{1}{b-a}\int_a^b \left(\frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2}\right)^{2k} dt \geq \left|\frac{1}{b-a}\int_a^b \left(\frac{g(t)+g(a+b-t)}{2} - \frac{g(a)+g(b)}{2}\right)\right|^{2k} dt = \left|\frac{1}{b-a}\int_a^b g(t)\,dt - \frac{g(a)+g(b)}{2}\right|^{2k}.$$

Therefore by (2.2) we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right|^{2k}\\
&\leq \frac{1}{(2k-1)\,2^{2k}}(b-a)^{2k-1}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\\
&\times\int_a^b\left(g'(t)-g'(a+b-t)\right)^{2k}dt
\end{aligned}$$

and by taking the power $1/(2k)$ we obtain (2.1). □

Remark 1. *For $k=1$ we derive the inequality*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right| \\
&\leq \frac{1}{2\pi}(b-a)\left(\frac{1}{b-a}\int_a^b\left[g'(t)-g'(a+b-t)\right]^2dt\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.3}$$

Proposition 2. *Let $g\in C^1([a,b],\mathbb{R})$. Then*

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right| \\
&\leq \frac{k}{(2k-1)^{\frac{1}{2k}}\,\pi}\sin\left(\frac{\pi}{2k}\right)(b-a)\\
&\times\left(\frac{1}{b-a}\int_a^b\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^{2k}dt\right)^{\frac{1}{2k}}
\end{aligned} \tag{2.4}$$

for $k\geq 1$.

Proof. If $g\in C^1([a,b],\mathbb{C})$, then by taking

$$f(t):=g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a},\ t\in[a,b]$$

we have $f(a)=f(b)=0$ and by (1.8) we have

$$\begin{aligned}
&\int_a^b\left(g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right)^{2k}dt\\
&\leq \frac{1}{2k-1}(b-a)^{2k}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\\
&\times\int_a^b\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^{2k}dt.
\end{aligned} \tag{2.5}$$

By Jensen's inequality for the power $2k$ we have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\left|g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right|^{2k}dt\\&\geq\left|\frac{1}{b-a}\int_a^b\left(g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right)dt\right|^{2k}\\&=\left|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right|^{2k}.\end{aligned}$$

By (2.5) we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right|^{2k}\\&\leq\frac{1}{2k-1}(b-a)^{2k-1}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\\&\times\int_a^b\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^{2k}dt\end{aligned}$$

and by taking the power $1/(2k)$ we obtain (2.4). □

Remark 2. *For $k=1$ we derive the inequality*

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right|\\&\leq\frac{1}{\pi}(b-a)\left(\frac{1}{b-a}\int_a^b\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^2dt\right)^{\frac{1}{2}}.\end{aligned}\tag{2.6}$$

We also have:

Proposition 3. *Let g be integrable on $[a,b]$. Then*

$$\begin{aligned}&\left|\frac{b+a}{2}\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt\right|\\&\leq\frac{k}{(2k-1)^{1/(2k)}\pi}(b-a)^2\sin\left(\frac{\pi}{2k}\right)\\&\times\left(\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^{2k}dt\right)^{1/(2k)}.\end{aligned}\tag{2.7}$$

Proof. Assume that $g:[a,b]\to\mathbb{C}$ is continuous, then by taking

$$f(t):=\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds,\ t\in[a,b]$$

we have $f(a)=f(b)=0$, and by (1.8) we have

$$\int_a^b \left|\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^{2k}dt \tag{2.8}$$
$$\le \frac{1}{2k-1}(b-a)^{2k}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}$$
$$\times\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^{2k}dt.$$

Observe that, integrating by parts, we have

$$\int_a^b\left(\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right)dt$$
$$=\int_a^b\left(\int_a^t g(s)\,ds\right)dt-\frac{b-a}{2}\int_a^b g(s)\,ds$$
$$=b\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt-\frac{b-a}{2}\int_a^b g(s)\,ds$$
$$=\frac{b+a}{2}\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt.$$

By Jensen's inequality we also have

$$\frac{1}{b-a}\int_a^b\left|\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^{2k}dt$$
$$\ge\left|\frac{1}{b-a}\int_a^b\left(\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right)\right|^{2k}$$
$$=\left|\frac{1}{b-a}\left(\frac{b+a}{2}\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt\right)\right|^{2k},$$

which gives that

$$\left|\frac{b+a}{2}\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt\right|^{2k}$$
$$\le(b-a)^{2k-1}\int_a^b\left|\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^{2k}dt.$$

By (2.8) we get

$$\begin{aligned}
&\left|\frac{b+a}{2}\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt\right|^{2k} \tag{2.9}\\
&\leq (b-a)^{2k-1}\int_a^b\left|\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^{2k}dt\\
&\leq \frac{1}{2k-1}(b-a)^{4k-1}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\\
&\times\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^{2k}dt.
\end{aligned}$$

By taking the power $1/(2k)$ we derive (2.7). □

Remark 3. *For $k=1$ we derive the inequality*

$$\begin{aligned}
&\left|\frac{b+a}{2}\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt\right| \tag{2.10}\\
&\leq \frac{1}{\pi}(b-a)^2\left(\frac{1}{b-a}\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2dt\right)^{1/2}\\
&=\frac{1}{\pi}(b-a)^2\left[\frac{1}{b-a}\int_a^b g^2(t)\,dt-\left(\frac{1}{b-a}\int_a^b g(s)\,ds\right)^2\right]^{1/2},
\end{aligned}$$

provided that g is integrable on $[a,b]$.

3. Weighted Inequalities

We have the following main result:

Theorem 4. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b). If $f\in C^1([a,b],\mathbb{R})$ is a function with the properties $f(a)=f(b)=0$, then*

$$\begin{aligned}
\int_a^b |f(x)|^{\lambda}h'(x)\,dx &\leq \frac{1}{2k-1}[h(b)-h(a)]^{2k}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k} \tag{3.1}\\
&\times\int_a^b\frac{|f'(x)|^{2k}}{[h'(x)]^{2k-1}}dx,
\end{aligned}$$

provided that the last integral in (3.1) is finite.

Proof. We write the inequality (1.8) for the function $g=f\circ h^{-1}$ on the interval $[h(a),h(b)]$ to get

$$\begin{aligned}
\int_{h(a)}^{h(b)}\left|\left(f\circ h^{-1}\right)(z)\right|^{2k}dz &\leq \frac{1}{2k-1}[h(b)-h(a)]^{2k}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k} \tag{3.2}\\
&\times\int_{h(a)}^{h(b)}\left|\left(f\circ h^{-1}\right)'(z)\right|^{2k}dz
\end{aligned}$$

since $\left(f\circ h^{-1}\right)(h(a))=f(a)=0$ and $\left(f\circ h^{-1}\right)(h(b))=f(b)=0$.

If $f:[c,d]\to\mathbb{R}$ is absolutely continuous on $[c,d]$, then $f\circ h^{-1}:[h(c),h(d)]\to\mathbb{R}$ is absolutely continuous on $[h(c),h(d)]$ and using the chain rule and the derivative of inverse functions we have

$$\left(f\circ h^{-1}\right)'(z)=\left(f'\circ h^{-1}\right)(z)\left(h^{-1}\right)'(z)=\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)} \tag{3.3}$$

for almost every (a.e.) $z\in[h(c),h(d)]$.

Using the inequality (3.2) we then get

$$\int_{h(a)}^{h(b)}\left|\left(f\circ h^{-1}\right)(z)\right|^{2k}dz\le\frac{1}{2k-1}[h(b)-h(a)]^{2k}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k} \times\int_{h(a)}^{h(b)}\left|\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right|^{2k}dz. \tag{3.4}$$

Observe also that, by the change of variable $x=h^{-1}(z)$, $z\in[h(a),h(b)]$, we have $z=h(x)$ that gives $dz=h'(x)\,dx$ and

$$\int_{h(a)}^{h(b)}\left|\left(f\circ h^{-1}\right)(z)\right|^{2\lambda}dz=\int_a^b|f(x)|^{2k}h'(x)\,dx.$$

We also have

$$\int_{h(a)}^{h(b)}\left|\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right|^{2k}dz=\int_a^b\left|\frac{f'(x)}{h'(x)}\right|^{2k}h'(x)\,dx=\int_a^b\frac{|f'(x)|^{2k}}{[h'(x)]^{2k-1}}dx.$$

Therefore, by (3.4) we derive the desired result (3.1). □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Corollary 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and $f\in C^1([a,b],\mathbb{R})$ satisfies the condition $f(a)=f(b)=0$, then*

$$\int_a^b|f(x)|^{2k}w(x)\,dx\le\frac{1}{2k-1}\left[\int_a^b w(s)\,ds\right]^{2k}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k} \times\int_a^b\frac{|f'(x)|^{2k}}{[w(x)]^{2k-1}}dx. \tag{3.5}$$

If $\int_a^b w(s)\,ds=1$, then we have the simpler inequality

$$\int_a^b|f(x)|^{2k}w(x)\,dx\le\frac{1}{2k-1}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\int_a^b\frac{|f'(x)|^{2k}}{[w(x)]^{2k-1}}dx. \tag{3.6}$$

Some examples are as follows:

a). If we take $w:[a,b]\subset(0,\infty)\to(0,\infty)$, $w(x)=\frac{1}{x}$, then we get from (3.5) that

$$\int_a^b \frac{|f(x)|^{2k}}{x}dx \le \frac{1}{2k-1}\left[\ln\left(\frac{b}{a}\right)\right]^{2k}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k} \times \int_a^b x^{2k-1}|f'(x)|^{2k}dx \tag{3.7}$$

for $k\ge 1$.

b). If we take $w:[a,b]\to(0,\infty)$, $w(x)=\exp(\alpha x)$, with $\alpha\in\mathbb{R},\alpha\neq 0$, then we get from (3.5) that

$$\begin{aligned}&\int_a^b |f(x)|^{2k}\exp(\alpha x)\,dx\\ &\le \frac{1}{2k-1}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}\left(\frac{\exp(\alpha b)-\exp(\alpha a)}{\alpha}\right)^{2k}\\ &\times\int_a^b |f'(x)|^{2k}\exp[-\alpha(2k-1)x]\,dx\end{aligned} \tag{3.8}$$

for $k\ge 1$.

4. Some Weighted Inequalities of Trapezoid Type

We have:

Theorem 5. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$ and $g\in C^1([a,b],\mathbb{R})$, then*

$$\begin{aligned}&\left|\int_a^b \frac{w(x)+w(a+b-x)}{2}g(x)\,dx-\frac{g(a)+g(b)}{2}\right|\\ &\le \frac{k}{2(2k-1)^{1/(2k)}\pi}\sin\left(\frac{\pi}{2k}\right)\\ &\times\left(\int_a^b |g'(x)-g'(a+b-x)|^{2k}[w(x)]^{2k-1}dx\right)^{1/(2k)},\end{aligned} \tag{4.1}$$

for $k\ge 1$.

In particular, if w is symmetrical, i.e. $w(a+b-t)=w(t)$ for any $t\in[a,b]$, then we have

$$\begin{aligned}&\left|\int_a^b w(x)\,g(x)\,dx-\frac{g(a)+g(b)}{2}\right|\\ &\le \frac{k}{2(2k-1)^{1/(2k)}\pi}\sin\left(\frac{\pi}{2k}\right)\\ &\times\left(\int_a^b |g'(x)-g'(a+b-x)|^{2k}[w(x)]^{2k-1}dx\right)^{1/(2k)}.\end{aligned} \tag{4.2}$$

Proof. Consider the function

$$f(x):=\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2},\ x\in[a,b],$$

we have $f(a)=f(b)=0$ and by (3.6) we have

$$\int_a^b \left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^{2k} w(x)\,dx \tag{4.3}$$
$$\leq \frac{1}{(2k-1)\,2^{2k}}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}$$
$$\times \int_a^b \left|g'(x)-g'(a+b-x)\right|^{2k}[w(x)]^{2k-1}\,dx.$$

By Jensen's integral inequality we have

$$\int_a^b \left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^{2k} w(x)\,dx$$
$$\geq \left|\int_a^b \left(\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right) w(x)\,dx\right|^{2k},$$

namely

$$\int_a^b \left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^{2k} w(x)\,dx \tag{4.4}$$
$$\geq \left|\int_a^b \frac{g(x)+g(a+b-x)}{2}w(x)\,dx-\frac{g(a)+g(b)}{2}\right|^{2k}.$$

Observe that, by the change of variable $s=a+b-x$, $x\in[a,b]$ we have that

$$\int_a^b g(a+b-x)\,w(x)\,dx=\int_a^b g(s)\,w(a+b-s)\,ds$$

and then

$$\int_a^b \frac{g(x)+g(a+b-x)}{2}w(x)\,dx=\int_a^b \frac{w(x)+w(a+b-x)}{2}g(x)\,dx.$$

Therefore (4.4) can be written as

$$\int_a^b \left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^{2k} w(x)\,dx \tag{4.5}$$
$$\geq \left|\int_a^b \frac{w(x)+w(a+b-x)}{2}g(x)\,dx-\frac{g(a)+g(b)}{2}\right|^{2k}.$$

By utilising (4.3) and (4.5) we derive

$$\left|\int_a^b \frac{w(x)+w(a+b-x)}{2}g(x)\,dx-\frac{g(a)+g(b)}{2}\right|^{2k}$$
$$\leq \int_a^b \left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^{2k} w(x)\,dx$$
$$\leq \frac{1}{(2k-1)\,2^{2k}}\left[\frac{k}{\pi}\sin\left(\frac{\pi}{2k}\right)\right]^{2k}$$
$$\times \int_a^b \left|g'(x)-g'(a+b-x)\right|^{2k}[w(x)]^{2k-1}\,dx.$$

By taking the power $1/(2k)$ we get the desired inequality (4.1). □

In 1906, Fejér [8], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 6 (Fejér's Inequality)**.** *Consider the integral* $\int_a^b h(x)\,w(x)\,dx$*, where* h *is a convex function in the interval* (a,b) *and* w *is a positive function in the same interval such that*

$$w(x) = w(a+b-x)\,, \text{ for any } x \in [a,b]$$

i.e., $y = w(x)$ *is a symmetric curve with respect to the straight line which contains the point* $\left(\frac{1}{2}(a+b),0\right)$ *and is normal to the* x*-axis. Under those conditions the following inequalities are valid:*

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{\int_a^b w(x)\,dx}\int_a^b h(x)\,w(x)\,dx \leq \frac{h(a)+h(b)}{2}. \tag{4.6}$$

If h *is concave on* (a,b)*, then the inequalities reverse in (4.6).*

Remark 4. *If* $g:[a,b]\to\mathbb{R}$ *is differentiable convex and* $g'_-(b)$ *and* $g'_+(a)$ *are finite and* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds = 1$ *and symmetrical, then by (4.2) we get the following reverse of the second inequality in (4.6)*

$$\begin{aligned} 0 &\leq \frac{g(a)+g(b)}{2} - \int_a^b w(x)\,g(x)\,dx \\ &\leq \frac{k}{2(2k-1)^{1/(2k)}\pi}\sin\left(\frac{\pi}{2k}\right) \\ &\times\left(\int_a^b \left|g'(x)-g'(a+b-x)\right|^{2k}[w(x)]^{2k-1}\,dx\right)^{1/(2k)} \\ &\leq \frac{k}{2(2k-1)^{1/(2k)}\pi}\sin\left(\frac{\pi}{2k}\right)\left[g'_-(b)-g'_+(a)\right]\left(\int_a^b [w(x)]^{2k-1}\,dx\right)^{1/(2k)} \end{aligned} \tag{4.7}$$

for $k \geq 1$.

Another trapezoid type weighted inequality is as follows:

Theorem 7. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds = 1$ *and* $g\in C^1([a,b],\mathbb{R})$*, then*

$$\begin{aligned} &\left|\frac{g(a)\,[b-E(w;[a,b])]+g(b)\,[E(w;[a,b])-a]}{b-a} - \int_a^b g(t)\,w(t)\,dt\right| \\ &\leq \frac{k}{(2k-1)^{1/(2k)}\pi}\sin\left(\frac{\pi}{2k}\right) \\ &\times\left(\int_a^b \left|g'(x)-\frac{g(b)-g(a)}{b-a}\right|^{2k}\frac{1}{[w(x)]^{2k-1}}dx\right)^{1/(2k)}, \end{aligned} \tag{4.8}$$

where

$$E(w;[a,b]) := \int_a^b t\,w(t)\,dt.$$

Proof. If $g \in C^1([a,b],\mathbb{R})$, then by taking

$$f(x) := g(x) - \frac{g(a)(b-x) + g(b)(x-a)}{b-a}, \ x \in [a,b]$$

we have $f(a) = f(b) = 0$ and by (3.6) we have

$$\begin{aligned} &\int_a^b \left| g(x) - \frac{g(a)(b-x) + g(b)(x-a)}{b-a} \right|^{2k} w(x)\,dx \\ &\le \frac{1}{2k-1} \left[\frac{k}{\pi} \sin\left(\frac{\pi}{2k}\right) \right]^{2k} \\ &\times \int_a^b \left| g'(x) - \frac{g(b) - g(a)}{b-a} \right|^{2k} \frac{1}{[w(x)]^{2k-1}} dx. \end{aligned} \tag{4.9}$$

By Jensen's integral inequality we have

$$\begin{aligned} &\int_a^b \left| g(x) - \frac{g(a)(b-x) + g(b)(x-a)}{b-a} \right|^{2k} w(x)\,dx \\ &\ge \left| \int_a^b \left(g(x) - \frac{g(a)(b-x) + g(b)(x-a)}{b-a} \right) w(x)\,dx \right|^{2k} \\ &= \left| \frac{g(a)[b - E(w;[a,b])] + g(b)[E(w;[a,b]) - a]}{b-a} \right. \\ &\left. - \int_a^b g(t) w(t)\,dt \right|^{2k}, \end{aligned}$$

then by (4.9) we get

$$\begin{aligned} &\left| \frac{g(a)[b - E(w;[a,b])] + g(b)[E(w;[a,b]) - a]}{b-a} \right. \\ &\left. - \int_a^b g(t) w(t)\,dt \right|^{2k} \\ &\le \int_a^b \left| g(x) - \frac{g(a)(b-x) + g(b)(x-a)}{b-a} \right|^{2k} w(x)\,dx \\ &\le \frac{1}{2k-1} \left[\frac{k}{\pi} \sin\left(\frac{\pi}{2k}\right) \right]^{2k} \\ &\times \int_a^b \left| g'(x) - \frac{g(b) - g(a)}{b-a} \right|^{2k} \frac{1}{[w(x)]^{2k-1}} dx. \end{aligned}$$

By taking the power $1/(2k)$ we deduce (4.8). □

The case of convex function is as follows:

Corollary 2. *If $g:[a,b]\to\mathbb{R}$ is continuously differentiable convex and $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$, then*

$$0\leq\frac{g(a)\left[b-E(w;[a,b])\right]+g(b)\left[E(w;[a,b])-a\right]}{b-a}-\int_a^b g(t)\,w(t)\,dt \tag{4.10}$$
$$\leq\frac{k}{(2k-1)^{1/(2k)}\pi}\sin\left(\frac{\pi}{2k}\right)\times\left(\int_a^b\left|g'(x)-\frac{g(b)-g(a)}{b-a}\right|^{2k}\frac{1}{[w(x)]^{2k-1}}dx\right)^{1/(2k)}.$$

The positivity follows by the fact that, for a convex function g on $[a,b]$ we have

$$\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\geq g(t)$$

for any $t\in[a,b]$. The rest is obvious by Theorem 7.

Remark 5. *With the assumption of Corollary 2 we have the simpler inequality*

$$0\leq\frac{g(a)\left[b-E(w;[a,b])\right]+g(b)\left[E(w;[a,b])-a\right]}{b-a}-\int_a^b g(t)\,w(t)\,dt \tag{4.11}$$
$$\leq\frac{k}{(2k-1)^{1/(2k)}\pi}\sin\left(\frac{\pi}{2k}\right)\times\max_{x\in(a,b)}\left|g'(x)-\frac{g(b)-g(a)}{b-a}\right|\left(\int_a^b\frac{1}{[w(x)]^{2k-1}}dx\right)^{1/(2k)},$$

provided $\int_a^b\frac{1}{[w(x)]^{2k-1}}dx<\infty$.

5. Some Inequalities for the Weighted Čebyšev Functional

Consider now the *weighted Čebyšev functional*

$$C_w(f,g):=\int_a^b w(t)f(t)g(t)\,dt-\int_a^b w(t)f(t)\,dt\int_a^b w(t)g(t)\,dt \tag{5.1}$$

where $f,\ g,\ w:[a,b]\to\mathbb{R}$ and $w(t)\geq0$ for a.e. $t\in[a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt=1$.

Theorem 8. *Assume that $w:[a,b]\to[0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$, then for $k\geq1$,*

$$|C_w(f,g)|\leq\frac{k}{(2k-1)^{1/(2k)}\pi}\sin\left(\frac{\pi}{2k}\right)\times\left(\int_a^b\left|\int_a^b f(s)\,w(s)\,ds-f(x)\right|^{2k}w(x)\,dx\right)^{1/(2k)}\times\left(\int_a^b\frac{|g'(x)|^{\frac{2k}{2k-1}}}{w(x)^{\frac{1}{2k-1}}}dx\right)^{\frac{2k-1}{2k}}, \tag{5.2}$$

provided that the involved integrals are finite.

Proof. Integrating by parts, we have

$$\begin{aligned}
&\int_a^b \left(\int_a^x f(t) w(t) dt - \int_a^x w(s) ds \int_a^b f(s) w(s) ds \right) g'(x) dx \\
&= \left[\left(\int_a^x f(t) w(t) dt - \int_a^x w(s) ds \int_a^b f(s) w(s) ds \right) g(x) \right|_a^b \\
&- \int_a^b g(x) \left(f(x) w(x) - w(x) \int_a^b f(s) w(s) ds \right) dx \\
&= - \int_a^b f(x) g(x) w(x) dx + \int_a^b f(s) w(s) ds \int_a^b g(x) w(x) dx,
\end{aligned}$$

which gives that

$$C_w(f,g) = \int_a^b \left(\int_a^x w(s) ds \int_a^b f(s) w(s) ds - \int_a^x f(t) w(t) dt \right) g'(x) dx. \tag{5.3}$$

If we take

$$h(x) := \int_a^x w(s) ds \int_a^b f(s) w(s) ds - \int_a^x f(t) w(t) dt, \;\; x \in [a,b]$$

we observe that $h(a) = h(b) = 0$ and $h \in C^1([a,b], \mathbb{C})$.

If we use (3.6) for $f = h$ defined above, then we get

$$\begin{aligned}
&\int_a^b \left| \int_a^x w(s) ds \int_a^b f(s) w(s) ds - \int_a^x f(t) w(t) dt \right|^{2k} w(x) dx \\
&\leq \frac{1}{2k-1} \left[\frac{k}{\pi} \sin\left(\frac{\pi}{2k} \right) \right]^{2k} \\
&\times \int_a^b \frac{\left| w(x) \int_a^b f(s) w(s) ds - f(x) w(x) \right|^{2k}}{[w(x)]^{2k-1}} dx,
\end{aligned}$$

namely

$$\begin{aligned}
&\int_a^b \left| \int_a^x w(s) ds \int_a^b f(s) w(s) ds - \int_a^x f(t) w(t) dt \right|^{2k} w(x) dx \\
&\leq \frac{1}{2k-1} \left[\frac{k}{\pi} \sin\left(\frac{\pi}{2k} \right) \right]^{2k} \\
&\times \int_a^b \left| \int_a^b f(s) w(s) ds - f(x) \right|^{2k} w(x) dx.
\end{aligned}$$

This implies that

$$\left(\int_a^b \left|\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right|^{2k} w(x)\,dx\right)^{1/(2k)} \tag{5.4}$$
$$\le \frac{k}{(2k-1)^{1/(2k)}\pi}\sin\left(\frac{\pi}{2k}\right)$$
$$\times\left(\int_a^b \left|\int_a^b f(s)\,w(s)\,ds - f(x)\right|^{2k} w(x)\,dx\right)^{1/(2k)}.$$

If we use Hölder's weighted inequality for $p = 2k > 2$ and $q = \frac{2k}{2k-1}$ then we get

$$|C_w(f,g)|$$
$$= \left|\int_a^b \left(\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right) g'(x)\,dx\right|$$
$$= \left|\int_a^b \left(\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right) \frac{g'(x)}{w(x)} w(x)\,dx\right|$$
$$\le \left|\int_a^b \left|\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right|^{2k} w(x)\,dx\right|^{1/(2k)}$$
$$\times\left(\int_a^b \left|\frac{g'(x)}{w(x)}\right|^{\frac{2k}{2k-1}} w(x)\,dx\right)^{\frac{2k-1}{2k}}$$
$$= \left|\int_a^b \left|\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right|^{2k} w(x)\,dx\right|^{1/(2k)}$$
$$\times\left(\int_a^b \frac{|g'(x)|^{\frac{2k}{2k-1}}}{w(x)^{\frac{1}{2k-1}}}dx\right)^{\frac{2k-1}{2k}}$$

and by (5.4) we derive the desired result (5.2). □

Remark 6. *For $k = 1$ we derive*

$$|C_w(f,g)| \le \frac{1}{\pi}\left(\int_a^b \left|\int_a^b f(s)\,w(s)\,ds - f(x)\right|^2 w(x)\,dx\right)^{1/2} \tag{5.5}$$
$$\times\left(\int_a^b \frac{|g'(x)|^2}{w(x)}dx\right)^{1/2},$$

provided that the involved integrals are finite.

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