WEIGHTED INTEGRAL INEQUALITIES OF AGARWAL AND PANG TYPE RELATED TO WIRTINGER'S RESULT

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Abstract. In this paper we show among others that, if $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$ and $f\in C^1([a,b],\mathbb{R})$ satisfies the conditions $f(a)=f(b)=0$ while $q:[a,b]\to[0,\infty)$ is continuous, then

$$\begin{aligned}&\int_a^b w(x)\,q(x)\,|f(x)|^\lambda dx\\&\le\frac{1}{2}\int_a^b w(x)\left(\int_a^x w(s)\,ds\int_x^b w(s)\,ds\right)^{(\lambda-1)/2}q(x)\,dx\int_a^b\frac{|f'(x)|^\lambda}{[w(x)]^{\lambda-1}}dx\\&\le\frac{1}{2^\lambda}\int_a^b w(x)\,q(x)\,dx\int_a^b\frac{|f'(x)|^\lambda}{[w(x)]^{\lambda-1}}dx\end{aligned}$$

for $\lambda\ge1$ and provided that the involved integrals are finite. Applications for trapezoid and Grüss' type inequalities are also given.

1. Introduction

It is well known that, see for instance [6], or [11], if $u\in C^1([a,b],\mathbb{R})$ satisfies $u(a)=u(b)=0$, then

$$\int_a^b u^2(t)\,dt\le\frac{(b-a)^2}{\pi^2}\int_a^b[u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a)=0$, then also

$$\int_a^b u^2(t)\,dt\le\frac{4(b-a)^2}{\pi^2}\int_a^b[u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t)=L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L\in\mathbb{R}$.

In the recent paper [7] we obtained the following weighted version of Wirtinger inequalities (1.1) and (1.2):

Theorem 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ and $f\in C^1([a,b],\mathbb{C})$ is a function with complex values and $f(a)=f(b)=0$, then*

$$\int_a^b|f(t)|^2w(t)\,dt\le\frac{1}{\pi^2}\left(\int_a^b w(s)\,ds\right)^2\int_a^b\frac{|f'(t)|^2}{w(t)}dt. \tag{1.3}$$

1991 *Mathematics Subject Classification.* 26D15; 26D10.

Key words and phrases. Wirtinger's inequality, Trapezoid inequality, Fejér's inequality, Grüss' inequality.

RGMIA Res. Rep. Coll. 24 (2021), Art. 44, 14 pp. Received 22/04/21

The equality holds in (1.3) iff

$$f(t)=K\sin\left[\frac{\pi\int_a^t w(s)\,ds}{\int_a^b w(s)\,ds}\right],\ K\in\mathbb{C}.$$

If $f(a)=0$, then

$$\int_a^b |f(t)|^2 w(t)\,dt\le\frac{4}{\pi^2}\left(\int_a^b w(s)\,ds\right)^2\int_a^b\frac{|f'(t)|^2}{w(t)}dt \tag{1.4}$$

with equality iff

$$f(t)=K\sin\left[\frac{\pi\int_a^t w(s)\,ds}{2\int_a^b w(s)\,ds}\right],\ K\in\mathbb{C}.$$

For some related Wirtinger type integral inequalities see [2], [3], [6]-[8] and [10]-[14].

Agarwal and Pang [1] proved the following Opial-Wirtinger's type inequality:

Theorem 2. *Let $\lambda\ge 1$ be a given real number and let p a nonnegative and continuous function on $[0,1]$. Further, let x be an absolutely continuous function on $[0,1]$ with $x(0)=x(1)=0$. Then*

$$\int_0^1 p(t)|x(t)|^\lambda dt\le\frac{1}{2}\int_0^1[t(1-t)]^{(\lambda-1)/2}p(t)\,dt\int_0^1|x'(t)|^\lambda dt. \tag{1.5}$$

Assume that g is absolutely continuous on $[a,b]$ with the property that $g(a)=g(b)=0$. Assume also that $q:[a,b]\to[0,\infty)$ continuous. By taking

$$x(t)=f((1-t)a+tb),\ p(t)=q((1-t)a+tb),\ t\in[0,1]$$

and observing that

$$x'(t)=(b-a)f'((1-t)a+tb),\ t\in(0,1)$$

we get from (1.5) that

$$\begin{aligned}&\int_0^1 q((1-t)a+tb)|f((1-t)a+tb)|^\lambda dt \\ &\le\frac{1}{2}(b-a)^\lambda\int_0^1[t(1-t)]^{(\lambda-1)/2}q((1-t)a+tb)\,dt \\ &\times\int_0^1|f'((1-t)a+tb)|^\lambda dt,\end{aligned} \tag{1.6}$$

and by the change of variable $x=(1-t)a+tb$, then we have $dx=(b-a)\,dt$, $t=\frac{x-a}{b-a}$ which gives by (1.6) that

$$\begin{aligned}&\frac{1}{b-a}\int_a^b q(x)|f(x)|^\lambda dx \\ &\le\frac{1}{2}(b-a)^\lambda\frac{1}{(b-a)^2}\int_a^b q(x)\left[\frac{(x-a)(b-x)}{(b-a)^2}\right]^{(\lambda-1)/2}dx\int_a^b|f'(x)|^\lambda dx,\end{aligned}$$

which is equivalent to

$$\int_a^b q(x)|f(x)|^\lambda dx\le\frac{1}{2}\int_a^b[(x-a)(b-x)]^{(\lambda-1)/2}q(x)\,dx\int_a^b|f'(x)|^\lambda dx, \tag{1.7}$$

provided that g is absolutely continuous on $[a,b]$ with the property that $g(a) = g(b) = 0$ and that $q : [a,b] \to [0,\infty)$ is continuous.

Since, by the elementary inequality

$$(x-a)(b-x) \le \frac{1}{4}(b-a)^2,$$

hence

$$\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2} q(x)\,dx \le \frac{1}{2^\lambda}(b-a)^{\lambda-1}\int_a^b q(x)\,dx,$$

and we have the chain of inequalities

$$\int_a^b q(x)|f(x)|^\lambda dx \le \frac{1}{2}\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2} q(x)\,dx \int_a^b |f'(x)|^\lambda dx \le \frac{1}{2^\lambda}(b-a)^{\lambda-1}\int_a^b q(x)\,dx \int_a^b |f'(x)|^\lambda dx, \tag{1.8}$$

for all $\lambda \ge 1$, provided that g is absolutely continuous on $[a,b]$ with the property that $g(a) = g(b) = 0$ and that $q : [a,b] \to [0,\infty)$ is continuous.

In this paper we show among others that, if $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds = 1$ and $f \in C^1([a,b],\mathbb{R})$ satisfies the conditions $f(a) = f(b) = 0$ while $q : [a,b] \to [0,\infty)$ is continuous, then

$$\begin{aligned}
&\int_a^b w(x)\,q(x)\,|f(x)|^\lambda dx \\
&\le \frac{1}{2}\int_a^b w(x)\left(\int_a^x w(s)\,ds\int_x^b w(s)\,ds\right)^{(\lambda-1)/2} q(x)\,dx \int_a^b \frac{|f'(x)|^\lambda}{[w(x)]^{\lambda-1}}dx \\
&\le \frac{1}{2^\lambda}\int_a^b w(x)\,q(x)\,dx \int_a^b \frac{|f'(x)|^\lambda}{[w(x)]^{\lambda-1}}dx
\end{aligned}$$

for $\lambda \ge 1$ and provided that the involved integrals are finite. Applications for trapezoid and Grüss' type inequalities are also given.

2. Some Weighted Inequalities of Trapezoid Type

We have the following weighted trapezoid inequality:

Theorem 3. *Assume that $q : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ with $\int_a^b q(s)\,ds = 1$ and $g \in C^1([a,b],\mathbb{C})$, then*

$$\begin{aligned}
&\left|\int_a^b \frac{q(x)+q(a+b-x)}{2} g(x)\,dx - \frac{g(a)+g(b)}{2}\right| \\
&\le \frac{1}{2^{1+1/\lambda}}\left(\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2} q(x)\,dx\right)^{1/\lambda} \\
&\quad\times\left(\int_a^b |g'(x)-g'(a+b-x)|^\lambda dx\right)^{1/\lambda} \\
&\le \frac{1}{4}(b-a)^{1-1/\lambda}\left(\int_a^b |g'(x)-g'(a+b-x)|^\lambda dx\right)^{1/\lambda}.
\end{aligned} \tag{2.1}$$

In particular, if q is symmetrical, i.e. $q(a+b-t)=q(t)$ for any $t\in[a,b]$, then we have

$$\left|\int_a^b q(x)\,g(x)\,dx-\frac{g(a)+g(b)}{2}\right| \tag{2.2}$$
$$\leq\frac{1}{2^{1+1/\lambda}}\left(\int_a^b[(x-a)(b-x)]^{(\lambda-1)/2}q(x)\,dx\right)^{1/\lambda}$$
$$\times\left(\int_a^b\left|g'(x)-g'(a+b-x)\right|^{\lambda}dx\right)^{1/\lambda}$$
$$\leq\frac{1}{4}(b-a)^{1-1/\lambda}\left(\int_a^b\left|g'(x)-g'(a+b-x)\right|^{\lambda}dx\right)^{1/\lambda}.$$

Proof. Consider the function

$$f(x):=\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2},\ x\in[a,b],$$

we have $f(a)=f(b)=0$ and by (1.8) we have

$$\int_a^b\left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^{\lambda}q(x)\,dx \tag{2.3}$$
$$\leq\frac{1}{2}\int_a^b[(x-a)(b-x)]^{(\lambda-1)/2}q(x)\,dx$$
$$\times\int_a^b\left|\frac{g'(x)-g'(a+b-x)}{2}\right|^{\lambda}dx$$
$$\leq\frac{1}{2^{\lambda}}(b-a)^{\lambda-1}\int_a^b q(x)\,dx\int_a^b\left|\frac{g'(x)-g'(a+b-x)}{2}\right|^{\lambda}dx.$$

By Jensen's integral inequality we have

$$\int_a^b\left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^{\lambda}q(x)\,dx$$
$$\geq\left|\int_a^b\left(\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right)q(x)\,dx\right|^{\lambda},$$

namely

$$\int_a^b\left|\frac{g(x)+g(a+b-x)}{2}-\frac{g(a)+g(b)}{2}\right|^{\lambda}q(x)\,dx \tag{2.4}$$
$$\geq\left|\int_a^b\frac{g(x)+g(a+b-x)}{2}q(x)\,dx-\frac{g(a)+g(b)}{2}\int_a^b q(x)\,dx\right|^{\lambda}.$$

Observe that, by the change of variable $s=a+b-x$, $x\in[a,b]$ we have that

$$\int_a^b g(a+b-x)\,q(x)\,dx=\int_a^b g(s)\,q(a+b-s)\,ds$$

and then

$$\int_a^b \frac{g(x)+g(a+b-x)}{2} q(x)\,dx = \int_a^b \frac{q(x)+q(a+b-x)}{2} g(x)\,dx.$$

Therefore (2.4) can be written as

$$\begin{aligned}&\int_a^b \left|\frac{g(x)+g(a+b-x)}{2} - \frac{g(a)+g(b)}{2}\right|^{\lambda} q(x)\,dx \\ &\geq \left|\int_a^b \frac{q(x)+q(a+b-x)}{2} g(x)\,dx - \frac{g(a)+g(b)}{2}\int_a^b q(x)\,dx\right|^{\lambda}.\end{aligned}$$

By making use of (2.3) we derive

$$\begin{aligned}&\left|\int_a^b \frac{q(x)+q(a+b-x)}{2} g(x)\,dx - \frac{g(a)+g(b)}{2}\int_a^b q(x)\,dx\right|^{\lambda} \\ &\leq \int_a^b \left|\frac{g(x)+g(a+b-x)}{2} - \frac{g(a)+g(b)}{2}\right|^{\lambda} q(x)\,dx \\ &\leq \frac{1}{2^{\lambda+1}}\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2} q(x)\,dx \\ &\times \int_a^b |g'(x)-g'(a+b-x)|^{\lambda}\,dx \\ &\leq \frac{1}{4^{\lambda}}(b-a)^{\lambda-1}\int_a^b q(x)\,dx \int_a^b |g'(x)-g'(a+b-x)|^{\lambda}\,dx.\end{aligned}$$

By taking the power $1/\lambda$ in this inequality, we get the desired result (2.1). □

In 1906, Fejér [9], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 4 (Fejér's Inequality)**.** *Consider the integral* $\int_a^b h(x)\,w(x)\,dx$*, where* h *is a convex function in the interval* (a,b) *and* w *is a positive function in the same interval such that*

$$w(x) = w(a+b-x)\,, \text{ for any } x \in [a,b]$$

i.e., $y = w(x)$ *is a symmetric curve with respect to the straight line which contains the point* $\left(\frac{1}{2}(a+b), 0\right)$ *and is normal to the* x*-axis. Under those conditions the following inequalities are valid:*

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{\int_a^b w(x)\,dx}\int_a^b h(x)\,w(x)\,dx \leq \frac{h(a)+h(b)}{2}. \tag{2.5}$$

If h *is concave on* (a,b)*, then the inequalities reverse in (2.5).*

Remark 1. *If* $g:[a,b]\to\mathbb{R}$ *is differentiable convex and* $g'_-(b)$ *and* $g'_+(a)$ *are finite and* $q:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b q(s)\,ds = 1$ *and symmetrical,* *then by (2.2) we get the following reverse of the second inequality in (2.5)*

$$\begin{aligned}0 &\leq \frac{g(a)+g(b)}{2}-\int_a^b q(x)\,g(x)\,dx \\ &\leq \frac{1}{2^{1+1/\lambda}}\left[g'_-(b)-g'_+(a)\right](b-a)^{1/\lambda} \\ &\times\left(\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2}\,q(x)\,dx\right)^{1/\lambda} \\ &\leq \frac{1}{4}(b-a)\left[g'_-(b)-g'_+(a)\right].\end{aligned} \tag{2.6}$$

Another trapezoid type weighted inequality is as follows:

Theorem 5. *Assume that $q:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b q(s)\,ds = 1$ and $g\in C^1([a,b],\mathbb{R})$, then*

$$\begin{aligned}&\left|\frac{g(a)\,[b-E(q;[a,b])]+g(b)\,[E(q;[a,b])-a]}{b-a}-\int_a^b g(t)\,q(t)\,dt\right| \\ &\leq \frac{1}{2^{1/\lambda}}\left(\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2}\,q(x)\,dx\right)^{1/\lambda} \\ &\times\left(\int_a^b \left|g'(x)-\frac{g(b)-g(a)}{b-a}\right|^{\lambda}dx\right)^{1/\lambda} \\ &\leq \frac{1}{2}(b-a)^{1-1/\lambda}\left(\int_a^b \left|g'(x)-\frac{g(b)-g(a)}{b-a}\right|^{\lambda}dx\right)^{1/\lambda},\end{aligned} \tag{2.7}$$

where

$$E(q;[a,b]) := \int_a^b t\,q(t)\,dt.$$

Proof. If $g\in C^1([a,b],\mathbb{R})$, then by taking

$$f(x) := g(x)-\frac{g(a)(b-x)+g(b)(x-a)}{b-a},\; x\in[a,b]$$

we have $f(a)=f(b)=0$ and by (1.8) we obtain

$$\begin{aligned}&\int_a^b \left|g(x)-\frac{g(a)(b-x)+g(b)(x-a)}{b-a}\right|^{\lambda} q(x)\,dx \\ &\leq \frac{1}{2}\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2}\,q(x)\,dx \\ &\times\int_a^b \left|g'(x)-\frac{g(b)-g(a)}{b-a}\right|^{\lambda}dx \\ &\leq \frac{1}{2^{\lambda}}(b-a)^{\lambda-1}\int_a^b \left|g'(x)-\frac{g(b)-g(a)}{b-a}\right|^{\lambda}dx.\end{aligned} \tag{2.8}$$

By Jensen's integral inequality we have

$$\begin{aligned}
&\int_a^b \left| g(x) - \frac{g(a)(b-x)+g(b)(x-a)}{b-a} \right|^\lambda q(x)\,dx \\
&\geq \left| \int_a^b \left(g(x) - \frac{g(a)(b-x)+g(b)(x-a)}{b-a} \right) q(x)\,dx \right|^\lambda \\
&= \left| \frac{g(a)[b-E(q;[a,b])]+g(b)[E(q;[a,b])-a]}{b-a} \right. \\
&\quad \left. - \int_a^b g(t)\,q(t)\,dt \right|^\lambda,
\end{aligned}$$

then by (2.8) we get

$$\begin{aligned}
&\left| \frac{g(a)[b-E(q;[a,b])]+g(b)[E(q;[a,b])-a]}{b-a} \right. \\
&\quad \left. - \int_a^b g(t)\,q(t)\,dt \right|^\lambda \\
&\leq \int_a^b \left| g(x) - \frac{g(a)(b-x)+g(b)(x-a)}{b-a} \right|^\lambda q(x)\,dx \\
&\leq \frac{1}{2} \int_a^b [(x-a)(b-x)]^{(\lambda-1)/2} q(x)\,dx \\
&\quad \times \int_a^b \left| g'(x) - \frac{g(b)-g(a)}{b-a} \right|^\lambda dx \\
&\leq \frac{1}{2^\lambda} (b-a)^{\lambda-1} \int_a^b \left| g'(x) - \frac{g(b)-g(a)}{b-a} \right|^\lambda dx.
\end{aligned}$$

By taking the power $1/\lambda$ in this inequality, we get the desired result (2.7). □

The case of convex function is as follows:

Corollary 1. *If $g:[a,b]\to\mathbb{R}$ is continuously differentiable convex and $q:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b q(s)\,ds = 1$, then*

$$\begin{aligned}
0 &\leq \frac{g(a)[b-E(q;[a,b])]+g(b)[E(q;[a,b])-a]}{b-a} \\
&\quad - \int_a^b g(t)\,q(t)\,dt \\
&\leq \frac{1}{2^{1/\lambda}} \left(\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2} q(x)\,dx \right)^{1/\lambda} \\
&\quad \times \left(\int_a^b \left| g'(x) - \frac{g(b)-g(a)}{b-a} \right|^\lambda dx \right)^{1/\lambda} \\
&\leq \frac{1}{2} (b-a)^{1-1/\lambda} \left(\int_a^b \left| g'(x) - \frac{g(b)-g(a)}{b-a} \right|^\lambda dx \right)^{1/\lambda}.
\end{aligned} \tag{2.9}$$

The positivity follows by the fact that, for a convex function g on $[a,b]$ we have

$$\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\geq g(t)$$

for any $t\in[a,b]$. The rest is obvious by Theorem 5.

Remark 2. *With the assumption of Corollary 1 we have the simpler inequality*

$$\begin{aligned}
0 &\leq \frac{g(a)[b-E(q;[a,b])]+g(b)[E(q;[a,b])-a]}{b-a} \\
&- \int_a^b g(t)q(t)\,dt \\
&\leq \frac{1}{2^{1/\lambda}}(b-a)^{1/\lambda}\left(\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2}q(x)\,dx\right)^{1/\lambda} \\
&\times \max_{x\in(a,b)}\left|g'(x)-\frac{g(b)-g(a)}{b-a}\right| \\
&\leq \frac{1}{2}(b-a)\max_{x\in(a,b)}\left|g'(x)-\frac{g(b)-g(a)}{b-a}\right|.
\end{aligned}\tag{2.10}$$

3. Some Inequalities for the Weighted Čebyšev Functional

Consider now the *weighted Čebyšev functional*

$$C_q(f,g):=\int_a^b q(t)f(t)g(t)\,dt-\int_a^b q(t)f(t)\,dt\int_a^b q(t)g(t)\,dt\tag{3.1}$$

where $f,\ g,\ q:[a,b]\to\mathbb{R}$ and $q(t)\geq 0$ for a.e. $t\in[a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b q(t)\,dt=1$.

Theorem 6. *Assume that f is integrable on $[a,b]$, g is absolutely continuous on $[a,b]$ and $q:[a,b]\to[0,\infty)$ is continuous on $[a,b]$ with $\int_a^b q(s)\,ds=1$, then for $\lambda>1$*

$$\begin{aligned}
&|C_q(f,g)| \\
&\leq \frac{1}{2^{1/\lambda}}\left(\int_a^b [(x-a)(b-x)]^{(\lambda-1)/2}q(x)\,dx\right)^{1/\lambda}\left(\int_a^b \frac{|g'(x)|^{\frac{\lambda}{\lambda-1}}}{q(x)^{\frac{1}{\lambda-1}}}dx\right)^{\frac{\lambda-1}{\lambda}} \\
&\times\left(\int_a^b\left|\int_a^b f(s)q(s)\,ds-f(x)\right|^{\lambda}q^{\lambda}(x)\,dx\right)^{1/\lambda} \\
&\leq \frac{1}{2}(b-a)^{1-1/\lambda}\left(\int_a^b \frac{|g'(x)|^{\frac{\lambda}{\lambda-1}}}{q(x)^{\frac{1}{\lambda-1}}}dx\right)^{\frac{\lambda-1}{\lambda}} \\
&\times\left(\int_a^b\left|\int_a^b f(s)q(s)\,ds-f(x)\right|^{\lambda}q^{\lambda}(x)\,dx\right)^{1/\lambda},
\end{aligned}\tag{3.2}$$

provided that the involved integrals are finite.

Proof. Integrating by parts, we have

$$\begin{aligned}
&\int_a^b \left(\int_a^x f(t)\,q(t)\,dt - \int_a^x q(s)\,ds \int_a^b f(s)\,q(s)\,ds \right) g'(x)\,dx \\
&= \left[\left(\int_a^x f(t)\,q(t)\,dt - \int_a^x q(s)\,ds \int_a^b f(s)\,q(s)\,ds \right) g(x) \right|_a^b \\
&- \int_a^b g(x) \left(f(x)\,q(x) - q(x) \int_a^b f(s)\,q(s)\,ds \right) dx \\
&= -\int_a^b f(x)\,g(x)\,q(x)\,dx + \int_a^b f(s)\,q(s)\,ds \int_a^b g(x)\,q(x)\,dx,
\end{aligned}$$

which gives that

$$C_q(f,g) = \int_a^b \left(\int_a^x q(s)\,ds \int_a^b f(s)\,q(s)\,ds - \int_a^x f(t)\,q(t)\,dt \right) g'(x)\,dx. \tag{3.3}$$

If we take

$$h(x) := \int_a^x q(s)\,ds \int_a^b f(s)\,q(s)\,ds - \int_a^x f(t)\,q(t)\,dt,\ x \in [a,b]$$

we observe that $h(a) = h(b) = 0$ and $h \in C^1([a,b],\mathbb{R})$.

By (1.8) we get for $f = h$ defined above,

$$\begin{aligned}
&\int_a^b q(x) \left| \int_a^x q(s)\,ds \int_a^b f(s)\,q(s)\,ds - \int_a^x f(t)\,q(t)\,dt \right|^\lambda dx \\
&\le \frac{1}{2} \int_a^b [(x-a)(b-x)]^{(\lambda-1)/2}\,q(x)\,dx \\
&\times \int_a^b \left| \int_a^b f(s)\,q(s)\,ds - f(x) \right|^\lambda q^\lambda(x)\,dx \\
&\le \frac{1}{2^\lambda} (b-a)^{\lambda-1} \int_a^b \left| \int_a^b f(s)\,q(s)\,ds - f(x) \right|^\lambda q^\lambda(x)\,dx.
\end{aligned}$$

This implies that

$$\left(\int_a^b q(x)\left|\int_a^x q(s)\,ds\int_a^b f(s)\,q(s)\,ds-\int_a^x f(t)\,q(t)\,dt\right|^{\lambda}dx\right)^{1/\lambda} \tag{3.4}$$
$$\leq \frac{1}{2^{1/\lambda}}\left(\int_a^b \left[(x-a)(b-x)\right]^{(\lambda-1)/2} q(x)\,dx\right)^{1/\lambda}$$
$$\times\left(\int_a^b\left|\int_a^b f(s)\,q(s)\,ds-f(x)\right|^{\lambda} q^{\lambda}(x)\,dx\right)^{1/\lambda}$$
$$\leq \frac{1}{2}(b-a)^{1-1/\lambda}\left(\int_a^b\left|\int_a^b f(s)\,q(s)\,ds-f(x)\right|^{\lambda} q^{\lambda}(x)\,dx\right)^{1/\lambda}.$$

If we use Hölder's weighted inequality for $p=\lambda>1$ and $q=\frac{\lambda}{\lambda-1}$ then we get

$$|C_q(f,g)|$$
$$=\left|\int_a^b\left(\int_a^x q(s)\,ds\int_a^b f(s)\,q(s)\,ds-\int_a^x f(t)\,q(t)\,dt\right)g'(x)\,dx\right|$$
$$=\left|\int_a^b\left(\int_a^x q(s)\,ds\int_a^b f(s)\,q(s)\,ds-\int_a^x f(t)\,q(t)\,dt\right)\frac{g'(x)}{q(x)}q(x)\,dx\right|$$

$$\leq\left|\int_a^b\left|\int_a^x q(s)\,ds\int_a^b f(s)\,q(s)\,ds-\int_a^x f(t)\,q(t)\,dt\right|^{\lambda} q(x)\,dx\right|^{1/\lambda}$$
$$\times\left(\int_a^b\left|\frac{g'(x)}{q(x)}\right|^{\frac{\lambda}{\lambda-1}} q(x)\,dx\right)^{\frac{\lambda-1}{\lambda}}$$
$$=\left|\int_a^b\left|\int_a^x q(s)\,ds\int_a^b f(s)\,q(s)\,ds-\int_a^x f(t)\,q(t)\,dt\right|^{\lambda} q(x)\,dx\right|^{1/\lambda}$$
$$\times\left(\int_a^b\frac{|g'(x)|^{\frac{\lambda}{\lambda-1}}}{q(x)^{\frac{1}{\lambda-1}}}dx\right)^{\frac{\lambda-1}{\lambda}}$$

and by (3.4) we derive the desired result (3.2). □

Remark 3. *For $\lambda = 2$ we derive*

$$
\begin{aligned}
&|C_q(f,g)| \\
&\leq \frac{1}{2^{1/2}}\left(\int_a^b [(x-a)(b-x)]^{1/2} q(x)\,dx\right)^{1/2}\left(\int_a^b \frac{|g'(x)|^2}{q(x)}dx\right)^{1/2} \\
&\times\left(\int_a^b \left|\int_a^b f(s)q(s)\,ds - f(x)\right|^2 q^2(x)\,dx\right)^{1/2} \\
&\leq \frac{1}{2}(b-a)^{1/2}\left(\int_a^b \frac{|g'(x)|^2}{q(x)}dx\right)^{1/2} \\
&\times\left(\int_a^b \left|\int_a^b f(s)q(s)\,ds - f(x)\right|^2 q^2(x)\,dx\right)^{1/2},
\end{aligned}
\tag{3.5}
$$

provided that the involved integrals are finite.

4. A Weighted Version

We can state the following weighted results:

Theorem 7. *Let $h : [a,b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b). If $f \in C^1([a,b],\mathbb{R})$ is a function with the properties $f(a) = f(b) = 0$ and $q : [a,b] \to [0,\infty)$ is continuous, then*

$$
\begin{aligned}
&\int_a^b q(x)|f(x)|^\lambda h'(x)\,dx \\
&\leq \frac{1}{2}\int_a^b ([h(x)-h(a)][h(b)-h(x)])^{(\lambda-1)/2} q(x)h'(x)\,dx \\
&\times\int_a^b \frac{|f'(x)|^\lambda}{[h'(x)]^{\lambda-1}}dx \\
&\leq \frac{1}{2^\lambda}[h(b)-h(a)]^{\lambda-1}\int_a^b q(x)h'(x)\,dx\int_a^b \frac{|f'(x)|^\lambda}{[h'(x)]^{\lambda-1}}dx
\end{aligned}
\tag{4.1}
$$

for $\lambda \geq 1$ and provided that the involved integrals are finite.

Proof. We write the inequality (1.8) for the functions $f \circ h^{-1}$, $q \circ h^{-1}$on the interval $[h(a), h(b)]$ and observe that $f \circ h^{-1}(h(a)) = f(a) = 0$, $f \circ h^{-1}(h(b)) = f(b) = 0$

and then,

$$\begin{aligned}(4.2)\qquad & \int_{h(a)}^{h(b)} q\circ h^{-1}(z)\left|f\circ h^{-1}(z)\right|^{\lambda} dz \\ & \leq \frac{1}{2}\int_{h(a)}^{h(b)} \left[(z-h(a))(h(b)-z)\right]^{(\lambda-1)/2} q\circ h^{-1}(z)\,dz \\ & \times \int_{h(a)}^{h(b)} \left|\left(f\circ h^{-1}\right)'(z)\right|^{\lambda} dz \\ & \leq \frac{1}{2^{\lambda}}(h(b)-h(a))^{\lambda-1}\int_{h(a)}^{h(b)} q\circ h^{-1}(z)\,dz\int_{h(a)}^{h(b)}\left|\left(f\circ h^{-1}\right)'(z)\right|^{\lambda} dz.\end{aligned}$$

If $f:[c,d]\to\mathbb{R}$ is absolutely continuous on $[c,d]$, then $f\circ h^{-1}:[h(c),h(d)]\to\mathbb{R}$ is absolutely continuous on $[h(c),h(d)]$ and using the chain rule and the derivative of inverse functions we have

$$(4.3)\qquad \left(f\circ h^{-1}\right)'(z)=\left(f'\circ h^{-1}\right)(z)\left(h^{-1}\right)'(z)=\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}$$

for almost every (a.e.) $z\in[h(c),h(d)]$.

Therefore (4.2) becomes

$$\begin{aligned}(4.4)\qquad & \int_{h(a)}^{h(b)} q\circ h^{-1}(z)\left|f\circ h^{-1}(z)\right|^{\lambda} dz \\ & \leq \frac{1}{2}\int_{h(a)}^{h(b)} \left[(z-h(a))(h(b)-z)\right]^{(\lambda-1)/2} q\circ h^{-1}(z)\,dz \\ & \times \int_{h(a)}^{h(b)} \left|\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right|^{\lambda} dz \\ & \leq \frac{1}{2^{\lambda}}(h(b)-h(a))^{\lambda-1}\int_{h(a)}^{h(b)} q\circ h^{-1}(z)\,dz\int_{h(a)}^{h(b)}\left|\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right|^{\lambda} dz.\end{aligned}$$

Observe also that, by the change of variable $x=h^{-1}(z)$, $z\in[h(a),h(b)]$, we have $z=h(x)$ that gives $dz=h'(x)\,dx$ and

$$\int_{h(a)}^{h(b)} q\circ h^{-1}(z)\left|\left(f\circ h^{-1}\right)(z)\right|^{\lambda} dz=\int_a^b q(x)\left|f(x)\right|^{\lambda} h'(x)\,dx.$$

We also have

$$\begin{aligned}& \int_{h(a)}^{h(b)} \left[(z-h(a))(h(b)-z)\right]^{(\lambda-1)/2} q\circ h^{-1}(z)\,dz \\ & =\int_a^b \left[(h(x)-h(a))(h(b)-h(x))\right]^{(\lambda-1)/2} q(x)\,h'(x)\,dx,\end{aligned}$$

$$\int_{h(a)}^{h(b)}\left|\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right|^{\lambda} dz=\int_a^b\left|\frac{f'(x)}{h'(x)}\right|^{\lambda} h'(x)\,dx=\int_a^b \frac{\left|f'(x)\right|^{\lambda}}{\left[h'(x)\right]^{\lambda-1}}dx$$

and

$$\int_{h(a)}^{h(b)} q\circ h^{-1}(z)\,dz=\int_a^b q(x)\,h'(x)\,dx.$$

By making use of (4.4) we then obtain the desired result (4.1). □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Corollary 2. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$ and $f\in C^1([a,b],\mathbb{R})$ satisfies the conditions $f(a)=f(b)=0$ while $q:[a,b]\to[0,\infty)$ is continuous, then*

$$\begin{aligned}(4.5)\qquad &\int_a^b w(x)\,q(x)\,|f(x)|^{\lambda}\,dx\\ &\le\frac{1}{2}\int_a^b w(x)\left(\int_a^x w(s)\,ds\int_x^b w(s)\,ds\right)^{(\lambda-1)/2} q(x)\,dx\int_a^b\frac{|f'(x)|^{\lambda}}{[w(x)]^{\lambda-1}}dx\\ &\le\frac{1}{2^{\lambda}}\int_a^b w(x)\,q(x)\,dx\int_a^b\frac{|f'(x)|^{\lambda}}{[w(x)]^{\lambda-1}}dx\end{aligned}$$

for $\lambda\ge1$ and provided that the involved integrals are finite.

In particular, for $\lambda=2$, we have

$$\begin{aligned}(4.6)\qquad &\int_a^b w(x)\,q(x)\,|f(x)|^{2}\,dx\\ &\le\frac{1}{2}\int_a^b w(x)\left(\int_a^x w(s)\,ds\int_x^b w(s)\,ds\right)^{1/2} q(x)\,dx\int_a^b\frac{|f'(x)|^{2}}{w(x)}dx\\ &\le\frac{1}{2^{\lambda}}\int_a^b w(x)\,q(x)\,dx\int_a^b\frac{|f'(x)|^{2}}{w(x)}dx\end{aligned}$$

provided that the involved integrals are finite.

Finally, we give two examples that can be derived from Theorem 7.

a). If we take $h(t)=\ln t$, $t\in[a,b]\subset(0,\infty)$, then by (4.1) we get

$$\begin{aligned}(4.7)\qquad &\int_a^b\frac{q(x)}{x}\,|f(x)|^{\lambda}\,dx\\ &\le\frac{1}{2}\int_a^b\frac{1}{x}\left[\ln\left(\frac{x}{a}\right)\ln\left(\frac{b}{x}\right)\right]^{(\lambda-1)/2} q(x)\,dx\int_a^b|f'(x)|^{\lambda}\,x^{\lambda-1}dx\\ &\le\frac{1}{2^{\lambda}}\left[\ln\left(\frac{b}{a}\right)\right]^{\lambda-1}\int_a^b\frac{q(x)}{x}dx\int_a^b|f'(x)|^{\lambda}\,x^{\lambda-1}dx,\end{aligned}$$

provided that $f\in C^1([a,b],\mathbb{R})$ is a function with the properties $f(a)=f(b)=0$ and $q:[a,b]\to[0,\infty)$ is continuous.

b). If we take $h(t) = \exp t$, $t \in [a,b]$, then by (4.1) we get

$$\int_a^b q(x)|f(x)|^\lambda \exp x dx \tag{4.8}$$
$$\leq \frac{1}{2}\int_a^b [(\exp x - \exp a)(\exp b - \exp x)]^{(\lambda-1)/2} q(x) \exp x dx$$
$$\times \int_a^b \frac{|f'(x)|^\lambda}{\exp[(\lambda-1)x]} dx$$
$$\leq \frac{1}{2^\lambda}(\exp b - \exp a)^{\lambda-1} \int_a^b q(x) \exp x dx \int_a^b \frac{|f'(x)|^\lambda}{\exp[(\lambda-1)x]} dx$$

provided that $f \in C^1([a,b],\mathbb{R})$ is a function with the properties $f(a) = f(b) = 0$ and $q : [a,b] \to [0,\infty)$ is continuous.

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