INTEGRAL INEQUALITIES FOR FUNCTIONS WITH VALUES IN HILBERT SPACES RELATED TO WIRTINGER'S RESULT

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ a complex Hilbert space. In this paper we show among others that, if $w:[a,b]\to(0,\infty)$ is a probability density function on $[a,b]$ and $g\in C^1([a,b],H)$ with $g(a)=g(b)=0$, then

$$\int_a^b \|g(t)\|^2 w(t)\,dt \le \frac{1}{\pi^2}\int_a^b \frac{\|g'(t)\|^2}{w(t)}dt.$$

Applications related to the trapezoid unweighted and weighted inequalities and of Grüss' type inequalities are also provided.

1. Introduction

It is well known that, see for instance [5], or [8], if $u\in C^1([a,b],\mathbb{R})$ satisfies $u(a)=u(b)=0$, then

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a)=0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t)=L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{C})$ is a function with complex values and $u(a)=u(b)=0$, then $\operatorname{Re}u(a)=\operatorname{Re}u(b)=0$ and $\operatorname{Im}u(a)=\operatorname{Im}u(b)=0$ and by writing (1.1) for $\operatorname{Re}u$ and $\operatorname{Im}u$ and adding the obtained inequalities, we get

$$\int_a^b |u(t)|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |u'(t)|^2\,dt \tag{1.3}$$

with the equality holding if and only if

$$u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$$

for some complex constant $K\in\mathbb{C}$.

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Similarly, if $u \in C^1([a,b],\mathbb{C})$ with $u(a)=0$, then by (1.2) we have

$$\int_a^b |u(t)|^2\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b |u'(t)|^2\,dt \tag{1.4}$$

and the equality holds if and only if

$$u(t) = L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$$

for some complex constant $L \in \mathbb{C}$.

For some related Wirtinger type integral inequalities see [1], [2], [5] and [7]-[11].

Let $(H;\langle\cdot,\cdot\rangle)$ a complex Hilbert space. In this paper we show among others that, if $w:[a,b]\to(0,\infty)$ is a probability density function on $[a,b]$ and $g \in C^1([a,b],H)$ with $g(a)=g(b)=0$, then

$$\int_a^b \|g(t)\|^2 w(t)\,dt \le \frac{1}{\pi^2}\int_a^b \frac{\|g'(t)\|^2}{w(t)}dt.$$

Applications related to the trapezoid unweighted and weighted inequalities and of Grüss' type inequalities are also provided.

2. General Results

We consider the complex Hilbert space $(H;\langle.,.\rangle)$. Now, assume that $f:[a,b]\to H$ is continuous on $[a,b]$ and strongly differentiable on (a,b). We define $u_m(t) := \|f(t)\|^m$ for $t\in[a,b]$ and $m\ge 1$.

Lemma 1. *Assume that $f:[a,b]\to H$ is continuous on $[a,b]$ and strongly differentiable on (a,b). Then*

$$u_m'(t) = \begin{cases} m\|f(t)\|^{m-2}\operatorname{Re}\langle f'(t),f(t)\rangle & \text{for all } t\in(a,b) \text{ and } m\ge 2, \\ \frac{m\operatorname{Re}\langle f'(t),f(t)\rangle}{\|f(t)\|^{2-m}} & \text{for } t\in(a,b) \text{ with } \|f(t)\|\ne 0 \text{ and } m\in[1,2). \end{cases} \tag{2.1}$$

Proof. Using the properties of inner product and norm, we get for x, $y\in H$

$$\|y\|^2 = \|y-x+x\|^2 = \|y-x\|^2 + 2\operatorname{Re}\langle y-x,x\rangle + \|x\|^2,$$

which gives

$$\|y\|^2 - \|x\|^2 = \|y-x\|^2 + 2\operatorname{Re}\langle y-x,x\rangle. \tag{2.2}$$

Let $t\in(a,b)$ and $h\ne 0$ small and such that $t+h\in(a,b)$. Then

$$\|f(t+h)\|^2 - \|f(t)\|^2 = \|f(t+h)-f(t)\|^2 + 2\operatorname{Re}\langle f(t+h)-f(t),f(t)\rangle$$

and dividing by $h\ne 0$ we get

$$\frac{\|f(t+h)\|^2-\|f(t)\|^2}{h} = h\left\|\frac{f(t+h)-f(t)}{h}\right\|^2 + 2\operatorname{Re}\left\langle\frac{f(t+h)-f(t)}{h},f(t)\right\rangle.$$

By taking the limit over $h \to 0$ and using the continuity of norm and inner product, then we get

$$\begin{aligned}
&\lim_{h\to 0}\frac{\|f(t+h)\|^2-\|f(t)\|^2}{h}\\
&=\lim_{h\to 0} h\left\|\lim_{h\to 0}\frac{f(t+h)-f(t)}{h}\right\|^2+2\operatorname{Re}\left\langle \lim_{h\to 0}\frac{f(t+h)-f(t)}{h}, f(t)\right\rangle\\
&=2\operatorname{Re}\left\langle f'(t), f(t)\right\rangle,
\end{aligned}$$

which shows that u_2 is differentiable and

$$u_2'(t)=2\operatorname{Re}\left\langle f'(t), f(t)\right\rangle,\ t\in(a,b).$$

Observe that

$$\begin{aligned}
\lim_{h\to 0}\frac{\|f(t+h)\|-\|f(t)\|}{h} &= \lim_{h\to 0}\frac{\|f(t+h)\|^2-\|f(t)\|^2}{h\left(\|f(t+h)\|+\|f(t)\|\right)}\\
&=\lim_{h\to 0}\frac{\|f(t+h)\|^2-\|f(t)\|^2}{h}\\
&\quad\times\frac{1}{\lim_{h\to 0}\left(\|f(t+h)\|+\|f(t)\|\right)}\\
&=\frac{2\operatorname{Re}\left\langle f'(t), f(t)\right\rangle}{2\|f(t)\|}=\frac{\operatorname{Re}\left\langle f'(t), f(t)\right\rangle}{\|f(t)\|}
\end{aligned}$$

provided that $\|f(t)\|\neq 0$ for $t\in(a,b)$.

This gives that

$$u_1'(t)=\frac{\operatorname{Re}\left\langle f'(t), f(t)\right\rangle}{\|f(t)\|},\ \text{provided that } \|f(t)\|\neq 0 \text{ for } t\in(a,b)..$$

Now, since $u_m(t)=(u_1(t))^m$ then for $t\in(a,b)$,

$$\begin{aligned}
u_m'(t) &= m(u_1(t))^{m-1}u_1'(t)=m\|f(t)\|^{m-1}\frac{\operatorname{Re}\left\langle f'(t), f(t)\right\rangle}{\|f(t)\|}\\
&=m\|f(t)\|^{m-2}\operatorname{Re}\left\langle f'(t), f(t)\right\rangle,
\end{aligned}$$

which proves (2.1). □

Corollary 1. *With the assumptions of Lemma 1 we have*

$$\begin{aligned}
(2.3)\quad & |u_m'(t)|\\
&=\begin{cases} m\|f(t)\|^{m-2}\left|\operatorname{Re}\left\langle f'(t), f(t)\right\rangle\right| & \text{for all } t\in(a,b) \text{ and } m\geq 2,\\[2ex] \frac{m\left|\operatorname{Re}\left\langle f'(t), f(t)\right\rangle\right|}{\|f(t)\|^{2-m}} & \text{for } t\in(a,b) \text{ with } \|f(t)\|\neq 0 \text{ and } m\in[1,2),\end{cases}\\
&\leq\begin{cases} m\|f(t)\|^{m-1}\|f'(t)\| & \text{for all } t\in(a,b) \text{ and } m\geq 2,\\[2ex] m\|f(t)\|^{m-1}\|f'(t)\| & \text{for } t\in(a,b) \text{ with } \|f(t)\|\neq 0 \text{ and } m\in[1,2).\end{cases}
\end{aligned}$$

The proof follows by the equality (2.1) and Schwarz inequality $\left|\operatorname{Re}\left\langle f'(t), f(t)\right\rangle\right|\leq\|f'(t)\|\|f(t)\|$, $t\in(a,b)$.

Theorem 1. *Assume that $f:[a,b]\to H$ is of class C^1 on $[a,b]$ and $f(a)=f(b)=0$. If $m\geq 2$, then*

$$\int_a^b \|f(t)\|^{2m}dt \leq \frac{m^2(b-a)^2}{\pi^2}\int_a^b \|f(t)\|^{2(m-2)}\left[\operatorname{Re}\langle f'(t),f(t)\rangle\right]^2 dt \leq \frac{m^2(b-a)^2}{\pi^2}\int_a^b \|f(t)\|^{2(m-1)}\|f'(t)\|^2 dt. \tag{2.4}$$

If $m\in[1,2)$ and $f(t)\neq 0$ for $t\in(a,b)$, then

$$\int_a^b \|f(t)\|^{2m}dt \leq \frac{m^2(b-a)^2}{\pi^2}\int_a^b \frac{|\operatorname{Re}\langle f'(t),f(t)\rangle|^2}{\|f(t)\|^{2(2-m)}}dt \leq \frac{m^2(b-a)^2}{\pi^2}\int_a^b \|f(t)\|^{2(m-1)}\|f'(t)\|^2 dt. \tag{2.5}$$

Proof. We consider $u(t)=u_m(t)=\|f(t)\|^m$, then $u(a)=u(b)=0$ and by (1.1) we get

$$\int_a^b u_m^2(t)\,dt \leq \frac{(b-a)^2}{\pi^2}\int_a^b [u_m'(t)]^2 dt. \tag{2.6}$$

If $m\geq 2$, then

$$u_m'(t)=m\|f(t)\|^{m-2}\operatorname{Re}\langle f'(t),f(t)\rangle,\ t\in(a,b)$$

and by (2.6) we get

$$\begin{aligned}\int_a^b \|f(t)\|^{2m}dt &\leq \frac{(b-a)^2}{\pi^2}\int_a^b \left[m\|f(t)\|^{m-2}\operatorname{Re}\langle f'(t),f(t)\rangle\right]^2 dt\\ &= \frac{m^2(b-a)^2}{\pi^2}\int_a^b \|f(t)\|^{2(m-2)}\left[\operatorname{Re}\langle f'(t),f(t)\rangle\right]^2 dt\\ &\leq \frac{m^2(b-a)^2}{\pi^2}\int_a^b \|f(t)\|^{2(m-1)}\|f'(t)\|^2 dt,\end{aligned}$$

which proves (2.4).

If $m\in[1,2)$ and $f(t)\neq 0$ for $t\in(a,b)$, then

$$u_m'(t)=m\frac{\operatorname{Re}\langle f'(t),f(t)\rangle}{\|f(t)\|^{2-m}}$$

and by (2.6) we get

$$\begin{aligned}\int_a^b \|f(t)\|^{2m}dt &\leq \frac{(b-a)^2}{\pi^2}\int_a^b \left[m\frac{\operatorname{Re}\langle f'(t),f(t)\rangle}{\|f(t)\|^{2-m}}\right]^2 dt\\ &= \frac{m^2(b-a)^2}{\pi^2}\int_a^b \frac{|\operatorname{Re}\langle f'(t),f(t)\rangle|^2}{\|f(t)\|^{2(2-m)}}dt\\ &\leq \frac{m^2(b-a)^2}{\pi^2}\int_a^b \|f(t)\|^{2(m-1)}\|f'(t)\|^2 dt,\end{aligned}$$

which proves (2.5). □

Remark 1. *Assume that $f:[a,b]\to H$ is of class C^1 on $[a,b]$ and $f(a)=f(b)=0$. Then*

$$\int_a^b \|f(t)\|^4\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [\operatorname{Re}\langle f'(t),f(t)\rangle]^2\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b \|f(t)\|^2\|f'(t)\|^2\,dt. \tag{2.7}$$

If $f(t)\neq 0$ for $t\in(a,b)$, then

$$\int_a^b \|f(t)\|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b \frac{|\operatorname{Re}\langle f'(t),f(t)\rangle|^2}{\|f(t)\|^2}dt \le \frac{(b-a)^2}{\pi^2}\int_a^b \|f'(t)\|^2\,dt. \tag{2.8}$$

Now, assume that there is a $c\in(a,b)$ such that $f(c)=0$ and $f(t)\neq 0$ for $t\in(a,c)\cup(c,b)$. From (2.8) we then get

$$\int_a^c \|f(t)\|^2\,dt \le \frac{(c-a)^2}{\pi^2}\int_a^c \|f'(t)\|^2\,dt$$

and

$$\int_c^b \|f(t)\|^2\,dt \le \frac{(b-c)^2}{\pi^2}\int_c^b \|f'(t)\|^2\,dt.$$

Then

$$\begin{aligned}\int_a^b \|f(t)\|^2\,dt &= \int_a^c \|f(t)\|^2\,dt+\int_c^b \|f(t)\|^2\,dt\\ &\le \frac{(c-a)^2}{\pi^2}\int_a^c \|f'(t)\|^2\,dt+\frac{(b-c)^2}{\pi^2}\int_c^b \|f'(t)\|^2\,dt\\ &\le \frac{(b-a)^2}{\pi^2}\left(\int_a^c \|f'(t)\|^2\,dt+\int_c^b \|f'(t)\|^2\,dt\right)\\ &= \frac{(b-a)^2}{\pi^2}\int_a^b \|f'(t)\|^2\,dt.\end{aligned}$$

This shows that the zeros of the function f on (a,b) do not affect the validity of the inequality on the interval $[a,b]$.

Therefore we can state the following result:

Corollary 2. *Assume that $f:[a,b]\to H$ is of class C^1 on $[a,b]$ and $f(a)=f(b)=0$. Then*

$$\int_a^b \|f(t)\|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b \|f'(t)\|^2\,dt. \tag{2.9}$$

By the use of inequality (1.2) we can also prove the following result:

Theorem 2. *Assume that $f:[a,b]\to H$ is of class C^1 on $[a,b]$ and $f(a)=0$. If $m\geq 2$, then*

$$\int_a^b \|f(t)\|^{2m}\,dt \leq \frac{4m^2(b-a)^2}{\pi^2}\int_a^b \|f(t)\|^{2(m-2)}\left[\operatorname{Re}\langle f'(t),f(t)\rangle\right]^2 dt \leq \frac{4m^2(b-a)^2}{\pi^2}\int_a^b \|f(t)\|^{2(m-1)}\|f'(t)\|^2\,dt. \tag{2.10}$$

If $m\in[1,2)$ and $f(t)\neq 0$ for $t\in(a,b)$, then

$$\int_a^b \|f(t)\|^{2m}\,dt \leq \frac{4m^2(b-a)^2}{\pi^2}\int_a^b \frac{\left|\operatorname{Re}\langle f'(t),f(t)\rangle\right|^2}{\|f(t)\|^{2(2-m)}}dt \leq \frac{4m^2(b-a)^2}{\pi^2}\int_a^b \|f(t)\|^{2(m-1)}\|f'(t)\|^2\,dt. \tag{2.11}$$

In a similar way as above, if $f:[a,b]\to H$ is of class C^1 on $[a,b]$ and $f(a)=0$, then

$$\int_a^b \|f(t)\|^2\,dt \leq \frac{4(b-a)^2}{\pi^2}\int_a^b \|f'(t)\|^2\,dt, \tag{2.12}$$

provided $f:[a,b]\to H$ is of class C^1 on $[a,b]$ and $f(a)=0$.

3. Some Simple Inequalities

We can state the following simple inequalities:

Proposition 1. *Let $g\in C^1([a,b],H)$. Then*

$$\left\|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right\| \leq \frac{(b-a)^{1/2}}{2\pi}\left(\int_a^b \|g'(t)-g'(a+b-t)\|^2\,dt\right)^{1/2}. \tag{3.1}$$

Proof. If $g\in C^1([a,b],\mathbb{R})$, then by taking

$$f(t):=\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2},\ t\in[a,b]$$

we have $f(a)=f(b)=0$ and by (2.9) we get

$$\int_a^b \left\|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right\|^2 dt \leq \frac{(b-a)^2}{\pi^2}\int_a^b \left\|\frac{g'(t)-g'(a+b-t)}{2}\right\|^2 dt = \frac{(b-a)^2}{4\pi^2}\int_a^b \|g'(t)-g'(a+b-t)\|^2\,dt. \tag{3.2}$$

The function $\|\cdot\|$ is convex, then by using Jensen's inequality we can also write that

$$\begin{aligned}(3.3)\quad &\frac{1}{b-a}\int_a^b\left\|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right\|^2dt\\ &\geq\left\|\frac{1}{b-a}\int_a^b\left(\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right)dt\right\|^2\\ &=\left\|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right\|^2,\end{aligned}$$

since

$$\int_a^b g(a+b-t)\,dt=\int_a^b g(t)\,dt.$$

Therefore, by (3.2) and (3.3) we get

$$\left\|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right\|^2\leq\frac{(b-a)}{4\pi^2}\int_a^b\|g'(t)-g'(a+b-t)\|^2\,dt.$$

By taking the square root we derive the desired inequality (3.1). □

Proposition 2. *Let $g\in C^1([a,b],H)$. Then*

$$\begin{aligned}(3.4)\quad &\left\|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right\|\\ &\leq\frac{(b-a)^{1/2}}{\pi}\left(\int_a^b\left\|g'(t)-\frac{g(b)-g(a)}{b-a}\right\|^2dt\right)^{1/2}.\end{aligned}$$

Proof. If $g\in C^1([a,b],H)$, then by taking

$$f(t):=g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a},\ \ t\in[a,b]$$

we have $f(a)=f(b)=0$ and by (2.9) we have

$$\begin{aligned}(3.5)\quad &\int_a^b\left\|g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right\|^2dt\\ &\leq\frac{(b-a)^2}{\pi^2}\int_a^b\left\|g'(t)-\frac{g(b)-g(a)}{b-a}\right\|^2dt.\end{aligned}$$

By Jensen's inequality for $\|\cdot\|^2$ we have

$$\begin{aligned}(3.6)\quad &\frac{1}{b-a}\int_a^b\left\|g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right\|^2dt\\ &\geq\left\|\frac{1}{b-a}\int_a^b\left(g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right)dt\right\|^2\\ &=\left\|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right\|^2.\end{aligned}$$

By utilising (3.5) and (3.6) we derive

$$\begin{aligned}&\left\|\frac{1}{b-a}\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}\right\|^2\\&\le\frac{1}{b-a}\int_a^b\left\|g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right\|^2dt\\&\le\frac{b-a}{\pi^2}\int_a^b\left\|g'(t)-\frac{g(b)-g(a)}{b-a}\right\|^2dt.\end{aligned}$$

By taking the square root, we derive the desired result (3.4). □

We also have:

Proposition 3. *Let $g\in C^1([a,b],H)$. Then*

$$\begin{aligned}&\left\|\frac{b+a}{2}\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt\right\|\\&\le\frac{(b-a)^2}{\pi}\left(\frac{1}{b-a}\int_a^b\|g(t)\|^2dt-\left\|\frac{1}{b-a}\int_a^b g(s)\,ds\right\|^2\right)^{1/2}.\end{aligned}\tag{3.7}$$

Proof. Assume that $g:[a,b]\to\mathbb{C}$ is continuous, then by taking

$$f(t):=\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds,\ t\in[a,b]$$

we have $f(a)=f(b)=0$, and by (2.9)

$$\begin{aligned}&\int_a^b\left\|\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right\|^2dt\\&\le\frac{(b-a)^2}{\pi^2}\int_a^b\left\|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right\|^2dt.\end{aligned}\tag{3.8}$$

By Jensen's inequality we also have

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\left\|\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right\|^2dt\\&\ge\left\|\frac{1}{b-a}\int_a^b\left(\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right)dt\right\|^2,\end{aligned}$$

namely

$$\begin{aligned}&\left\|\int_a^b\left(\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right)dt\right\|^2\\&\le(b-a)\int_a^b\left\|\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right\|^2dt\end{aligned}\tag{3.9}$$

Observe that, integrating by parts, we have

$$\begin{aligned}
(3.10)\qquad & \int_a^b \left(\int_a^t g(s)\,ds - \frac{t-a}{b-a} \int_a^b g(s)\,ds \right) dt \\
&= \int_a^b \left(\int_a^t g(s)\,ds \right) dt - \frac{b-a}{2} \int_a^b g(s)\,ds \\
&= b \int_a^b g(s)\,ds - \int_a^b t g(t)\,dt - \frac{b-a}{2} \int_a^b g(s)\,ds \\
&= \frac{b+a}{2} \int_a^b g(s)\,ds - \int_a^b t g(t)\,dt.
\end{aligned}$$

Therefore by (3.8)-(3.10) we derive

$$\begin{aligned}
(3.11)\qquad & \left\| \frac{b+a}{2} \int_a^b g(s)\,ds - \int_a^b t g(t)\,dt \right\|^2 \\
&\le (b-a) \int_a^b \left\| \int_a^t g(s)\,ds - \frac{t-a}{b-a} \int_a^b g(s)\,ds \right\|^2 dt \\
&\le \frac{(b-a)^3}{\pi^2} \int_a^b \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt \\
&= \frac{(b-a)^4}{\pi^2} \frac{1}{b-a} \int_a^b \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt.
\end{aligned}$$

By taking the square root in (3.11) and taking into account that

$$\begin{aligned}
& \frac{1}{b-a} \int_a^b \left\| g(t) - \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 dt \\
&= \frac{1}{b-a} \int_a^b \left[\|g(t)\|^2 - 2\frac{1}{b-a} \operatorname{Re} \left\langle g(t), \int_a^b g(s)\,ds \right\rangle \right. \\
&\left. + \left\| \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 \right] dt \\
&= \frac{1}{b-a} \int_a^b \|g(t)\|^2\,dt - 2\frac{1}{(b-a)^2} \operatorname{Re} \left\langle \int_a^b g(t)\,dt, \int_a^b g(s)\,ds \right\rangle \\
&+ \left\| \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2 \\
&= \frac{1}{b-a} \int_a^b \|g(t)\|^2\,dt - \left\| \frac{1}{b-a} \int_a^b g(s)\,ds \right\|^2,
\end{aligned}$$

we deduce the desired result. □

In [4] we showed among other that, if $g:[a,b]\rightarrow H$ is integrable and there exists the vectors $x,\ X\in H$ such that

$$\operatorname{Re}\left\langle X-g\left(t\right),g\left(t\right)-x\right\rangle \geq 0,\ \text{ for a.e. }\ t\in\left[a,b\right] \tag{3.12}$$

or, equivalently

$$\left\Vert g\left(t\right)-\frac{X+x}{2}\right\Vert \leq\frac{1}{2}\left\Vert X-x\right\Vert \ \text{ for a.e. }\ t\in\left[a,b\right], \tag{3.13}$$

then

$$\frac{1}{b-a}\int_{a}^{b}\left\Vert g\left(t\right)\right\Vert^{2}dt-\left\Vert\frac{1}{b-a}\int_{a}^{b}g\left(s\right)ds\right\Vert^{2}\leq\frac{1}{4}\left\Vert X-x\right\Vert^{2}.$$

Corollary 3. *Let $g\in C^{1}([a,b],H)$ and the vectors $x,\ X\in H$ such that either the condition (3.12) or the condition (3.13) holds, then*

$$\left\Vert\frac{b+a}{2}\int_{a}^{b}g\left(s\right)ds-\int_{a}^{b}tg\left(t\right)dt\right\Vert\leq\frac{\left(b-a\right)^{2}}{2\pi}\left\Vert X-x\right\Vert. \tag{3.14}$$

4. Weighted Inequalities

We have the following composite inequalities:

Theorem 3. *Let $h:[a,b]\rightarrow\left[h\left(a\right),h\left(b\right)\right]$ be a continuous strictly increasing function that is of class C^{1} on $\left(a,b\right)$. Assume that $g:[a,b]\rightarrow H$ is of class C^{1} on $[a,b]$ and $g\left(a\right)=g\left(b\right)=0$. If $m\geq 2$, then*

$$\begin{aligned}\int_{a}^{b}\left\Vert g\left(t\right)\right\Vert^{2m}h'\left(t\right)dt & \\ \leq\frac{m^{2}\left[h\left(b\right)-h\left(a\right)\right]^{2}}{\pi^{2}}&\int_{a}^{b}\left\Vert g\left(t\right)\right\Vert^{2(m-2)}\frac{\left[\operatorname{Re}\left\langle g'\left(t\right),g\left(t\right)\right\rangle\right]^{2}}{h'\left(t\right)}dt \\ \leq\frac{m^{2}\left[h\left(b\right)-h\left(a\right)\right]^{2}}{\pi^{2}}&\int_{a}^{b}\left\Vert g\left(t\right)\right\Vert^{2(m-1)}\frac{\left\Vert g'\left(t\right)\right\Vert^{2}}{h'\left(t\right)}dt.\end{aligned} \tag{4.1}$$

If $m\in[1,2)$ and $g\left(t\right)\neq 0$ for $t\in\left(a,b\right)$, then

$$\begin{aligned}\int_{a}^{b}\left\Vert g\left(t\right)\right\Vert^{2m}h'\left(t\right)dt &\leq\frac{m^{2}\left[h\left(b\right)-h\left(a\right)\right]^{2}}{\pi^{2}}\int_{a}^{b}\frac{\left|\operatorname{Re}\left\langle g'\left(t\right),g\left(t\right)\right\rangle\right|^{2}}{\left\Vert g\left(t\right)\right\Vert^{2(2-m)}h'\left(t\right)}dt \\ &\leq\frac{m^{2}\left[h\left(b\right)-h\left(a\right)\right]^{2}}{\pi^{2}}\int_{a}^{b}\left\Vert g\left(t\right)\right\Vert^{2(m-1)}\frac{\left\Vert g'\left(t\right)\right\Vert^{2}}{h'\left(t\right)}dt.\end{aligned} \tag{4.2}$$

Proof. Let $m\geq 2$ and consider the function $f=g\circ h^{-1}$ defined on $\left[h\left(a\right),h\left(b\right)\right]$. Then $f\left(h\left(a\right)\right)=g\circ h^{-1}\left(h\left(a\right)\right)=g\left(a\right)=0$ and $f\left(h\left(b\right)\right)=g\circ h^{-1}\left(h\left(b\right)\right)=g\left(b\right)=0$.

If we write the inequality (2.4) for $f=g\circ h^{-1}$ on $[h(a),h(b)]$, then we get

$$\begin{aligned}(4.3)\qquad &\int_{h(a)}^{h(b)}\left\|g\circ h^{-1}(z)\right\|^{2m}dz\\ &\leq\frac{m^{2}(h(b)-h(a))^{2}}{\pi^{2}}\\ &\times\int_{h(a)}^{h(b)}\left\|g\circ h^{-1}(z)\right\|^{2(m-2)}\left[\operatorname{Re}\left\langle\left(g\circ h^{-1}\right)'(z),g\circ h^{-1}(z)\right\rangle\right]^{2}dz\\ &\leq\frac{m^{2}(h(b)-h(a))^{2}}{\pi^{2}}\int_{a}^{b}\left\|g\circ h^{-1}(z)\right\|^{2(m-1)}\left\|\left(g\circ h^{-1}\right)'(z)\right\|^{2}dz.\end{aligned}$$

If $g:(c,d)\to H$ is strongly differentiable on (c,d), then $g\circ h^{-1}:(h(c),h(d))\to H$ is strongly differentiable on $(h(c),h(d))$ and using the chain rule and the derivative of inverse functions we have

$$(4.4)\qquad\left(g\circ h^{-1}\right)'(z)=\left(g'\circ h^{-1}\right)(z)\left(h^{-1}\right)'(z)=\frac{\left(g'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}$$

for every $z\in(h(c),h(d))$.

The inequality (4.3) then becomes

$$\begin{aligned}(4.5)\qquad &\int_{h(a)}^{h(b)}\left\|g\circ h^{-1}(z)\right\|^{2m}dz\\ &\leq\frac{m^{2}(h(b)-h(a))^{2}}{\pi^{2}}\\ &\times\int_{h(a)}^{h(b)}\left\|g\circ h^{-1}(z)\right\|^{2(m-2)}\frac{\left[\operatorname{Re}\left\langle\left(g'\circ h^{-1}\right)(z),g\circ h^{-1}(z)\right\rangle\right]^{2}}{\left[\left(h'\circ h^{-1}\right)(z)\right]^{2}}dz\\ &\leq\frac{m^{2}(h(b)-h(a))^{2}}{\pi^{2}}\\ &\times\int_{h(a)}^{h(b)}\frac{\left\|g\circ h^{-1}(z)\right\|^{2(m-1)}\left\|\left(g'\circ h^{-1}\right)(z)\right\|^{2}}{\left[\left(h'\circ h^{-1}\right)(z)\right]^{2}}dz.\end{aligned}$$

Observe also that, by the change of variable $t=h^{-1}(z)$, $z\in[h(a),h(b)]$, we have $z=h(t)$ that gives $dz=h'(t)\,dt$ and

$$\int_{h(a)}^{h(b)}\left\|g\circ h^{-1}(z)\right\|^{2m}dz=\int_{a}^{b}\|g(t)\|^{2m}h'(t)\,dt,$$

$$\begin{aligned}&\int_{h(a)}^{h(b)}\left\|g\circ h^{-1}(z)\right\|^{2(m-2)}\frac{\left[\operatorname{Re}\left\langle\left(g'\circ h^{-1}\right)(z),g\circ h^{-1}(z)\right\rangle\right]^{2}}{\left[\left(h'\circ h^{-1}\right)(z)\right]^{2}}dz\\ &=\int_{a}^{b}\|g(t)\|^{2(m-2)}\frac{\left[\operatorname{Re}\langle g'(t),g(t)\rangle\right]^{2}}{\left[h'(t)\right]^{2}}h'(t)\,dt\\ &=\int_{a}^{b}\|g(t)\|^{2(m-2)}\frac{\left[\operatorname{Re}\langle g'(t),g(t)\rangle\right]^{2}}{h'(t)}dt\end{aligned}$$

and

$$\int_{h(a)}^{h(b)}\frac{\left\|g\circ h^{-1}(z)\right\|^{2(m-1)}\left\|\left(g'\circ h^{-1}\right)(z)\right\|^{2}}{\left[\left(h'\circ h^{-1}\right)(z)\right]^{2}}dz=\int_{a}^{b}\|g(t)\|^{2(m-2)}\frac{\|g'(t)\|^{2}}{h'(t)}dt$$

and by (4.5) we derive (4.1). □

Corollary 4. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b). Assume that $g:[a,b]\to H$ is of class C^1 on $[a,b]$ and $g(a)=g(b)=0$. Then*

$$\int_a^b \|g(t)\|^2 h'(t)\,dt \le \frac{[h(b)-h(a)]^2}{\pi^2}\int_a^b \frac{\|f'(t)\|^2}{h'(t)}dt. \tag{4.6}$$

If $w:[a,b]\to\mathbb{R}$ is continuous, positive on the interval $[a,b]$ and $\int_a^b w(s)\,ds=1$, namely w is a probability density function, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Proposition 4. *Assume that $w:[a,b]\to(0,\infty)$ is a probability density function on $[a,b]$ and $g\in C^1([a,b],H)$ with $g(a)=g(b)=0$. If $m\ge 2$, then*

$$\begin{aligned}\int_a^b \|g(t)\|^{2m} w(t)\,dt &\le \frac{m^2}{\pi^2}\int_a^b \|g(t)\|^{2(m-2)}\frac{[\operatorname{Re}\langle g'(t),g(t)\rangle]^2}{w(t)}dt \\ &\le \frac{m^2}{\pi^2}\int_a^b \|g(t)\|^{2(m-1)}\frac{\|g'(t)\|^2}{w(t)}dt.\end{aligned} \tag{4.7}$$

If $m\in[1,2)$ and $g(t)\neq 0$ for $t\in(a,b)$, then

$$\begin{aligned}\int_a^b \|g(t)\|^{2m} w(t)\,dt &\le \frac{m^2}{\pi^2}\int_a^b \frac{|\operatorname{Re}\langle g'(t),g(t)\rangle|^2}{\|g(t)\|^{2(2-m)}w(t)}dt \\ &\le \frac{m^2}{\pi^2}\int_a^b \|g(t)\|^{2(m-1)}\frac{\|g'(t)\|^2}{w(t)}dt.\end{aligned} \tag{4.8}$$

In particular, we have

Corollary 5. *Assume that $w:[a,b]\to(0,\infty)$ is a probability density function on $[a,b]$ and $f\in C^1([a,b],H)$ with $g(a)=g(b)=0$, then*

$$\int_a^b \|g(t)\|^2 w(t)\,dt \le \frac{1}{\pi^2}\int_a^b \frac{\|g'(t)\|^2}{w(t)}dt. \tag{4.9}$$

We can prove the following result as well:

Proposition 5. *Assume that $w:[a,b]\to(0,\infty)$ is a probability density function on $[a,b]$ and $g\in C^1([a,b],H)$ with $g(a)=0$. If $m\ge 2$, then*

$$\begin{aligned}\int_a^b \|g(t)\|^{2m} w(t)\,dt &\le \frac{4m^2}{\pi^2}\int_a^b \|g(t)\|^{2(m-2)}\frac{[\operatorname{Re}\langle g'(t),g(t)\rangle]^2}{w(t)}dt \\ &\le \frac{4m^2}{\pi^2}\int_a^b \|g(t)\|^{2(m-1)}\frac{\|g'(t)\|^2}{w(t)}dt.\end{aligned} \tag{4.10}$$

If $m\in[1,2)$ and $g(t)\neq 0$ for $t\in(a,b)$, then

$$\begin{aligned}\int_a^b \|g(t)\|^{2m} w(t)\,dt &\le \frac{4m^2}{\pi^2}\int_a^b \frac{|\operatorname{Re}\langle g'(t),g(t)\rangle|^2}{\|g(t)\|^{2(2-m)}w(t)}dt \\ &\le \frac{4m^2}{\pi^2}\int_a^b \|g(t)\|^{2(m-1)}\frac{\|g'(t)\|^2}{w(t)}dt.\end{aligned} \tag{4.11}$$

In particular, we have

Corollary 6. *Assume that* $w:[a,b]\to(0,\infty)$ *is a probability density function on* $[a,b]$ *and* $g\in C^1([a,b],H)$ *with* $g(a)=0$*, then*

$$\int_a^b \|g(t)\|^2 w(t)\,dt \le \frac{4}{\pi^2}\int_a^b \frac{\|g'(t)\|^2}{w(t)}dt. \tag{4.12}$$

We observe that if in (4.12) we replace g by $g-g(a)$, then we get the inequality

$$\int_a^b \|g(t)-g(a)\|^2 w(t)\,dt \le \frac{4}{\pi^2}\int_a^b \frac{\|g'(t)\|^2}{w(t)}dt. \tag{4.13}$$

where $w:[a,b]\to(0,\infty)$ is a probability density function on $[a,b]$ and $g\in C^1([a,b],H)$.

If g is a function with values in the Hilbert space H and $g(a)=0$, then the inequality (4.12) can be stated on the infinite interval $[a,\infty)$ as follows

$$\int_a^\infty \|g(t)\|^2 w(t)\,dt \le \frac{4}{\pi^2}\int_a^\infty \frac{\|g'(t)\|^2}{w(t)}dt, \tag{4.14}$$

provided that $w:[a,\infty)\to(0,\infty)$ is a probability density function on $[a,\infty)$ and $g\in C^1([a,\infty),H)$.

In probability theory and statistics, the *beta prime distribution* (also known as *inverted beta distribution* or *beta distribution of the second kind*) is an absolutely continuous probability distribution defined for $x\ge 0$ with two parameters α and β, having the probability density function:

$$w_{\alpha,\beta}(x) := \frac{x^{\alpha-1}(1+x)^{-\alpha-\beta}}{B(\alpha,\beta)},$$

where B is *Beta function*

$$B(\alpha,\beta) := \int_0^1 t^{\alpha-1}(1-t)^{\beta-1},\ \alpha,\ \beta>1.$$

The cumulative distribution function is

$$W_{\alpha,\beta}(x) = I_{\frac{x}{1+x}}(\alpha,\beta),$$

where I is the *regularized incomplete beta function* defined by

$$I_z(\alpha,\beta) := \frac{B(z;\alpha,\beta)}{B(\alpha,\beta)}.$$

Here $B(\cdot;\alpha,\beta)$ is the *incomplete beta function* defined by

$$B(z;\alpha,\beta) := \int_0^z t^{\alpha-1}(1-t)^{\beta-1},\ \alpha,\ \beta,\ z>0.$$

Now, if we replace w by $w_{\alpha,\beta}$ in (4.14), then we get

$$\begin{aligned}&\int_0^\infty \|g(t)\|^2 t^{\alpha-1}(1+t)^{-\alpha-\beta}\,dt \\ &\le \frac{4}{\pi^2}B^2(\alpha,\beta)\int_0^\infty \|g'(t)\|^2 t^{-\alpha+1}(1+t)^{\alpha+\beta}\,dt\end{aligned} \tag{4.15}$$

for $\alpha,\ \beta>1$ provided $g\in C^1([a,\infty),H)$, $g(0)=0$ and the integrals above exist.

5. Some Weighted Inequalities of Trapezoid Type

We have:

Theorem 4. *Assume that* $w:[a,b]\to(0,\infty)$ *is a probability density function on* $[a,b]$ *and* $g\in C^1([a,b],H)$, *then*

$$\left\|\int_a^b \frac{w(t)+w(a+b-t)}{2}g(t)\,dt-\frac{g(a)+g(b)}{2}\right\| \leq \frac{1}{2\pi}\left(\int_a^b \frac{\|g'(t)-g'(a+b-t)\|^2}{w(t)}dt\right)^{1/2} \leq \frac{1}{2\pi}\max_{t\in[a,b]}\|g'(t)-g'(a+b-t)\|\left(\int_a^b \frac{1}{w(t)}dt\right)^{1/2}, \tag{5.1}$$

provided $\int_a^b \frac{1}{w(t)}dt<\infty$.

In particular, if w *is symmetrical, i.e.* $w(a+b-t)=w(t)$ *for any* $t\in[a,b]$, *then we have*

$$\left\|\int_a^b w(t)\,g(t)\,dt-\frac{g(a)+g(b)}{2}\right\| \leq \frac{1}{2\pi}\left(\int_a^b \frac{\|g'(t)-g'(a+b-t)\|^2}{w(t)}dt\right)^{1/2} \leq \frac{1}{2\pi}\max_{t\in[a,b]}\|g'(t)-g'(a+b-t)\|\left(\int_a^b \frac{1}{w(t)}dt\right)^{1/2}. \tag{5.2}$$

Proof. Consider the function

$$f(t):=\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2},\ t\in[a,b],$$

we have $f(a)=f(b)=0$ and by (4.9) we have

$$\int_a^b \left\|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right\|^2 w(t)\,dt \leq \frac{1}{4\pi^2}\int_a^b \frac{\|g'(t)-g'(a+b-t)\|^2}{w(t)}dt. \tag{5.3}$$

By the weighted Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\int_a^b \left\|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right\|^2 w(t)\,dt \geq \left\|\int_a^b w(t)\left(\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right)dt\right\|^2 = \left\|\int_a^b w(t)\left(\frac{g(t)+g(a+b-t)}{2}\right)dt-\frac{g(a)+g(b)}{2}\right\|^2. \tag{5.4}$$

Observe that, by the change of variable $s = a + b - t$, $t \in [a, b]$ we have that

$$\int_a^b w(t)\, g(a+b-t)\, dt = \int_a^b w(a+b-s)\, g(s)\, ds$$

and then

$$\int_a^b w(t) \frac{g(t) + g(a+b-t)}{2} dt = \int_a^b \frac{w(t) + w(a+b-t)}{2} g(t)\, dt.$$

By making use of (5.3) and (5.4) we get the first inequality in (5.1). The second inequality in (4.1) is obvious. □

Remark 2. *If g' is K-Lipschitzian on $[a,b]$, i.e. $\|g'(t) - g'(s)\| \le K|t-s|$ for any $[a,b]$, then by (4.1) we get*

$$\left| \int_a^b \frac{w(t) + w(a+b-t)}{2} g(t)\, dt - \frac{g(a) + g(b)}{2} \right| \le \frac{1}{\pi} K \left(\int_a^b \frac{\left(t - \frac{a+b}{2}\right)^2}{w(t)} dt \right)^{1/2} \le \frac{1}{2\pi} K (b-a) \left(\int_a^b \frac{1}{w(t)} dt \right)^{1/2}, \tag{5.5}$$

provided $\int_a^b \frac{1}{w(t)} dt < \infty$.

If $g : [a,b] \to H$ is strongly twice differentiable with $\|g''\|_{[a,b],\infty} := \sup_{t\in(a,b)} \|g''(s)\| < \infty$ and $w : [a,b] \to (0,\infty)$ is symmetrical, then

$$\left\| \int_a^b w(t)\, g(t)\, dt - \frac{g(a) + g(b)}{2} \right\| \le \frac{1}{\pi} \|g''\|_{[a,b],\infty} \left(\int_a^b \frac{\left(t - \frac{a+b}{2}\right)^2}{w(t)} dt \right)^{1/2} \le \frac{1}{2\pi} \|g''\|_{[a,b],\infty} (b-a) \left(\int_a^b \frac{1}{w(t)} dt \right)^{1/2}. \tag{5.6}$$

Another trapezoid type weighted inequality is as follows:

Theorem 5. *Assume that $w : [a,b] \to (0,\infty)$ is a probability density function on $[a,b]$ and $g \in C^1([a,b], H)$, then*

$$\left\| \frac{[b - E(w;[a,b])]\, g(a) + [E(w;[a,b]) - a]\, g(b)}{b-a} - \int_a^b g(t)\, w(t)\, dt \right\| \le \frac{1}{\pi} \left(\int_a^b \left\| g'(t) - \frac{g(b) - g(a)}{b-a} \right\|^2 \frac{1}{w(t)} dt \right)^{1/2} \le \frac{1}{\pi} \max_{t\in[a,b]} \left\{ \left\| g'(t) - \frac{g(b)-g(a)}{b-a} \right\| \right\} \left(\int_a^b \frac{1}{w(t)} dt \right)^{1/2}, \tag{5.7}$$

provided $\int_a^b \frac{1}{w(t)} dt < \infty$, where

$$E(w;[a,b]) := \int_a^b t\, w(t)\, dt.$$

Proof. If $g \in C^1([a,b],\mathbb{C})$, then by taking

$$f(t) := g(t) - \frac{(b-t)g(a) + (t-a)g(b)}{b-a}, \ t \in [a,b]$$

we have $f(a) = f(b) = 0$ and by (4.9) we have

$$\int_a^b \left\| g(t) - \frac{(b-t)g(a) + (t-a)g(b)}{b-a} \right\|^2 w(t)\,dt \le \frac{1}{\pi^2} \int_a^b \left\| g'(t) - \frac{g(b)-g(a)}{b-a} \right\|^2 \frac{1}{w(t)} dt. \tag{5.8}$$

By the weighted Cauchy-Bunyakovsky-Schwarz (CBS) integral inequality we have

$$\begin{aligned} &\int_a^b \left\| g(t) - \frac{(b-t)g(a) + (t-a)g(b)}{b-a} \right\|^2 w(t)\,dt \\ &\ge \left\| \int_a^b w(t) g(t)\,dt - \int_a^b w(t) \frac{(b-t)g(a) + (t-a)g(b)}{b-a} dt \right\|^2 \\ &= \left\| \int_a^b w(t) g(t)\,dt - \left(\int_a^b \frac{(b-t)}{b-a} w(t)\,dt \right) g(a) - \left(\int_a^b \frac{(t-a)}{b-a} w(t)\,dt \right) g(b) \right\|^2 \\ &= \left\| \int_a^b g(t) w(t)\,dt - \frac{[b - E(w;[a,b])] g(a) + [E(w;[a,b]) - a] g(b)}{b-a} \right\|^2 . \end{aligned} \tag{5.9}$$

By using (5.8) and (5.9) we get the first inequality in (5.7).

The second inequality in (5.7) is obvious. □

6. Some Inequalities for the Weighted Čebyšev Functional

Consider now the *weighted Čebyšev functional*

$$C_w(\alpha, g) := \int_a^b w(t)\alpha(t) g(t)\,dt - \int_a^b w(t)\alpha(t)\,dt \int_a^b w(t) g(t)\,dt$$

where $w : [a,b] \to \mathbb{R}$ and $w(t) \ge 0$ for a.e. $t \in [a,b]$, $\alpha : [a,b] \to \mathbb{C}$ and $g : [a,b] \to H$ are functions such that the involved integrals exist and $\int_a^b w(t)\,dt = 1$.

Theorem 6. *Assume that $w : [a,b] \to (0,\infty)$ is a probability density function on $[a,b]$, α is absolutely continuous on $[a,b]$ and $g \in C^1([a,b],H)$, then*

$$\|C_w(\alpha, g)\| \le \frac{b-a}{\pi} \left(\int_a^b |\alpha'(t)|^2 dt \right)^{1/2} \times \left(\int_a^b \left\| g(t) - \int_a^b g(s) w(s)\,ds \right\|^2 w^2(t)\,dt \right)^{1/2}. \tag{6.1}$$

Proof. Integrating by parts, we have

$$\begin{aligned}
&\int_a^b \alpha'(t)\left(\int_a^t w(s)\,g(s)\,ds - \int_a^t w(s)\,ds\int_a^b g(s)\,w(s)\,ds\right)dt \\
&= \left[\alpha(t)\left(\int_a^t w(s)\,g(s)\,ds - \int_a^t w(s)\,ds\int_a^b g(s)\,w(s)\,ds\right)\right|_a^b \\
&- \int_a^b \alpha(t)\left(w(t)\,g(t) - w(t)\int_a^b g(s)\,w(s)\,ds\right)dt \\
&= -\int_a^b w(t)\,\alpha(t)\,g(t)\,dt + \int_a^b \alpha(t)\,w(t)\,dt\int_a^b w(s)\,g(s)\,ds,
\end{aligned}$$

which gives that

$$C_w(\alpha,g) = \int_a^b \alpha'(t)\left(\int_a^t w(s)\,ds\int_a^b w(s)\,g(s)\,ds - \int_a^t w(s)\,g(s)\,ds\right)dt. \tag{6.2}$$

Using (CBS) integral inequality we have

$$\begin{aligned}
\|C_w(\alpha,g)\|^2 &\le \int_a^b |\alpha'(t)|^2\,dt \\
&\quad\times \int_a^b \left\|\int_a^t w(s)\,ds\int_a^b w(s)\,g(s)\,ds - \int_a^t w(s)\,g(s)\,ds\right\|^2 dt.
\end{aligned} \tag{6.3}$$

If we take

$$h(t) := \int_a^t w(s)\,ds\int_a^b w(s)\,g(s)\,ds - \int_a^t w(s)\,g(s)\,ds$$

we observe that $h(a) = h(b) = 0$ and $h \in C^1([a,b],H)$.

By (2.9) we then get

$$\begin{aligned}
&\int_a^b \left\|\int_a^t w(s)\,ds\int_a^b w(s)\,g(s)\,ds - \int_a^t w(s)\,g(s)\,ds\right\|^2 dt \\
&\le \frac{(b-a)^2}{\pi^2}\int_a^b \left\|w(t)\,g(t) - w(t)\int_a^b w(s)\,g(s)\,ds\right\|^2 dt \\
&= \frac{(b-a)^2}{\pi^2}\int_a^b \left\|g(t) - \int_a^b w(s)\,g(s)\,ds\right\|^2 w^2(t)\,dt
\end{aligned} \tag{6.4}$$

and by (6.3) and (6.4) we derive (6.1). □

Remark 3. *If we take $w \equiv 1/(b-a)$ in (6.1), then we get the unweighted Grüss' type inequality*

$$\|C_w(\alpha,g)\| \le \frac{\sqrt{b-a}}{\pi}\left(\int_a^b |\alpha'(t)|^2\,dt\right)^{1/2} \times \left(\frac{1}{b-a}\int_a^b \left\|g(t)-\frac{1}{b-a}\int_a^b w(s)\,g(s)\,ds\right\|^2 dt\right)^{1/2} = \frac{\sqrt{b-a}}{\pi}\left(\int_a^b |\alpha'(t)|^2\,dt\right)^{1/2} \times \left(\frac{1}{b-a}\int_a^b \|g(t)\|^2\,dt - \left\|\frac{1}{b-a}\int_a^b g(s)\,ds\right\|^2\right)^{1/2}. \tag{6.5}$$

We also have:

Theorem 7. *Assume that $w:[a,b]\to(0,\infty)$ is a probability density function on $[a,b]$, $\alpha\in L_2([a,b],\mathbb{C})$ and $g\in C^1([a,b],H)$, then*

$$\|C_w(\alpha,g)\| \le \frac{2}{\pi}\left(\int_a^b w(t)\,|\alpha(t)|^2\,dt - \left|\int_a^b w(s)\,\alpha(s)\,ds\right|^2\right)^{1/2} \times \left(\int_a^b \frac{\|g'(t)\|^2}{w(t)}dt\right)^{1/2}. \tag{6.6}$$

Proof. We use the following Sonin type identity

$$C_w(\alpha,g) = \int_a^b \left(\alpha(t) - \int_a^b w(s)\,\alpha(s)\,ds\right)(g(t)-g(a))\,w(t)\,dt, \tag{6.7}$$

which can be proved directly on calculating the integral from the right hand side.

By using the weighted (CBS) integral inequality, we have

$$\begin{aligned} &\|C_w(\alpha,g)\| \\ &\le \int_a^b \left|\alpha(t) - \int_a^b w(s)\,\alpha(s)\,ds\right| \|g(t)-g(a)\|\,w(t)\,dt \\ &\le \left(\int_a^b \left|\alpha(t) - \int_a^b w(s)\,\alpha(s)\,ds\right|^2 w(t)\,dt\right)^{1/2} \\ &\times \left(\int_a^b \|g(t)-g(a)\|^2\,w(t)\,dt\right)^{1/2}. \end{aligned} \tag{6.8}$$

By (4.13) we get

$$\begin{aligned}\|C_w(\alpha,g)\| &\leq \left(\int_a^b \left|\alpha(t) - \int_a^b w(s)\,\alpha(s)\,ds\right|^2 w(t)\,dt\right)^{1/2} \\ &\quad\times \frac{2}{\pi}\left(\int_a^b \frac{\|g'(t)\|^2}{w(t)}dt\right)^{1/2} \\ &= \frac{2}{\pi}\left(\int_a^b |\alpha(t)|^2 w(t)\,dt - \left|\int_a^b w(s)\,\alpha(s)\,ds\right|^2\right)^{1/2} \\ &\quad\times \left(\int_a^b \frac{\|g'(t)\|^2}{w(t)}dt\right)^{1/2},\end{aligned}$$

which proves (6.6). □

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