GENERALIZATIONS OF STEKLOFF AND ALMANSI INEQUALITIES FOR FUNCTIONS WITH VALUES IN HILBERT SPACES AND APPLICATIONS

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Abstract. In this paper we establish some weighted versions of Stekloff and Almansi inequalities for functions with values in Hilbert spaces. Applications for bounding the weighted *Čebyšev functional* are also given.

1. Introduction

It is well known that, see for instance [4], or [7], if $u \in C^1([a,b],\mathbb{R})$, namely u is continuous on $[a,b]$ and has a derivative that is continuous on (a,b) and satisfies $u(a)=u(b)=0$, then the following *Wirtinger type inequality* is valid

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K\in\mathbb{R}$.

If $u \in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a)=0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t)=L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L\in\mathbb{R}$.

For some related Wirtinger type integral inequalities see [2]-[5] and [6]-[14].

In 1901, W. Stekloff, [12], proved that, if $u \in C^1([a,b],\mathbb{R})$ and $\int_a^b u(t)\,dt = 0$, then

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(x)]^2\,dx. \tag{1.3}$$

In addition, if $u(a)=u(b)$, then, as proved by E. Almansi in 1905, [1], the inequality (1.3) can be improved as follows

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{4\pi^2}\int_a^b [u'(x)]^2\,dx. \tag{1.4}$$

We can state the following result for complex functions $h:[a,b]\to\mathbb{C}$.

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Theorem 1. *If* $h \in C^1([a,b],\mathbb{C})$ *and* $\int_a^b h(t)\,dt = 0$, *then*

$$\int_a^b |h(x)|^2\,dx \le \frac{(b-a)^2}{\pi^2}\int_a^b |h'(x)|^2\,dx. \tag{1.5}$$

In addition, if $h(a) = h(b)$, *then*

$$\int_a^b |h(x)|^2\,dx \le \frac{(b-a)^2}{4\pi^2}\int_a^b |h'(x)|^2\,dx. \tag{1.6}$$

The proof follows by (1.3) and (1.4) applied for $u = \operatorname{Re} h$ and $u = \operatorname{Im} h$ and by adding the corresponding inequalities.

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. If $\{e_\alpha\}_{\alpha\in\mathcal{U}}$ ($\mathcal{U}$ is a certain index set), is a complete orthonormal system in a Hilbert space H, then for any element $x \in H$, *Parseval's equality* holds:

$$\|x\|^2 = \sum_{\alpha\in\mathcal{U}} |\langle x, e_\alpha\rangle|^2 \tag{1.7}$$

and the sum on the right-hand side is to be understood as $\sup_{\mathcal{U}_0}\sum_{\alpha\in\mathcal{U}_0}|\langle x, e_\alpha\rangle|^2$where the supremum is taken over all finite subsets $\mathcal{U}_0$ of $\mathcal{U}$.

Assume that H is a separable Hilbert space and $x, y \in H$. If $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H and if $a_n = \langle x, e_n\rangle$ and $b_n = \langle y, e_n\rangle$ are the Fourier coefficients of x and y, then

$$\langle x, y\rangle = \sum_{n=1}^{\infty} a_n\overline{b_n}, \tag{1.8}$$

the so-called *generalized Parseval equality*.

2. Main Results

We consider the complex Hilbert space $(H;\langle .,.\rangle)$.

Theorem 2. *Assume that* $f : [a,b] \to H$ *is of class* C^1 *on* $[a,b]$ *and* $\int_a^b f(t)\,dt = 0$, *then*

$$\int_a^b \|f(t)\|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b \|f'(x)\|^2\,dx. \tag{2.1}$$

In addition, if $f(a) = f(b)$, *then*

$$\int_a^b \|f(t)\|^2\,dt \le \frac{(b-a)^2}{4\pi^2}\int_a^b \|f'(x)\|^2\,dx. \tag{2.2}$$

Proof. Assume that $\{e_\alpha\}_{\alpha\in\mathcal{U}}$ is a complete orthonormal system in the Hilbert space H. For $\alpha \in \mathcal{U}$, consider the function $h_\alpha(t) = \langle f(t), e_\alpha\rangle$, $t \in [a,b]$. Then h_α is of class C^1 on $[a,b]$, $h'_\alpha(t) = \langle f'(t), e_\alpha\rangle$ and

$$\int_a^b h_\alpha(t) = \int_a^b \langle f(t), e_\alpha\rangle\,dt = \left\langle \int_a^b f(t)\,dt, e_\alpha\right\rangle = 0.$$

By using inequality (1.5) we get

$$\int_a^b |\langle f(t), e_\alpha\rangle|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |\langle f'(t), e_\alpha\rangle|^2\,dt, \tag{2.3}$$

for all $\alpha \in \mathcal{U}$.

By summing in inequality (2.3) over $\alpha \in \mathcal{U}$, then we get

$$\sum_{\alpha\in\mathcal{U}} \int_a^b |\langle f(t), e_\alpha\rangle|^2 dt \le \frac{(b-a)^2}{\pi^2} \sum_{\alpha\in\mathcal{U}} \int_a^b |\langle f'(t), e_\alpha\rangle|^2 dt,$$

namely

$$\int_a^b \left(\sum_{\alpha\in\mathcal{U}} |\langle f(t), e_\alpha\rangle|^2 \right) dt \le \frac{(b-a)^2}{\pi^2} \int_a^b \left(\sum_{\alpha\in\mathcal{U}} |\langle f'(t), e_\alpha\rangle|^2 \right) dt. \tag{2.4}$$

By Parseval's equality (1.7) we get

$$\sum_{\alpha\in\mathcal{U}} |\langle f(t), e_\alpha\rangle|^2 = \|f(t)\|^2, \; t \in [a,b]$$

and

$$\sum_{\alpha\in\mathcal{U}} |\langle f'(t), e_\alpha\rangle|^2 = \|f'(t)\|^2, \; t \in (a,b).$$

Therefore by (2.4) we deduce (2.1).

Now, if $f(a) = f(b)$, then $h_\alpha(a) = h_\alpha(b) = 0$ for $\alpha \in \mathcal{U}$ and by (1.6) we derive (2.2). □

The following composite version also holds:

Theorem 3. *Let $h : [a,b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b).*

(i) *If $f \in C^1([a,b], H)$ with $\frac{f'}{\sqrt{h'(t)}} \in L_2([a,b], H)$ and $\int_a^b f(t) h'(t) dt = 0$, then*

$$\int_a^b \|f(t)\|^2 h'(t) dt \le \frac{[h(b) - h(a)]^2}{\pi^2} \int_a^b \frac{\|f'(t)\|^2}{h'(t)} dt. \tag{2.5}$$

(ii) *In addition, if $f(a) = f(b)$, then we have the better inequality*

$$\int_a^b \|f(t)\|^2 h'(t) dt \le \frac{[h(b) - h(a)]^2}{4\pi^2} \int_a^b \frac{\|f'(t)\|^2}{h'(t)} dt. \tag{2.6}$$

Proof. (i) We write the inequality (2.1) for the function $f \circ h^{-1}$ on the interval $[h(a), h(b)]$ to get

$$\int_{h(a)}^{h(b)} \left\| \left(f \circ h^{-1}\right)(z) \right\|^2 dz \le \frac{(h(b) - h(a))^2}{\pi^2} \int_{h(a)}^{h(b)} \left\| \left(f \circ h^{-1}\right)'(z) \right\|^2 dz, \tag{2.7}$$

provided

$$\int_{h(a)}^{h(b)} f \circ h^{-1}(z) dz = 0.$$

If $f : [c,d] \to H$ is strongly differentiable on (c,d), then $f \circ h^{-1} : (h(c), h(d)) \to H$ is strongly differentiable on $(h(c), h(d))$ and using the chain rule and the derivative of inverse functions we have

$$\left(f \circ h^{-1}\right)'(z) = \left(f' \circ h^{-1}\right)(z) \left(h^{-1}\right)'(z) = \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \tag{2.8}$$

for every $z \in (h(c), h(d))$.

Using the inequality (2.7) we then get

$$\int_{h(a)}^{h(b)} \left\| \left(f\circ h^{-1}\right) (z)\right\|^{2} dz \leq \frac{\left(h\left(b\right) -h\left(a\right) \right)^{2}}{\pi^{2}} \int_{h(a)}^{h(b)} \left\| \frac{\left(f^{\prime}\circ h^{-1}\right) (z)}{\left(h^{\prime}\circ h^{-1}\right) (z)}\right\|^{2} dz, \tag{2.9}$$

provided $\int_{h(a)}^{h(b)} f\circ h^{-1}\left(z\right) dz=0.$

Observe also that, by the change of variable $t=h^{-1}\left(z\right) ,\ z\in \left[h\left(a\right) ,h\left(b\right) \right] ,$ we have $z=h\left(t\right)$ that gives $dz=h^{\prime}\left(t\right) dt,$

$$\int_{h(a)}^{h(b)} f\circ h^{-1}\left(z\right) dx=\int_{a}^{b} f\left(t\right) h^{\prime}\left(t\right) dt=0$$

and

$$\int_{h(a)}^{h(b)} \left\| \left(f\circ h^{-1}\right) (z)\right\|^{2} dz=\int_{a}^{b} \left\| f\left(t\right) \right\|^{2} h^{\prime}\left(t\right) dt. \tag{2.10}$$

We also have

$$\int_{h(a)}^{h(b)} \left\| \frac{\left(f^{\prime}\circ h^{-1}\right) (z)}{\left(h^{\prime}\circ h^{-1}\right) (z)}\right\|^{2} dz=\int_{a}^{b} \left\| \frac{f^{\prime}\left(t\right) }{h^{\prime}\left(t\right) }\right\|^{2} h^{\prime}\left(t\right) dt=\int_{a}^{b} \frac{\left\| f^{\prime}\left(t\right) \right\|^{2}}{h^{\prime}\left(t\right) }dt.$$

By making use of (2.9) we get (2.5).

(ii) The inequality (2.6) follows by (2.2) in a similar way. □

If $w:[a,b]\rightarrow \mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\rightarrow \lbrack 0,\infty),$ $W\left(x\right) :=\int_{a}^{x} w\left(s\right) ds$ is strictly increasing and differentiable on $\left(a,b\right)$. We have $W^{\prime}\left(x\right) =w\left(x\right)$ for any $x\in \left(a,b\right)$.

Corollary 1. *Assume that* $w:[a,b]\rightarrow (0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_{a}^{b} w\left(s\right) ds=1$ *and* $f\in C^{1}([a,b],H)$.

(i) *If* $\frac{f^{\prime}}{\sqrt{w}}\in L_{2}\left(\left[a,b\right] ,H\right)$ *and* $\int_{a}^{b} f\left(t\right) w\left(t\right) dt=0,$ *then*

$$\int_{a}^{b} \left\| f\left(t\right) \right\|^{2} w\left(t\right) dt\leq \frac{1}{\pi^{2}}\int_{a}^{b} \frac{\left\| f^{\prime}\left(t\right) \right\|^{2}}{w\left(t\right) }dt. \tag{2.11}$$

(ii) *In addition, if* $f\left(a\right) =f\left(b\right) ,$ *then we have the better inequality*

$$\int_{a}^{b} \left\| f\left(t\right) \right\|^{2} w\left(t\right) dt\leq \frac{1}{4\pi^{2}}\int_{a}^{b} \frac{\left\| f^{\prime}\left(t\right) \right\|^{2}}{w\left(t\right) }dt. \tag{2.12}$$

3. Some Inequalities for the Weighted Čebyšev Functional

Consider now the *weighted Čebyšev inner product functional*

$$D_{w}\left(f,g\right) :=\int_{a}^{b} w\left(t\right) \left\langle f\left(t\right) ,g\left(t\right) \right\rangle dt-\left\langle \int_{a}^{b} w\left(t\right) f\left(t\right) dt,\int_{a}^{b} w\left(t\right) g\left(t\right) dt\right\rangle \tag{3.1}$$

where $f,$ $g:[a,b]\rightarrow H$ and $w:[a,b]\rightarrow (0,\infty)$ are continuous on $[a,b]$ with $\int_{a}^{b} w\left(s\right) ds=1.$

We need the following result that is of interest in itself:

Lemma 1. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds = 1$ *and* $h\in C^1([a,b],H)$ *with* $\frac{h'}{\sqrt{w}}\in L_2([a,b],H)$*. Then*

$$0\le D_w(h,h)=\int_a^b w(t)\|h(t)\|^2dt-\left\|\int_a^b w(t)h(t)\,dt\right\|^2 \le \frac{1}{\pi^2}\int_a^b w(t)\,dt\int_a^b\frac{\|h'(t)\|^2}{w(t)}dt. \tag{3.2}$$

In addition, if $h(a)=h(b)$*, then*

$$0\le D_w(h,h)\le\frac{1}{4\pi^2}\int_a^b\frac{\|h'(t)\|^2}{w(t)}dt. \tag{3.3}$$

Proof. Let

$$f(t):=h(t)-\int_a^b w(s)h(s)\,ds,$$

then

$$\int_a^b f(t)w(t)\,dt=\int_a^b w(t)\left(h(t)-\int_a^b w(s)h(s)\,ds\right)dt =\int_a^b w(t)h(t)\,dt-\int_a^b w(t)\,dt\int_a^b w(s)h(s)\,ds=0.$$

From (2.11) we have

$$\int_a^b w(t)\left\|h(t)-\int_a^b w(s)h(s)\,ds\right\|^2dt\le\frac{1}{\pi^2}\int_a^b\frac{\|h'(t)\|^2}{w(t)}dt \tag{3.4}$$

and since

$$\begin{aligned}
&\int_a^b w(t)\left\|h(t)-\int_a^b w(s)h(s)\,ds\right\|^2dt\\
&=\int_a^b w(t)\left[\|h(t)\|^2-2\operatorname{Re}\left\langle h(t),\int_a^b w(s)h(s)\,ds\right\rangle+\left\|\int_a^b w(s)h(s)\,ds\right\|^2\right]\\
&=\int_a^b w(t)\|h(t)\|^2dt-2\operatorname{Re}\left\langle\int_a^b w(t)h(t)\,dt,\int_a^b w(s)h(s)\,ds\right\rangle\\
&+\left\|\int_a^b w(s)h(s)\,ds\right\|^2\\
&=\int_a^b w(t)\|h(t)\|^2dt-2\left\|\int_a^b w(s)h(s)\,ds\right\|^2+\left\|\int_a^b w(s)h(s)\,ds\right\|^2\\
&=\int_a^b w(t)\|h(t)\|^2dt-\left\|\int_a^b w(t)h(t)\,dt\right\|^2,
\end{aligned}$$

hence the inequality (3.2) is proved.

The inequality (3.3) follows by (2.12) in a similar way and we omit the details. □

We have the following Grüss' type inequality:

Theorem 4. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds=1$ *and* $f,\,g\in C^1([a,b],H)$ *with* $\frac{f'}{\sqrt{w}},\ \frac{g'}{\sqrt{w}}\in L_2([a,b],H)$. *Then*

$$
|D_w(f,g)|\le [D_w(f,f)]^{1/2}[D_w(g,g)]^{1/2} \tag{3.5}
$$
$$
\le\begin{cases}
\frac{1}{\pi^2}\int_a^b w(t)\,dt\left(\int_a^b\frac{\|f'(t)\|^2}{w(t)}dt\right)^{1/2}\left(\int_a^b\frac{\|g'(t)\|^2}{w(t)}dt\right)^{1/2}, \\[2ex]
\frac{1}{2\pi^2}\int_a^b w(t)\,dt\left(\int_a^b\frac{\|f'(t)\|^2}{w(t)}dt\right)^{1/2}\left(\int_a^b\frac{\|g'(t)\|^2}{w(t)}dt\right)^{1/2} \\
\text{if either } f(a)=f(b)\ \text{ or } g(a)=g(b), \\[2ex]
\frac{1}{4\pi^2}\int_a^b w(t)\,dt\left(\int_a^b\frac{\|f'(t)\|^2}{w(t)}dt\right)^{1/2}\left(\int_a^b\frac{\|g'(t)\|^2}{w(t)}dt\right)^{1/2} \\
\text{if } f(a)=f(b)\ \text{ and } g(a)=g(b).
\end{cases}
$$

Proof. Observe that we have the following identity for the inner product

$$
D_w(f,g)=\int_a^b w(t)\left\langle f(t)-\int_a^b w(s)f(s)\,ds,\,g(t)-\int_a^b w(s)g(s)\,ds\right\rangle dt.
$$

By using Schwarz inequality in Hilbert spaces, we have

$$
\begin{aligned}
|D_w(f,g)|&\le\int_a^b w(t)\left|\left\langle f(t)-\int_a^b w(s)f(s)\,ds,\,g(t)-\int_a^b w(s)g(s)\,ds\right\rangle\right| dt\\
&\le\int_a^b w(t)\left\|f(t)-\int_a^b w(s)f(s)\,ds\right\|\left\|g(t)-\int_a^b w(s)g(s)\,ds\right\| dt.
\end{aligned}
$$

Further, by utilising Cauchy-Bunyakovsky-Schwarz weighted integral, we have

$$
\begin{aligned}
&\int_a^b w(t)\left\|f(t)-\int_a^b w(s)f(s)\,ds\right\|\left\|g(t)-\int_a^b w(s)g(s)\,ds\right\| dt\\
&\le\left(\int_a^b w(t)\left\|f(t)-\int_a^b w(s)f(s)\,ds\right\|^2 dt\right)^{1/2}\\
&\times\left(\int_a^b w(t)\left\|g(t)-\int_a^b w(s)g(s)\,ds\right\|^2 dt\right)^{1/2}\\
&=[D_w(f,f)]^{1/2}[D_w(g,g)]^{1/2},
\end{aligned}
$$

which proves the first inequality in (3.5).

The second part follows by Lemma 1 for $h=f$ and $h=g$. □

For $w\equiv 1/(b-a)$ we consider the unweighted *Čebyšev functional*

$$
D(f,g):=\frac{1}{b-a}\int_a^b\langle f(t),g(t)\rangle\,dt-\left\langle\frac{1}{b-a}\int_a^b f(t)\,dt,\frac{1}{b-a}\int_a^b g(t)\,dt\right\rangle.
$$

We have the following particular result:

Corollary 2. *Assume that* $f,\ g\in C^{1}([a,b],\mathbb{C})$ *with* $f',\ g'\in L_{2}\left([a,b],H\right).$ *Then*

$$
\begin{aligned}
(3.6)\qquad &\left|D\left(f,g\right)\right|\leq\left[D\left(f,f\right)\right]^{1/2}\left[D\left(g,g\right)\right]^{1/2}\\
&\leq\begin{cases}
\frac{1}{\pi^{2}}\left(b-a\right)\left(\int_{a}^{b}\left\|f'\left(t\right)\right\|^{2}dt\right)^{1/2}\left(\int_{a}^{b}\left\|g'\left(t\right)\right\|^{2}dt\right)^{1/2},\\
\\
\frac{1}{2\pi^{2}}\left(b-a\right)\left(\int_{a}^{b}\left\|f'\left(t\right)\right\|^{2}dt\right)^{1/2}\left(\int_{a}^{b}\left\|g'\left(t\right)\right\|^{2}dt\right)^{1/2}\\
\text{if either } f\left(a\right)=f\left(b\right)\ \text{ or } g\left(a\right)=g\left(b\right),\\
\\
\frac{1}{4\pi^{2}}\left(b-a\right)\left(\int_{a}^{b}\left\|f'\left(t\right)\right\|^{2}dt\right)^{1/2}\left(\int_{a}^{b}\left\|g'\left(t\right)\right\|^{2}dt\right)^{1/2}\\
\text{if } f\left(a\right)=f\left(b\right)\ \text{ and } g\left(a\right)=g\left(b\right).
\end{cases}
\end{aligned}
$$

Remark 1. *The first inequality in (3.6) for the case of real functions was obtained by Lupaş in 1973,* [8].

If we assume that $g\in C^{1}\left([a,b],H\right)$ and take

$$
h\left(t\right)=\tilde{g}\left(t\right):=\frac{1}{2}\left[g\left(a+b-t\right)-g\left(t\right)\right],\ \ t\in\left[a,b\right],
$$

then $\int_{a}^{b}h\left(t\right)dt=0,$

$$
\begin{aligned}
\int_{a}^{b}\left\|h\left(t\right)\right\|^{2}dt&=\frac{1}{4}\int_{a}^{b}\left\|g\left(a+b-t\right)-g\left(t\right)\right\|^{2}dt\\
&=\frac{1}{4}\left[\int_{a}^{b}\left\|g\left(a+b-t\right)\right\|^{2}-2\operatorname{Re}\left\langle g\left(a+b-t\right),g\left(t\right)\right\rangle+\left\|g\left(t\right)\right\|^{2}\right]dt\\
&=\frac{1}{2}\left[\int_{a}^{b}\left|g\left(t\right)\right|^{2}dt-\int_{a}^{b}\operatorname{Re}\left\langle g\left(a+b-t\right),g\left(t\right)\right\rangle dt\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{a}^{b}\left\|h'\left(t\right)\right\|^{2}dt&=\frac{1}{4}\int_{a}^{b}\left\|g'\left(a+b-t\right)+g'\left(t\right)\right\|^{2}dt\\
&=\frac{1}{2}\left[\int_{a}^{b}\left\|g'\left(t\right)\right\|^{2}dt+\int_{a}^{b}\operatorname{Re}\left\langle g'\left(a+b-t\right),g'\left(t\right)\right\rangle dt\right]
\end{aligned}
$$

and by (2.1) we get

$$
\begin{aligned}
(3.7)\qquad 0&\leq\left[\int_{a}^{b}\left|g\left(t\right)\right|^{2}dt-\int_{a}^{b}\operatorname{Re}\left\langle g\left(a+b-t\right),g\left(t\right)\right\rangle dt\right]\\
&\leq\frac{\left(b-a\right)^{2}}{\pi^{2}}\left[\int_{a}^{b}\left\|g'\left(t\right)\right\|^{2}dt+\int_{a}^{b}\operatorname{Re}\left\langle g'\left(a+b-t\right),g'\left(t\right)\right\rangle dt\right].
\end{aligned}
$$

If we assume that $g\in C^{1}\left([a,b],H\right)$ with $\int_{a}^{b}g\left(t\right)dt=0$, and if we take

$$
h\left(t\right)=\breve{g}\left(t\right):=\frac{1}{2}\left[g\left(a+b-t\right)+g\left(t\right)\right]
$$

we have $h(a) = h(b)$ and by (2.2) we derive

$$0 \leq \frac{1}{2}\left[\int_a^b \|g(t)\|^2 dt + \int_a^b \operatorname{Re}\langle g(a+b-t), g(t)\rangle dt\right] \tag{3.8}$$
$$\leq \frac{(b-a)^2}{16\pi^2}\int_a^b \|g'(t) - g'(a+b-t)\|^2 dt$$
$$= \frac{(b-a)^2}{8\pi^2}\left[\int_a^b \|g'(t)\|^2 - \int_a^b \operatorname{Re}\langle g(a+b-t), g(t)\rangle dt\right],$$

namely

$$0 \leq \int_a^b \|g(t)\|^2 dt + \int_a^b \operatorname{Re}\langle g(a+b-t), g(t)\rangle dt \tag{3.9}$$
$$\leq \frac{(b-a)^2}{4\pi^2}\left[\int_a^b \|g'(t)\|^2 - \int_a^b \operatorname{Re}\langle g(a+b-t), g(t)\rangle dt\right].$$

If g' is Lipschitzian with the constant K, namely $\|g'(t) - g'(s)\| \leq K|t-s|$ for all $t, s \in [a,b]$, then

$$\frac{1}{2}\|g'(t) - g'(a+b-t)\| \leq K\left|t - \frac{a+b}{2}\right|, \ t \in [a,b].$$

By the first inequality in (3.8) we get

$$0 \leq \int_a^b \|g'(t)\|^2 dt + \int_a^b \operatorname{Re}\langle g'(a+b-t), g'(t)\rangle dt$$
$$\leq \frac{(b-a)^2}{2\pi^2}\int_a^b \left\|\frac{g'(t) - g'(a+b-t)}{2}\right\|^2 dt \leq \frac{(b-a)^2}{2\pi^2}K^2\int_a^b \left|t - \frac{a+b}{2}\right|^2 dt$$

and since

$$\int_a^b \left|t - \frac{a+b}{2}\right|^2 dt = \frac{(b-a)^3}{12},$$

hence we obtain the inequality

$$0 \leq \int_a^b \|g'(t)\|^2 dt + \int_a^b \operatorname{Re}\langle g'(a+b-t), g'(t)\rangle dt \leq \frac{(b-a)^5}{24\pi^2}K^2. \tag{3.10}$$

Moreover, if we write the inequality (3.2) for $\breve{g}$ that is symmetrical on $[a,b]$, then we get

$$0 \leq \frac{1}{b-a}\int_a^b \|\breve{g}(x)\|^2 dx - \left\|\frac{1}{b-a}\int_a^b g(x)\,dx\right\|^2 \leq \frac{b-a}{4\pi^2}\int_a^b \left\|\widetilde{g'}(x)\right\|^2 dx. \tag{3.11}$$

In addition, if g' is Lipschitzian with the constant $K > 0$, then we obtain from (3.11) that

$$0 \leq \frac{1}{b-a}\int_a^b \|\breve{g}(x)\|^2 dx - \left\|\frac{1}{b-a}\int_a^b g(x)\,dx\right\|^2 \leq \frac{1}{48\pi^2}K^2(b-a)^4. \tag{3.12}$$

Finally if g is *symmetrical* on $[a,b]$, namely $g(a+b-t)=g(t)$ for any $t\in[a,b]$, g' is Lipschitzian with the constant $K>0$, then we get from (3.12) that

$$0\le\frac{1}{b-a}\int_a^b\|g(x)\|^2dx-\left\|\frac{1}{b-a}\int_a^b g(x)\,dx\right\|^2\le\frac{1}{48\pi^2}K^2(b-a)^4. \tag{3.13}$$

We observe that

$$\begin{aligned}
&D\left(\breve{f},g\right)\\
&=\frac{1}{b-a}\int_a^b\left\langle\breve{f}(t),g(t)\right\rangle dt-\left\langle\frac{1}{b-a}\int_a^b\breve{f}(t)\,dt,\frac{1}{b-a}\int_a^b g(t)\,dt\right\rangle\\
&=\frac{1}{b-a}\int_a^b\left\langle\frac{f(t)+f(a+b-t)}{2},g(t)\right\rangle dt\\
&-\left\langle\frac{1}{b-a}\int_a^b\breve{f}(t)\,dt,\frac{1}{b-a}\int_a^b g(t)\,dt\right\rangle\\
&=\frac{1}{b-a}\int_a^b\left\langle f(t),\frac{g(t)+g(a+b-t)}{2}\right\rangle dt\\
&-\left\langle\frac{1}{b-a}\int_a^b\breve{f}(t)\,dt,\frac{1}{b-a}\int_a^b g(t)\,dt\right\rangle\\
&=D(f,\breve{g})
\end{aligned}$$

and

$$\begin{aligned}
&D\left(\breve{f},\breve{g}\right)\\
&=\frac{1}{b-a}\int_a^b\left\langle\breve{f}(t),\breve{g}(t)\right\rangle dt-\left\langle\frac{1}{b-a}\int_a^b\breve{f}(t)\,dt,\frac{1}{b-a}\int_a^b \breve{g}(t)\,dt\right\rangle\\
&=\frac{1}{b-a}\int_a^b\left\langle\breve{f}(t),\frac{g(t)+g(a+b-t)}{2}\right\rangle dt\\
&-\left\langle\frac{1}{b-a}\int_a^b\breve{f}(t)\,dt,\frac{1}{b-a}\int_a^b g(t)\,dt\right\rangle\\
&=C\left(\breve{f},g\right).
\end{aligned}$$

Proposition 1. *Assume that* $f,\,g\in C^1([a,b],H)$.

(i) *If* f' *is Lipschitzian with the constant* K *and* $g'\in L_2([a,b],H)$, *then*

$$\left|D\left(\breve{f},g\right)\right|\le\frac{\sqrt{3}}{12\pi^2}K(b-a)^{5/2}\left(\int_a^b\|g'(t)\|^2dt\right)^{1/2}. \tag{3.14}$$

(ii) *If* f' *is Lipschitzian with the constant* K *and* g' *is Lipschitzian with the constant* $L>0$, *then*

$$\left|D\left(\breve{f},g\right)\right|\le\frac{1}{48\pi^2}KL(b-a)^4. \tag{3.15}$$

The inequality (3.14) follows by the second inequality in (3.6) for the functions $\breve{f}$ and g while the inequality (3.15) follows by the third inequality in (3.6) for the functions $\breve{f}$ and $\breve{g}$.

Corollary 3. *Assume that* $f,\, g \in C^1([a,b], H)$.

(i) *If* f *is symmetrical on* $[a,b]$, f' *is Lipschitzian with the constant* K *and* $g' \in L_2([a,b], H)$, *then*

$$|D(f,g)| \le \frac{\sqrt{3}}{12\pi^2} K (b-a)^{5/2} \left(\int_a^b \|g'(t)\|^2 dt \right)^{1/2}. \tag{3.16}$$

(ii) *If* f *and* g *are symmetrical on* $[a,b]$, f' *is Lipschitzian with the constant* K *and* g' *is Lipschitzian with the constant* $L > 0$, *then*

$$|D(f,g)| \le \frac{1}{48\pi^2} KL (b-a)^4. \tag{3.17}$$

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