ABSOLUTE VALUE INTEGRAL INEQUALITIES FOR FUNCTIONS WITH OPERATOR VALUES IN HILBERT SPACES RELATED TO WIRTINGER'S RESULT

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^{*}-algebra of bounded linear operators on H. In this paper we show among others that, if $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ such that $\int_a^b w(s)\,ds=1$ and $A\in C^1([a,b],\mathcal{B}(H))$ with $\frac{A'}{\sqrt{w}}\in L_2([a,b],\mathcal{B}(H))$ and $A(a)=A(b)=0$, then

$$\int_a^b w(t)\,|A(t)|^2\,dt\le\frac{1}{\pi^2}\int_a^b\frac{|A'(t)|^2}{w(t)}dt$$

in the operator order of $\mathcal{B}(H)$. Applications related to the trapezoid unweighted and weighted inequalities and of Grüss' type inequalities are also provided.

1. Introduction

It is well known that, see for instance [4], or [7], if $u\in C^1([a,b],\mathbb{R})$ satisfies $u(a)=u(b)=0$, then

$$\int_a^b u^2(t)\,dt\le\frac{(b-a)^2}{\pi^2}\int_a^b[u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a)=0$, then also

$$\int_a^b u^2(t)\,dt\le\frac{4(b-a)^2}{\pi^2}\int_a^b[u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t)=L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{C})$ is a function with complex values and $u(a)=u(b)=0$, then $\operatorname{Re}u(a)=\operatorname{Re}u(b)=0$ and $\operatorname{Im}u(a)=\operatorname{Im}u(b)=0$ and by writing (1.1) for $\operatorname{Re}u$ and $\operatorname{Im}u$ and adding the obtained inequalities, we get

$$\int_a^b|u(t)|^2\,dt\le\frac{(b-a)^2}{\pi^2}\int_a^b|u'(t)|^2\,dt \tag{1.3}$$

with the equality holding if and only if

$$u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$$

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for some complex constant $K \in \mathbb{C}$.

Similarly, if $u \in C^1([a,b],\mathbb{C})$ with $u(a)=0$, then by (1.2) we have

$$\int_a^b |u(t)|^2 dt \le \frac{4(b-a)^2}{\pi^2} \int_a^b |u'(t)|^2 dt \tag{1.4}$$

and the equality holds if and only if

$$u(t) = L \sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$$

for some complex constant $L \in \mathbb{C}$.

For some related Wirtinger type integral inequalities see [1], [3], [4] and [6]-[10].

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. If $\{e_\alpha\}_{\alpha\in\mathcal{U}}$ ($\mathcal{U}$ is a certain index set), is a complete orthonormal system in a Hilbert space H, then for any element $x \in H$, *Parseval's equality* holds:

$$\|x\|^2 = \sum_{\alpha\in\mathcal{U}} |\langle x, e_\alpha\rangle|^2 \tag{1.5}$$

and the sum on the right-hand side is to be understood as $\sup_{\mathcal{U}_0} \sum_{\alpha\in\mathcal{U}_0} |\langle x, e_\alpha\rangle|^2$ where the supremum is taken over all finite subsets $\mathcal{U}_0$ of $\mathcal{U}$.

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. For $A : [a,b] \to B(H)$ strongly measurable and Bochner squared integrable, namely $A \in L_2([a,b],\mathcal{B}(H))$, we have

$$\frac{1}{b-a}\int_a^b |A(t)|^2 dt \ge \left|\frac{1}{b-a}\int_a^b A(t)\,dt\right|^2 \tag{1.6}$$

in the operator order of $\mathcal{B}(H)$.

Indeed, since

$$\begin{aligned} 0 \le |A(t)-A(s)|^2 &= (A(t)-A(s))^*(A(t)-A(s)) \\ &= |A(t)|^2 - A^*(s)A(t) - A^*(t)A(s) + |A(s)|^2, \end{aligned}$$

hence

$$|A(t)|^2 + |A(s)|^2 \ge A^*(s)A(t) + A^*(t)A(s)$$

for all $t,s \in [a,b]$.

Integrating over $s,t \in [a,b]$, we get

$$\int_a^b\int_a^b \left[|A(t)|^2 + |A(s)|^2\right] dtds \ge \int_a^b\int_a^b \left[A^*(s)A(t) + A^*(t)A(s)\right] dtds \tag{1.7}$$

in the operator order of $\mathcal{B}(H)$.

Observe that

$$\begin{aligned} \int_a^b\int_a^b \left[|A(t)|^2 + |A(s)|^2\right] dtds &= (b-a)\int_a^b |A(t)|^2 dt + (b-a)\int_a^b |A(s)|^2 ds \\ &= 2(b-a)\int_a^b |A(t)|^2 dt \end{aligned}$$

and

$$\begin{aligned}&\int_a^b \int_a^b \left[A^*(s)A(t)+A^*(t)A(s)\right]dtds\\&=\int_a^b A^*(s)\,ds\int_a^b A(t)\,dt+\int_a^b A^*(t)\,dt\int_a^b A(s)\,ds\\&=2\left(\int_a^b A(t)\,dt\right)^*\int_a^b A(t)\,dt,\end{aligned}$$

and by (1.7) we derive (1.6).

In a similar way, if $w:[a,b]\to[0,\infty)$ with $\int_a^b w(t)\,dt=1$, then

$$\int_a^b w(t)\left|A(t)\right|^2 dt \ge \left|\int_a^b w(t)A(t)\,dt\right|^2, \tag{1.8}$$

provided that $A\in L_{2,w}([a,b],\mathcal{B}(H)):=\left\{A:[a,b]\to B(H),\ \int_a^b w(t)\left\|A(t)\right\|^2 dt<\infty\right\}$.

2. Main Results

We have the following inequality of Wirtinger type in the operator order of $\mathcal{B}(H)$:

Theorem 1. *Assume that $A:[a,b]\to\mathcal{B}(H)$ is of class C^1 on $[a,b]$ and $A(a)=A(b)=0$, then*

$$\int_a^b \left|A(t)\right|^2 dt \le \frac{(b-a)^2}{\pi^2}\int_a^b \left|A'(t)\right|^2 dt. \tag{2.1}$$

If only $A(a)=0$, then

$$\int_a^b \left|A(t)\right|^2 dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b \left|A'(t)\right|^2 dt. \tag{2.2}$$

Proof. Let $x\in H$. Assume that $\{e_\alpha\}_{\alpha\in\mathcal{U}}$ is a complete orthonormal system in the Hilbert space H. For $\alpha\in\mathcal{U}$, consider the function $h_\alpha(t)=\langle A(t)x,e_\alpha\rangle$, $t\in[a,b]$. Then h_α is of class C^1 on $[a,b]$, $h_\alpha'(t)=\langle A'(t)x,e_\alpha\rangle$ and

$$h_\alpha(a)=\langle A(a)x,e_\alpha\rangle=0=\langle A(b)x,e_\alpha\rangle=h_\alpha(b).$$

By using inequality (1.3) we get

$$\int_a^b \left|\langle A(t)x,e_\alpha\rangle\right|^2 dt \le \frac{(b-a)^2}{\pi^2}\int_a^b \left|\langle A'(t)x,e_\alpha\rangle\right|^2 dt, \tag{2.3}$$

for all $\alpha\in\mathcal{U}$.

By summing in inequality (2.3) over $\alpha\in\mathcal{U}$, then we get

$$\sum_{\alpha\in\mathcal{U}}\int_a^b \left|\langle A(t)x,e_\alpha\rangle\right|^2 dt \le \frac{(b-a)^2}{\pi^2}\sum_{\alpha\in\mathcal{U}}\int_a^b \left|\langle A'(t)x,e_\alpha\rangle\right|^2 dt,$$

namely

$$\int_a^b \left(\sum_{\alpha\in\mathcal{U}}\left|\langle A(t)x,e_\alpha\rangle\right|^2\right) dt \le \frac{(b-a)^2}{\pi^2}\int_a^b \left(\sum_{\alpha\in\mathcal{U}}\left|\langle A'(t)x,e_\alpha\rangle\right|^2\right) dt. \tag{2.4}$$

By Parseval's equality (1.5) we get

$$\sum_{\alpha\in\mathcal{U}}\left|\langle A(t)x,e_\alpha\rangle\right|^2=\|A(t)x\|^2,\ t\in[a,b]$$

and

$$\sum_{\alpha\in\mathcal{U}}\left|\langle A'(t)x,e_\alpha\rangle\right|^2=\|A'(t)x\|^2,\ t\in(a,b).$$

Therefore by (2.4) we deduce

$$\int_a^b\|A(t)x\|^2dt\le\frac{(b-a)^2}{\pi^2}\int_a^b\|A'(t)x\|^2dt \tag{2.5}$$

for all $x\in H$.

Now, observe that

$$\begin{aligned}\int_a^b\|A(t)x\|^2dt&=\int_a^b\langle A(t)x,A(t)x\rangle dt=\int_a^b\langle A^*(t)A(t)x,x\rangle dt\\&=\int_a^b\left\langle|A(t)|^2x,x\right\rangle dt=\left\langle\left(\int_a^b|A(t)|^2dt\right)x,x\right\rangle\end{aligned}$$

and

$$\begin{aligned}\int_a^b\|A'(t)x\|^2dt&=\int_a^b\langle A'(t)x,A'(t)x\rangle dt=\int_a^b\left\langle(A'(t))^*A'(t)x,x\right\rangle dt\\&=\int_a^b\left\langle|A'(t)|^2x,x\right\rangle dt=\left\langle\left(\int_a^b|A'(t)|^2dt\right)x,x\right\rangle\end{aligned}$$

and by (2.5) we get

$$\left\langle\left(\int_a^b|A(t)|^2dt\right)x,x\right\rangle\le\frac{(b-a)^2}{\pi^2}\left\langle\left(\int_a^b|A'(t)|^2dt\right)x,x\right\rangle$$

for all $x\in H$, which is equivalent to (2.1).

The inequality (2.2) follows in a similar way from (1.2). □

Recall the Löwner–Heinz inequality which says that $A\ge B\ge0$ implies $A^\alpha\ge B^\alpha$ for all $\alpha\in[0,1]$. By taking the power $1/2$ in (2.1) we derive

$$\left(\int_a^b|A(t)|^2dt\right)^{1/2}\le\frac{b-a}{\pi}\left(\int_a^b|A'(t)|^2dt\right)^{1/2}, \tag{2.6}$$

provided that $A:[a,b]\to\mathcal{B}(H)$ is of class C^1 on $[a,b]$ and $A(a)=A(b)=0$.

If $A(a)=0$, then also

$$\left(\int_a^b|A(t)|^2dt\right)^{1/2}\le\frac{2(b-a)}{\pi}\left(\int_a^b|A'(t)|^2dt\right)^{1/2}. \tag{2.7}$$

We can state the following simple inequalities:

Proposition 1. *Let $B \in C^1([a,b], \mathcal{B}(H))$. Then*

$$\left| \frac{1}{b-a} \int_a^b B(t)\,dt - \frac{B(a)+B(b)}{2} \right|^2 \leq \frac{b-a}{4\pi^2} \int_a^b \left| B'(t) - B'(a+b-t) \right|^2 dt. \tag{2.8}$$

Proof. If $B \in C^1([a,b], \mathcal{B}(H))$, then by taking

$$A(t) := \frac{B(t)+B(a+b-t)}{2} - \frac{B(a)+B(b)}{2},\ t \in [a,b]$$

we have $A(a) = A(b) = 0$ and by (2.1) we get

$$\begin{aligned} &\int_a^b \left| \frac{B(t)+B(a+b-t)}{2} - \frac{B(a)+B(b)}{2} \right|^2 dt \\ &\leq \frac{(b-a)^2}{\pi^2} \int_a^b \left| \frac{B'(t)-B'(a+b-t)}{2} \right|^2 dt \\ &= \frac{(b-a)^2}{4\pi^2} \int_a^b \left| B'(t)-B'(a+b-t) \right|^2 dt. \end{aligned} \tag{2.9}$$

By the Cauchy-Bunyakowsky-Schwarz inequality (1.6) we have

$$\begin{aligned} &\frac{1}{b-a} \int_a^b \left| \frac{B(t)+B(a+b-t)}{2} - \frac{B(a)+B(b)}{2} \right|^2 dt \\ &\geq \left| \frac{1}{b-a} \int_a^b \left(\frac{B(t)+B(a+b-t)}{2} - \frac{B(a)+B(b)}{2} \right) dt \right|^2 \\ &= \left| \frac{1}{b-a} \int_a^b B(t)\,dt - \frac{B(a)+B(b)}{2} \right|^2, \end{aligned} \tag{2.10}$$

since

$$\int_a^b B(a+b-t)\,dt = \int_a^b B(t)\,dt.$$

Therefore, by (2.9) and (2.10) we get

$$\left| \frac{1}{b-a} \int_a^b B(t)\,dt - \frac{B(a)+B(b)}{2} \right|^2 \leq \frac{(b-a)}{4\pi^2} \int_a^b \left| B'(t) - B'(a+b-t) \right|^2 dt,$$

which proves (2.8). □

Proposition 2. *Let $B \in C^1([a,b], \mathcal{B}(H))$. Then*

$$\left| \frac{1}{b-a} \int_a^b B(t)\,dt - \frac{B(a)+B(b)}{2} \right|^2 \leq \frac{b-a}{\pi^2} \int_a^b \left| B'(t) - \frac{B(b)-B(a)}{b-a} \right|^2 dt. \tag{2.11}$$

Proof. If $B \in C^1([a,b], \mathcal{B}(H))$, then by taking

$$A(t) := B(t) - \frac{B(a)(b-t)+B(b)(t-a)}{b-a},\ t \in [a,b]$$

we have $A(a)=A(b)=0$ and by (2.1) we have

$$\int_a^b \left|B(t)-\frac{B(a)(b-t)+B(b)(t-a)}{b-a}\right|^2 dt \leq \frac{(b-a)^2}{\pi^2}\int_a^b \left|B'(t)-\frac{B(b)-B(a)}{b-a}\right|^2 dt. \tag{2.12}$$

By the Cauchy-Bunyakowsky-Schwarz inequality (1.6) we have

$$\begin{aligned} &\frac{1}{b-a}\int_a^b \left|B(t)-\frac{B(a)(b-t)+B(b)(t-a)}{b-a}\right|^2 dt \\ &\geq \left|\frac{1}{b-a}\int_a^b \left(B(t)-\frac{B(a)(b-t)+B(b)(t-a)}{b-a}\right) dt\right|^2 \\ &= \left|\frac{1}{b-a}\int_a^b B(t)\,dt-\frac{B(a)+B(b)}{2}\right|^2. \end{aligned} \tag{2.13}$$

By utilising (2.12) and (2.13) we derive

$$\begin{aligned} &\left|\frac{1}{b-a}\int_a^b B(t)\,dt-\frac{B(a)+B(b)}{2}\right|^2 \\ &\leq \frac{1}{b-a}\int_a^b \left|B(t)-\frac{B(a)(b-t)+B(b)(t-a)}{b-a}\right|^2 dt \\ &\leq \frac{b-a}{\pi^2}\int_a^b \left|B'(t)-\frac{B(b)-B(a)}{b-a}\right|^2 dt, \end{aligned}$$

which proves the desired result (2.11). □

We also have:

Proposition 3. *Let $B\in C^1([a,b],\mathcal{B}(H))$. Then*

$$\begin{aligned} &\left|\frac{b+a}{2}\int_a^b B(s)\,ds-\int_a^b tB(t)\,dt\right|^2 \\ &\leq \frac{(b-a)^4}{\pi^2}\left(\frac{1}{b-a}\int_a^b |B(t)|^2\,dt-\left|\frac{1}{b-a}\int_a^b B(s)\,ds\right|^2\right). \end{aligned} \tag{2.14}$$

Proof. Assume that $B:[a,b]\to\mathbb{C}$ is continuous, then by taking

$$A(t):=\int_a^t B(s)\,ds-\frac{t-a}{b-a}\int_a^b B(s)\,ds,\ t\in[a,b]$$

we have $A(a)=A(b)=0$, and by (2.1)

$$\begin{aligned} &\int_a^b \left|\int_a^t B(s)\,ds-\frac{t-a}{b-a}\int_a^b B(s)\,ds\right|^2 dt \\ &\leq \frac{(b-a)^2}{\pi^2}\int_a^b \left|B(t)-\frac{1}{b-a}\int_a^b B(s)\,ds\right|^2 dt. \end{aligned} \tag{2.15}$$

By (1.6) we also have

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\left|\int_a^t B(s)\,ds-\frac{t-a}{b-a}\int_a^b B(s)\,ds\right|^2 dt\\
&\geq\left|\frac{1}{b-a}\int_a^b\left(\int_a^t B(s)\,ds-\frac{t-a}{b-a}\int_a^b B(s)\,ds\right)dt\right|^2,
\end{aligned}$$

namely

$$\begin{aligned}
&\left|\int_a^b\left(\int_a^t B(s)\,ds-\frac{t-a}{b-a}\int_a^b B(s)\,ds\right)dt\right|^2\\
&\leq(b-a)\int_a^b\left|\int_a^t B(s)\,ds-\frac{t-a}{b-a}\int_a^b B(s)\,ds\right|^2 dt
\end{aligned}\tag{2.16}$$

Observe that, integrating by parts, we have

$$\begin{aligned}
&\int_a^b\left(\int_a^t B(s)\,ds-\frac{t-a}{b-a}\int_a^b B(s)\,ds\right)dt\\
&=\int_a^b\left(\int_a^t B(s)\,ds\right)dt-\frac{b-a}{2}\int_a^b B(s)\,ds\\
&=b\int_a^b B(s)\,ds-\int_a^b tB(t)\,dt-\frac{b-a}{2}\int_a^b B(s)\,ds\\
&=\frac{b+a}{2}\int_a^b B(s)\,ds-\int_a^b tB(t)\,dt.
\end{aligned}\tag{2.17}$$

Therefore by (2.15)-(2.17) we derive

$$\begin{aligned}
&\left|\frac{b+a}{2}\int_a^b B(s)\,ds-\int_a^b tB(t)\,dt\right|^2\\
&\leq(b-a)\int_a^b\left|\int_a^t B(s)\,ds-\frac{t-a}{b-a}\int_a^b B(s)\,ds\right|^2 dt\\
&\leq\frac{(b-a)^3}{\pi^2}\int_a^b\left|B(t)-\frac{1}{b-a}\int_a^b B(s)\,ds\right|^2 dt\\
&=\frac{(b-a)^4}{\pi^2}\frac{1}{b-a}\int_a^b\left|B(t)-\frac{1}{b-a}\int_a^b B(s)\,ds\right|^2 dt.
\end{aligned}\tag{2.18}$$

By taking into account that

$$\frac{1}{b-a}\int_a^b \left|B(t)-\frac{1}{b-a}\int_a^b B(s)\,ds\right|^2 dt$$
$$=\frac{1}{b-a}\int_a^b \left[|B(t)|^2 - B^*(t)\frac{1}{b-a}\int_a^b B(s)\,ds\right.$$
$$\left.-B(t)\left(\frac{1}{b-a}\int_a^b B(s)\,ds\right)^* + \left|\frac{1}{b-a}\int_a^b B(s)\,ds\right|^2\right] dt$$
$$=\frac{1}{b-a}\int_a^b |B(t)|^2 - \frac{1}{b-a}\int_a^b B^*(t)\,dt\frac{1}{b-a}\int_a^b B(s)\,ds$$
$$-\frac{1}{b-a}\int_a^b B(t)\,dt\left(\frac{1}{b-a}\int_a^b B(s)\,ds\right)^* + \left|\frac{1}{b-a}\int_a^b B(s)\,ds\right|^2$$
$$=\frac{1}{b-a}\int_a^b |B(t)|^2\,dt - \left|\frac{1}{b-a}\int_a^b B(s)\,ds\right|^2,$$

we deduce the desired result. □

3. Weighted Inequalities

The following composite version also holds:

Theorem 2. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b).*

(i) *If $A\in C^1([a,b],\mathcal{B}(H))$ with $\frac{A'}{\sqrt{h'(t)}}\in L_2([a,b],\mathcal{B}(H))$ and $A(a)=A(b)=0$, then*

$$\int_a^b h'(t)\,|A(t)|^2\,dt \le \frac{[h(b)-h(a)]^2}{\pi^2}\int_a^b \frac{|A'(t)|^2}{h'(t)}dt. \tag{3.1}$$

(ii) *If $A(a)=0$, then*

$$\int_a^b h'(t)\,|A(t)|^2\,dt \le \frac{4[h(b)-h(a)]^2}{\pi^2}\int_a^b \frac{|A'(t)|^2}{h'(t)}dt. \tag{3.2}$$

Proof. (i) We write the inequality (2.1) for the function $A\circ h^{-1}$ on the interval $[h(a),h(b)]$ for which $A\circ h^{-1}(h(a))=A(a)=0$, $A\circ h^{-1}(h(b))=A(b)=0$ to get

$$\int_{h(a)}^{h(b)} \left|\left(A\circ h^{-1}\right)(z)\right|^2 dz \le \frac{(h(b)-h(a))^2}{\pi^2}\int_{h(a)}^{h(b)} \left|\left(A\circ h^{-1}\right)'(z)\right|^2 dz. \tag{3.3}$$

If $A:[c,d]\to H$ is strongly differentiable on (c,d), then $A\circ h^{-1}:(h(c),h(d))\to H$ is strongly differentiable on $(h(c),h(d))$ and using the chain rule and the derivative of inverse functions we have

$$\left(A\circ h^{-1}\right)'(z)=\left(A'\circ h^{-1}\right)(z)\left(h^{-1}\right)'(z)=\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)} \tag{3.4}$$

for every $z\in(h(c),h(d))$.

Using the inequality (3.3) we then get

$$\int_{h(a)}^{h(b)}\left|\left(A\circ h^{-1}\right)(z)\right|^{2}dz\leq\frac{(h(b)-h(a))^{2}}{\pi^{2}}\int_{h(a)}^{h(b)}\left|\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right|^{2}dz. \tag{3.5}$$

Observe also that, by the change of variable $t=h^{-1}(z)$, $z\in[h(a),h(b)]$, we have $z=h(t)$ that gives $dz=h'(t)\,dt$,

$$\int_{h(a)}^{h(b)}A\circ h^{-1}(z)\,dx=\int_{a}^{b}A(t)\,h'(t)\,dt$$

and

$$\int_{h(a)}^{h(b)}\left|\left(A\circ h^{-1}\right)(z)\right|^{2}dz=\int_{a}^{b}|A(t)|^{2}h'(t)\,dt. \tag{3.6}$$

We also have

$$\int_{h(a)}^{h(b)}\left|\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right|^{2}dz=\int_{a}^{b}\left|\frac{A'(t)}{h'(t)}\right|^{2}h'(t)\,dt=\int_{a}^{b}\frac{|A'(t)|^{2}}{h'(t)}dt.$$

By making use of (3.5) we get (3.1).

(ii) The inequality (3.2) follows by (2.2) in a similar way. □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Corollary 1. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds=1$ *and* $A\in C^1([a,b],\mathcal{B}(H))$.

(i) *If* $\frac{A'}{\sqrt{w}}\in L_2([a,b],\mathcal{B}(H))$ *and and* $A(a)=A(b)=0$, *then*

$$\int_{a}^{b}w(t)\,|A(t)|^{2}dt\leq\frac{1}{\pi^{2}}\int_{a}^{b}\frac{|A'(t)|^{2}}{w(t)}dt. \tag{3.7}$$

(ii) *In addition, if* $A(a)=0$, *then we have the better inequality*

$$\int_{a}^{b}w(t)\,|A(t)|^{2}dt\leq\frac{4}{\pi^{2}}\int_{a}^{b}\frac{|A'(t)|^{2}}{w(t)}dt. \tag{3.8}$$

We have:

Theorem 3. *Assume that* $w:[a,b]\to(0,\infty)$ *is a probability density function on* $[a,b]$ *and* $B\in C^1([a,b],\mathcal{B}(H))$, *then*

$$\begin{aligned}&\left|\int_{a}^{b}\frac{w(t)+w(a+b-t)}{2}B(t)\,dt-\frac{B(a)+B(b)}{2}\right|^{2}\\&\leq\frac{1}{4\pi^{2}}\int_{a}^{b}\frac{|B'(t)-B'(a+b-t)|^{2}}{w(t)}dt.\end{aligned} \tag{3.9}$$

In particular, if w is symmetrical, i.e. $w(a+b-t)=w(t)$ for any $t\in[a,b]$, then we have

$$\left|\int_a^b w(t)\,B(t)\,dt-\frac{B(a)+B(b)}{2}\right|^2 \leq \frac{1}{4\pi^2}\int_a^b \frac{|B'(t)-B'(a+b-t)|^2}{w(t)}dt. \tag{3.10}$$

Proof. Consider the function

$$A(t):=\frac{B(t)+B(a+b-t)}{2}-\frac{B(a)+B(b)}{2},\ t\in[a,b],$$

we have $A(a)=A(b)=0$ and by (3.7) we have

$$\int_a^b w(t)\left|\frac{B(t)+B(a+b-t)}{2}-\frac{B(a)+B(b)}{2}\right|^2 dt \leq \frac{1}{4\pi^2}\int_a^b \frac{|B'(t)-B'(a+b-t)|^2}{w(t)}dt. \tag{3.11}$$

By the weighted Cauchy-Bunyakovsky-Schwarz integral inequality (1.8) we have

$$\begin{aligned}\int_a^b w(t)\left|\frac{B(t)+B(a+b-t)}{2}-\frac{B(a)+B(b)}{2}\right|^2 dt &\geq \left|\int_a^b w(t)\left(\frac{B(t)+B(a+b-t)}{2}-\frac{B(a)+B(b)}{2}\right)dt\right|^2\\ &=\left|\int_a^b w(t)\left(\frac{B(t)+B(a+b-t)}{2}\right)dt-\frac{B(a)+B(b)}{2}\right|^2.\end{aligned} \tag{3.12}$$

Observe that, by the change of variable $s=a+b-t$, $t\in[a,b]$ we have that

$$\int_a^b w(t)\,B(a+b-t)\,dt=\int_a^b w(a+b-s)\,B(s)\,ds$$

and then

$$\int_a^b \frac{B(t)+B(a+b-t)}{2}w(t)\,dt=\int_a^b \frac{w(t)+w(a+b-t)}{2}B(t)\,dt.$$

By making use of (3.11) and (3.12) we get the inequality in (3.9). □

Remark 1. *If B' is K-Lipschitzian on $[a,b]$, i.e. $\|B'(t)-B'(s)\|\leq K|t-s|$ for any $[a,b]$, then $|B'(t)-B'(s)|\leq K|t-s|$ in the order of $\mathcal{B}(H)$ by (3.9) we get*

$$\left|\int_a^b \frac{w(t)+w(a+b-t)}{2}B(t)\,dt-\frac{B(a)+B(b)}{2}\right|^2 \leq \frac{1}{\pi^2}K^2\int_a^b \frac{\left(t-\frac{a+b}{2}\right)^2}{w(t)}dt \leq \frac{1}{4\pi^2}K^2(b-a)^2\left(\int_a^b \frac{1}{w(t)}dt\right), \tag{3.13}$$

provided $\int_a^b \frac{1}{w(t)}dt<\infty$.

If $B:[a,b]\to\mathcal{B}(H)$ is strongly twice differentiable with $\|B''\|_{[a,b],\infty}:=\sup_{t\in(a,b)}\|B''(s)\|<\infty$ and $w:[a,b]\to(0,\infty)$ is symmetrical, then

$$\left|\int_a^b w(t)\,B(t)\,dt-\frac{B(a)+B(b)}{2}\right| \le \frac{1}{\pi}\|B''\|_{[a,b],\infty}\left(\int_a^b\frac{\left(t-\frac{a+b}{2}\right)^2}{w(t)}dt\right)^{1/2} \le \frac{1}{2\pi}\|B''\|_{[a,b],\infty}(b-a)\left(\int_a^b\frac{1}{w(t)}dt\right)^{1/2}, \tag{3.14}$$

provided $\int_a^b\frac{1}{w(t)}dt<\infty$.

Another trapezoid type weighted inequality is as follows:

Theorem 4. *Assume that $w:[a,b]\to(0,\infty)$ is a probability density function on $[a,b]$ and $B\in C^1([a,b],\mathcal{B}(H))$, then*

$$\left|\frac{[b-E(w;[a,b])]\,B(a)+[E(w;[a,b])-a]\,B(b)}{b-a}-\int_a^b w(t)\,B(t)\,dt\right|^2 \le \frac{1}{\pi^2}\int_a^b\frac{1}{w(t)}\left|B'(t)-\frac{B(b)-B(a)}{b-a}\right|^2dt, \tag{3.15}$$

where

$$E(w;[a,b]):=\int_a^b tw(t)\,dt.$$

Proof. If $B\in C^1([a,b],\mathcal{B}(H))$, then by taking

$$A(t):=B(t)-\frac{(b-t)\,B(a)+(t-a)\,B(b)}{b-a},\ t\in[a,b]$$

we have $A(a)=A(b)=0$ and by (3.7) we have

$$\int_a^b w(t)\left|B(t)-\frac{(b-t)\,B(a)+(t-a)\,B(b)}{b-a}\right|^2dt \le \frac{1}{\pi^2}\int_a^b\frac{1}{w(t)}\left|B'(t)-\frac{B(b)-B(a)}{b-a}\right|^2dt. \tag{3.16}$$

By the weighted Cauchy-Bunyakovsky-Schwarz integral inequality (1.8) we have

$$\begin{aligned}(3.17)\quad & \int_a^b w(t)\left|B(t)-\frac{(b-t)B(a)+(t-a)B(b)}{b-a}\right|^2 dt \\ & \geq \left|\int_a^b w(t)B(t)\,dt-\int_a^b w(t)\frac{(b-t)B(a)+(t-a)B(b)}{b-a}dt\right|^2 \\ & = \left|\int_a^b w(t)B(t)\,dt-\left(\int_a^b \frac{(b-t)}{b-a}w(t)\,dt\right)B(a)\right. \\ & \left.-\left(\int_a^b \frac{(t-a)}{b-a}w(t)\,dt\right)B(b)\right| \\ & = \left|\int_a^b w(t)B(t)\,dt-\frac{[b-E(w;[a,b])]B(a)+[E(w;[a,b])-a]B(b)}{b-a}\right|^2 .\end{aligned}$$

By using (3.16) and (3.17) we get the inequality in (3.15). □

4. Some Inequalities for the Weighted Čebyšev Functional

Consider now the *weighted Čebyšev functional*

$$C_w(\alpha,g):=\int_a^b w(t)\alpha(t)B(t)\,dt-\int_a^b w(t)\alpha(t)\,dt\int_a^b w(t)B(t)\,dt$$

where $w:[a,b]\to\mathbb{R}$ and $w(t)\geq 0$ for a.e. $t\in[a,b]$, $\alpha:[a,b]\to\mathbb{C}$ and $B:[a,b]\to\mathcal{B}(H)$ are functions such that the involved integrals exist and $\int_a^b w(t)\,dt=1$.

We have for $\alpha:[a,b]\to\mathbb{C}$ and $A:[a,b]\to\mathcal{B}(H)$,

$$\begin{aligned} 0 & \leq \left|\overline{\alpha(t)}A(s)-\overline{\alpha(s)}A(t)\right|^2 \\ & = |\alpha(t)|\,|A(s)|^2-\alpha(s)\overline{\alpha(t)}A^*(t)A(s) \\ & -\alpha(t)\overline{\alpha(s)}A^*(s)A(t)+|\alpha(s)|^2|A(t)|^2 ,\end{aligned}$$

which gives that

$$\begin{aligned} & |\alpha(t)|^2|A(s)|^2+|\alpha(s)|^2|A(t)|^2 \\ & \geq \alpha(s)\overline{\alpha(t)}A^*(t)A(s)+\alpha(t)\overline{\alpha(s)}A^*(s)A(t)\end{aligned}$$

for all $s,t\in[a,b]$.

Integrating over t and s on $[a,b]$, then we get

$$\begin{aligned} & \int_a^b|\alpha(t)|^2dt\int_a^b|A(s)|^2ds+\int_a^b|\alpha(s)|^2ds\int_a^b|A(t)|^2dt \\ & \geq \int_a^b\overline{\alpha(t)}A^*(t)\,dt\int_a^b\alpha(s)A(s)\,ds+\int_a^b\overline{\alpha(s)}A^*(s)\,ds\int_a^b\alpha(t)A(t)\,dt \\ & = 2\left|\int_a^b\alpha(s)A(s)\,ds\right|^2 ,\end{aligned}$$

which proves that

$$\int_a^b |\alpha(t)|^2 dt \int_a^b |A(t)|^2 dt \geq \left| \int_a^b \alpha(t) A(t) dt \right|^2, \tag{4.1}$$

provided that $\alpha \in L_2([a,b], \mathbb{C})$ and $A \in L_2([a,b], \mathcal{B}(H))$.

Theorem 5. *Let $w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$ with $\int_a^b w(t)\, dt = 1$, $\alpha \in L_2([a,b], \mathbb{C})$ and $B \in L_2([a,b], \mathcal{B}(H))$, then*

$$\begin{aligned} &|C_w(\alpha, B)|^2 \\ &\leq \frac{(b-a)^2}{\pi^2} \int_a^b |\alpha'(t)|^2 dt \int_a^b w^2(t) \left| \int_a^b B(s) w(s) ds - B(t) \right|^2 dt. \end{aligned} \tag{4.2}$$

Proof. Integrating by parts, we have

$$\begin{aligned} &\int_a^b \alpha'(t) \left(\int_a^t w(s) B(s) ds - \int_a^t w(s) ds \int_a^b w(s) B(s) ds \right) dt \\ &= \left[\alpha(t) \left(\int_a^t w(s) B(s) ds - \int_a^t w(s) ds \int_a^b w(s) B(s) ds \right) \right|_a^b \\ &- \int_a^b \alpha(t) \left(w(t) B(t) - w(t) \int_a^b w(s) B(s) ds \right) dt \\ &= - \int_a^b \alpha(t) w(t) B(t) dt + \int_a^b \alpha(t) w(t) dt \int_a^b w(s) B(s) ds, \end{aligned}$$

which gives that

$$\begin{aligned} &C_w(\alpha, B) \\ &= \int_a^b \alpha\prime(t) \left(\int_a^t w(s) ds \int_a^b w(s) B(s) ds - \int_a^t w(s) B(s) ds \right) dt. \end{aligned} \tag{4.3}$$

By utilising (4.1) we have

$$\begin{aligned} &|C_w(\alpha, B)|^2 \\ &= \left| \int_a^b \alpha'(t) \left(\int_a^t w(s) ds \int_a^b w(s) B(s) ds - \int_a^t w(s) B(s) ds \right) dt \right|^2 \\ &\leq \int_a^b |\alpha'(t)|^2 dt \int_a^b \left| \int_a^t w(s) ds \int_a^b w(s) B(s) ds - \int_a^t w(s) B(s) ds \right|^2 dt. \end{aligned} \tag{4.4}$$

Consider $A(t) = \int_a^t w(s) ds \int_a^b w(s) B(s) ds - \int_a^t w(s) B(s) ds$, $t \in [a,b]$. Observe that $A(b) = A(a) = 0$ and

$$A'(t) = w(t) \int_a^b w(s) B(s) ds - w(t) B(t) = w(t) \left(\int_a^b w(s) B(s) ds - B(t) \right).$$

Then by (2.1)

$$\int_a^b \left|\int_a^t w(s)\,ds \int_a^b w(s)\,B(s)\,ds - \int_a^t w(s)\,B(s)\,ds\right|^2 dt$$
$$\leq \frac{(b-a)^2}{\pi^2}\int_a^b w^2(t)\left|\int_a^b w(s)\,B(s)\,ds - B(t)\right|^2 dt,$$

and by (4.4) we derive (4.2). □

We also have:

Theorem 6. *Assume that* $w:[a,b]\to(0,\infty)$ *is a probability density function on* $[a,b]$, $\alpha\in L_2([a,b],\mathbb{C})$ *and* $B\in C^1([a,b],\mathcal{B}(H))$, *then*

$$|C_w(\alpha,B)|^2 \leq \frac{4}{\pi^2}\left(\int_a^b |\alpha(t)|^2 w(t)\,dt - \left|\int_a^b w(s)\,\alpha(s)\,ds\right|^2\right) \times \int_a^b \frac{|B'(t)|^2}{w(t)}dt. \tag{4.5}$$

Proof. We use the following Sonin type identity

$$C_w(\alpha,B) = \int_a^b \left(\alpha(t) - \int_a^b w(s)\,\alpha(s)\,ds\right)(B(t)-B(a))\,w(t)\,dt, \tag{4.6}$$

which can be proved directly on calculating the integral from the right hand side.

By using the weighted (CBS) integral inequality (4.1), we have

$$|C_w(\alpha,B)|^2 \leq \left(\int_a^b \left|\alpha(t) - \int_a^b w(s)\,\alpha(s)\,ds\right|^2 w(t)\,dt\right) \times \left(\int_a^b w(t)\,|B(t)-B(a)|^2\,dt\right). \tag{4.7}$$

By (3.8) we get

$$|C_w(\alpha,B)|^2 \leq \left(\int_a^b \left|\alpha(t) - \int_a^b w(s)\,\alpha(s)\,ds\right|^2 w(t)\,dt\right) \times \frac{4}{\pi^2}\left(\int_a^b \frac{|B'(t)|^2}{w(t)}dt\right)$$
$$= \frac{4}{\pi^2}\left(\int_a^b |\alpha(t)|^2 w(t)\,dt - \left|\int_a^b w(s)\,\alpha(s)\,ds\right|^2\right) \times \int_a^b \frac{|B'(t)|^2}{w(t)}dt,$$

which proves (4.5). □

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