OSTROWSKI TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

SILVESTRU SEVER DRAGOMIR[1,2]

Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. In this paper we show among others that, if $s\in(a,b)$, the function $A:[a,b]\to B(H)$ is continuous on $[a,b]$ and strongly differentiable on $(a,s)\cup(s,b)$ with $A'\in L_2([a,b],\mathcal{B}(H))$, then

$$\left|(b-a)A(s)-\int_a^b A(t)\,dt\right|^2 \le (b-a)\left[\frac{1}{12}(b-a)^2+\left(s-\frac{a+b}{2}\right)^2\right]\int_a^b |A'(t)|^2\,dt$$

in the operator order of $\mathcal{B}(H)$. Some examples for the operator exponential and inverse functions are also provided.

1. Introduction

In 1998, Dragomir and Wang proved the following Ostrowski type inequality for p-norm [9].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then we have the inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{(q+1)^{1/q}}\left[\left(\frac{x-a}{b-a}\right)^{q+1}+\left(\frac{b-x}{b-a}\right)^{q+1}\right]^{1/q}(b-a)^{1/q}\|f'\|_{[a,b],p}, \tag{1.1}$$

for all $x\in[a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p}:=\left(\int_a^b |g(t)|^p\,dt\right)^{1/p}.$$

From (1.1) we get the following midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\frac{1}{2(q+1)^{1/q}}(b-a)^{1/q}\|f'\|_{[a,b],p}, \tag{1.2}$$

and $\frac{1}{2}$ is a best possible constant.

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For $p=q=2$ we derive the $L_2[a,b]$-inequality

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^2 \leq \frac{1}{3}\left[\left(\frac{x-a}{b-a}\right)^3+\left(\frac{b-x}{b-a}\right)^3\right](b-a)\left\|f'\right\|_{[a,b],2}^2, \tag{1.3}$$

for all $x\in[a,b]$ and the midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|^2 \leq \frac{1}{12}(b-a)\left\|f'\right\|_{[a,b]}^2. \tag{1.4}$$

For a survey on scalar Ostrowski's inequality, see [8]. For recent papers on this inequality see also [1]-[2] and [10]-[12].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\leq|A|+|B|$, so the classical arguments using this inequality can not be used. In order to obtain the corresponding version for the operator modulus we need the following preparations.

Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$. We have for $\alpha:[a,b]\to\mathbb{C}$ and $A:[a,b]\to\mathcal{B}(H)$,

$$\begin{aligned}0&\leq\left|\overline{\alpha(t)}A(s)-\overline{\alpha(s)}A(t)\right|^2=|\alpha(t)|\,|A(s)|^2-\alpha(s)\overline{\alpha(t)}A^*(t)A(s)\\&-\alpha(t)\overline{\alpha(s)}A^*(s)A(t)+|\alpha(s)|^2|A(t)|^2,\end{aligned}$$

which gives that

$$|\alpha(t)|^2|A(s)|^2+|\alpha(s)|^2|A(t)|^2\geq\alpha(s)\overline{\alpha(t)}A^*(t)A(s)+\alpha(t)\overline{\alpha(s)}A^*(s)A(t)$$

for all $s,\,t\in[a,b]$.

Now, multiply this with $w(s)w(t)\geq 0$ to get

$$\begin{aligned}&w(t)|\alpha(t)|^2w(s)|A(s)|^2+w(s)|\alpha(s)|^2w(t)|A(t)|^2\\&\geq w(t)\overline{\alpha(t)}A^*(t)w(s)\alpha(s)A(s)+w(s)\overline{\alpha(s)}A^*(s)w(t)\alpha(t)A(t)\end{aligned}$$

for all $s,\,t\in[a,b]$.

Integrating over t and s on $[a,b]$, then we get

$$\begin{aligned}&\int_a^b w(t)|\alpha(t)|^2dt\int_a^b|A(s)|^2ds+\int_a^b|\alpha(s)|^2ds\int_a^b w(t)|A(t)|^2dt\\&\geq\int_a^b w(t)\overline{\alpha(t)}A^*(t)\,dt\int_a^b\alpha(s)A(s)\,ds\\&+\int_a^b w(s)\overline{\alpha(s)}A^*(s)\,ds\int_a^b\alpha(t)A(t)\,dt\\&=2\left|\int_a^b w(s)\alpha(s)A(s)\,ds\right|^2,\end{aligned}$$

which proves that

$$\int_a^b w(t)|\alpha(t)|^2 dt \int_a^b w(t)|A(t)|^2 dt \geq \left| \int_a^b w(t)\alpha(t) A(t) dt \right|^2, \tag{1.5}$$

provided that $\alpha \in L_{2,w}([a,b],\mathbb{C})$ and

$$A \in L_{2,w}([a,b],\mathcal{B}(H)) := \left\{ A : [a,b] \to B(H), \int_a^b w(t)\|A(t)\|^2 dt < \infty \right\}.$$

In a similar way we can prove the following discrete inequality

$$\sum_{k=1}^n w_k |z_k|^2 \sum_{k=1}^n w_k |A_k|^2 \geq \left| \sum_{k=1}^n w_k z_k A_k \right|^2, \tag{1.6}$$

Stan gordonwhere $z_k \in \mathbb{C}$, $A_k \in \mathcal{B}(H)$, $w_k \geq 0$ for $k \in \{1,...,n\}$ and $\sum_{k=1}^n w_k = 1$.

2. Main Results

We have the following Ostrowski type inequality for the modulus of operators:

Theorem 2. *Assume that for $s \in (a,b)$, the function $A : [a,b] \to B(H)$ is continuous on $[a,b]$ and strongly differentiable on $(a,s) \cup (s,b)$ with $A' \in L_2([a,b],\mathcal{B}(H))$. Then*

$$\begin{aligned} &\left| (b-a) A(s) - \int_a^b A(t) dt \right|^2 \\ &\leq (b-a) \left[\frac{1}{12}(b-a)^2 + \left(s - \frac{a+b}{2} \right)^2 \right] \int_a^b |A'(t)|^2 dt. \end{aligned} \tag{2.1}$$

Proof. Using integration by parts for Bochner's integral, we have

$$\int_a^s (t-a) A'(t) dt = (s-a) A(s) - \int_a^s A(t) ds$$

and

$$\int_s^b (t-b) A'(t) dt = (b-s) A(s) - \int_s^b A(t) ds,$$

and by adding this equalities we obtain the *Montgomery type identity* for the Bochner's integral

$$(b-a) A(s) - \int_a^b A(t) dt = \int_a^s (t-a) A'(t) dt + \int_s^b (t-b) A'(t) dt. \tag{2.2}$$

Consider the kernel

$$p_s(t) := \begin{cases} t-a \text{ if } t \in [a,s] \\ \\ t-b \text{ if } t \in (s,b]. \end{cases}$$

Then by (2.2) we get the representation

$$(b-a) A(s) - \int_a^b A(t) dt = \int_a^b p_s(t) A'(t) dt.$$

Since $p_s \in L_2([a,b],\mathbb{R})$ and $A' \in L_2([a,b],\mathcal{B}(H))$, then by (1.5) we get

$$\begin{aligned}
(2.3)\qquad & \left|(b-a)A(s) - \int_a^b A(t)\,ds\right|^2 \\
&= \left|\int_a^b p_s(t)A'(t)\,dt\right|^2 \le \int_a^b [p_s(t)]^2\,dt \int_a^b |A'(t)|^2\,dt \\
&= \left[\int_a^s (t-a)^2\,dt + \int_s^b (t-b)^2\,dt\right]\int_a^b |A'(t)|^2\,dt \\
&= \frac{1}{3}\left[(s-a)^3 + (b-s)^3\right]\int_a^b |A'(t)|^2\,dt.
\end{aligned}$$

Observe that, by simple calculations we get

$$\frac{1}{3}\left[(s-a)^3 + (b-s)^3\right] = (b-a)\left[\frac{1}{12}(b-a)^2 + \left(s - \frac{a+b}{2}\right)^2\right].$$

Therefore by (2.3) we derive the desired result (2.1). □

Corollary 1. *Assume that the function $A : [a,b] \to B(H)$ is continuous on $[a,b]$ and strongly differentiable on $\left(a, \frac{a+b}{2}\right) \cup \left(\frac{a+b}{2}, b\right)$ with $A' \in L_2([a,b],\mathcal{B}(H))$. Then*

$$(2.4)\qquad \left|(b-a)A\left(\frac{a+b}{2}\right) - \int_a^b A(t)\,dt\right|^2 \le \frac{1}{12}(b-a)^3 \int_a^b |A'(t)|^2\,dt.$$

The constant $\frac{1}{12}$ is best possible in (2.4).

Proof. It is enough to prove the best constant in the case of scalar case.

Consider the function

$$A_0(t) := \begin{cases} \frac{1}{2}(t-a)^2, & t \in \left[a, \frac{a+b}{2}\right], \\ \frac{1}{2}(t-b)^2, & t \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

Then

$$A_0\left(\frac{a+b}{2}\right) = \frac{1}{8}(b-a)^2$$

and

$$A_0'(t) := \begin{cases} t-a, & t \in \left(a, \frac{a+b}{2}\right), \\ t-b, & t \in \left(\frac{a+b}{2}, b\right). \end{cases}$$

We have

$$\int_a^b A_0(t)\,dt = \frac{1}{2}\int_a^{\frac{a+b}{2}} (t-a)^2\,dt + \frac{1}{2}\int_{\frac{a+b}{2}}^b (t-b)^2\,dt = \frac{(b-a)^3}{24},$$

$$\int_a^b |A'(t)|^2\,dt = \int_a^{\frac{a+b}{2}} (t-a)^2\,dt + \int_{\frac{a+b}{2}}^b (t-b)^2\,dt = \frac{(b-a)^3}{12}$$

and

$$(b-a)A\left(\frac{a+b}{2}\right) - \int_a^b A(t)\,dt = \frac{1}{8}(b-a)^3 - \frac{(b-a)^3}{24} = \frac{(b-a)^3}{12}.$$

Therefore

$$\left|(b-a)\,A\left(\frac{a+b}{2}\right)-\int_a^b A(t)\,dt\right|^2=\frac{(b-a)^6}{144}$$

and

$$\frac{1}{12}(b-a)^3\int_a^b\left|A'(t)\right|^2dt=\frac{(b-a)^6}{144},$$

which proves the sharpness of the constant $\frac{1}{12}$. □

Remark 1. *Recall the Löwner–Heinz inequality which says that $A\geq B\geq 0$ implies $A^\alpha\geq B^\alpha$ for all $\alpha\in[0,1]$. By taking the power $1/2$ in (2.1) we derive*

$$\left|(b-a)\,A(s)-\int_a^b A(t)\,dt\right| \leq (b-a)^{1/2}\left[\frac{1}{12}(b-a)^2+\left(s-\frac{a+b}{2}\right)^2\right]^{1/2}\left(\int_a^b\left|A'(t)\right|^2dt\right)^{1/2}, \tag{2.5}$$

while from (2.4) we get

$$\left|(b-a)\,A\left(\frac{a+b}{2}\right)-\int_a^b A(t)\,dt\right|\leq\frac{\sqrt{3}}{6}(b-a)^{3/2}\left(\int_a^b\left|A'(t)\right|^2dt\right)^{1/2}. \tag{2.6}$$

The constant $\frac{\sqrt{3}}{6}$ is best possible in (2.6).

The following alternative result also holds:

Theorem 3. *Assume that for $s\in(a,b)$, the function $A:[a,b]\to\mathcal{B}(H)$ is continuous on $[a,b]$ and strongly differentiable on $(a,s)\cup(s,b)$ with $A'\in L_2([a,b],\mathcal{B}(H))$. Then*

$$\begin{aligned}&\left|(b-a)\,A(s)-\int_a^b A(t)\,dt\right|^2 \\ &\leq\left[(s-a)^4+(b-s)^4\right] \\ &\times\left[\left|\int_0^1\tau A'((1-\tau)\,a+\tau s)\,d\tau\right|^2+\left|\int_0^1(1-\tau)\,A'((1-\tau)\,s+\tau b)\,d\tau\right|^2\right] \\ &\leq\frac{1}{3}\left[(s-a)^4+(b-s)^4\right] \\ &\times\int_0^1\left[\left|A'((1-\tau)\,a+\tau s)\right|^2+\left|A'((1-\tau)\,s+\tau b)\right|^2\right]d\tau.\end{aligned} \tag{2.7}$$

Proof. We use the following change of variable $t=(1-\tau)\,c+\tau d$, $dt=(d-c)\,d\tau$ and

$$\int_c^d A(t)\,dt=(d-c)\int_0^1 A((1-\tau)\,c+\tau d)\,d\tau.$$

Therefore

$$\int_a^s (t-a)\,A'(t)\,dt = (s-a)\int_0^1 ((1-\tau)\,a+\tau s-a)\,A'((1-\tau)\,a+\tau s)\,d\tau$$
$$= (s-a)^2\int_0^1 \tau A'((1-\tau)\,a+\tau s)\,d\tau$$

and

$$\int_s^b (t-b)\,A'(t)\,dt = (b-s)\int_0^1 ((1-\tau)\,s+\tau b-b)\,A'((1-\tau)\,s+\tau b)$$
$$= -(b-s)^2\int_0^1 (1-\tau)\,A'((1-\tau)\,s+\tau b)\,d\tau.$$

By (2.2) we derive the following operator identity of interest

$$(b-a)\,A(s)-\int_a^b A(t)\,ds = (s-a)^2\int_0^1 \tau A'((1-\tau)\,a+\tau s)\,d\tau - (b-s)^2\int_0^1 (1-\tau)\,A'((1-\tau)\,s+\tau b)\,d\tau. \tag{2.8}$$

If we take the modulus and use the elementary inequality which follows by (1.6)

$$\left(|z_1|^2+|z_2|^2\right)\left(|A_1|^2+|A_2|^2\right) \geq |z_1A_1+z_2A_2|^2,$$

then we get

$$\begin{aligned}
&\left|(b-a)\,A(s)-\int_a^b A(t)\,dt\right|^2\\
&= \Bigg|(s-a)^2\int_0^1 \tau A'((1-\tau)\,a+\tau s)\,d\tau\\
&-(b-s)^2\int_0^1 (1-\tau)\,A'((1-\tau)\,s+\tau b)\,d\tau\Bigg|^2\\
&\leq \left[(s-a)^4+(b-s)^4\right]\\
&\times\left[\left|\int_0^1 \tau A'((1-\tau)\,a+\tau s)\,d\tau\right|^2+\left|\int_0^1 (1-\tau)^2 A'((1-\tau)\,s+\tau b)\,d\tau\right|^2\right],
\end{aligned}$$

which proves the first inequality in (2.7).

By (1.5) we have

$$\left|\int_0^1 \tau A'((1-\tau)\,a+\tau s)\,d\tau\right|^2 \leq \int_0^1 \tau^2 d\tau\int_0^1 |A'((1-\tau)\,a+\tau s)|^2\,d\tau$$
$$= \frac{1}{3}\int_0^1 |A'((1-\tau)\,a+\tau s)|^2\,d\tau$$

and

$$\left|\int_0^1 (1-\tau)^2 A'((1-\tau)\,s+\tau b)\,d\tau\right|^2 \leq \int_0^1 (1-\tau)^2\,d\tau\int_0^1 |A'((1-\tau)\,s+\tau b)|^2\,d\tau$$
$$= \frac{1}{3}\int_0^1 |A'((1-\tau)\,s+\tau b)|^2\,d\tau,$$

which proves the last part of (2.7). □

The following representation result holds.

Lemma 1. *Let $B:[a,b]\to\mathcal{B}(H)$ be a Bochner integrable on $[a,b]$. Then for any $\lambda\in[0,1]$ we have the representation*

$$\int_0^1 B[(1-t)a+tb]\,dt=(1-\lambda)\int_0^1 B[(1-t)((1-\lambda)a+\lambda b)+tb]\,dt + \lambda\int_0^1 B[(1-t)a+t((1-\lambda)a+\lambda b)]\,dt. \tag{2.9}$$

In particular,

$$\int_0^1 B[(1-t)a+tb]\,dt = \frac{1}{2}\int_0^1\left(B\left[(1-t)\frac{a+b}{2}+tb\right]+B\left[(1-t)a+t\frac{a+b}{2}\right]\right)dt. \tag{2.10}$$

Proof. For $\lambda=0$ and $\lambda=1$ the equality (2.9) is obvious.

Let $\lambda\in(0,1)$. Observe that

$$\int_0^1 B[(1-t)(\lambda b+(1-\lambda)a)+tb]\,dt = \int_0^1 B[((1-t)\lambda+t)b+(1-t)(1-\lambda)a]\,dt$$

and

$$\int_0^1 B[t(\lambda b+(1-\lambda)a)+(1-t)a]\,dt=\int_0^1 B[t\lambda b+(1-\lambda t)a]\,dt.$$

If we make the change of variable $u:=(1-t)\lambda+t$ then we have $1-u=(1-t)(1-\lambda)$ and $du=(1-\lambda)\,dt$. Then

$$\int_0^1 B[((1-t)\lambda+t)b+(1-t)(1-\lambda)a]\,dt=\frac{1}{1-\lambda}\int_\lambda^1 B[ub+(1-u)a]\,du.$$

If we make the change of variable $u:=\lambda t$ then we have $du=\lambda dt$ and

$$\int_0^1 B[t\lambda b+(1-\lambda t)a]\,dt=\frac{1}{\lambda}\int_0^\lambda B[ub+(1-u)a]\,du.$$

Therefore

$$\begin{aligned}&(1-\lambda)\int_0^1 B[(1-t)(\lambda b+(1-\lambda)a)+tb]\,dt\\ &+\lambda\int_0^1 B[t(\lambda b+(1-\lambda)a)+(1-t)a]\,dt\\ &=\int_\lambda^1 B[ub+(1-u)a]\,du+\int_0^\lambda B[ub+(1-u)a]\,du\\ &=\int_0^1 B[ub+(1-u)a]\,du\end{aligned}$$

and the identity (2.9) is proved. □

Corollary 2. *Assume that the function $A:[a,b]\to\mathcal{B}(H)$ is continuous on $[a,b]$ and strongly differentiable on $\left(a,\frac{a+b}{2}\right)\cup\left(\frac{a+b}{2},b\right)$ with $A'\in L_2([a,b],\mathcal{B}(H))$. Then*

$$\begin{aligned}(2.11)\qquad &\left|(b-a)A\left(\frac{a+b}{2}\right)-\int_a^b A(t)\,dt\right|^2\\ &\leq\frac{1}{8}(b-a)^4\left[\left|\int_0^1\tau A'\left((1-\tau)a+\tau\frac{a+b}{2}\right)d\tau\right|^2\right.\\ &\left.+\left|\int_0^1(1-\tau)A'\left((1-\tau)\frac{a+b}{2}+\tau b\right)d\tau\right|^2\right]\\ &\leq\frac{1}{12}(b-a)^4\int_0^1|A'((1-\tau)a+\tau b)|^2\,dt=\frac{1}{12}(b-a)^3\int_0^1|A'(t)|^2\,dt.\end{aligned}$$

The constants $\frac{1}{8}$ and $\frac{1}{12}$ are best possible in (2.4).

Proof. From (2.7) we have for $s=\frac{a+b}{2}$ that

$$\begin{aligned}(2.12)\qquad &\left|(b-a)A\left(\frac{a+b}{2}\right)-\int_a^b A(t)\,dt\right|^2\\ &\leq\frac{1}{8}(b-a)^4\left[\left|\int_0^1\tau A'\left((1-\tau)a+\tau\frac{a+b}{2}\right)d\tau\right|^2\right.\\ &\left.+\left|\int_0^1(1-\tau)A'\left((1-\tau)\frac{a+b}{2}+\tau b\right)d\tau\right|^2\right]\\ &\leq\frac{1}{24}(b-a)^4\\ &\times\int_0^1\left[\left|A'\left((1-\tau)a+\tau\frac{a+b}{2}\right)\right|^2+\left|A'\left((1-\tau)\frac{a+b}{2}+\tau b\right)\right|^2\right]d\tau.\end{aligned}$$

By (2.10) we also have

$$\begin{aligned}&\frac{1}{2}\int_0^1\left[\left|A'\left((1-\tau)a+\tau\frac{a+b}{2}\right)\right|^2+\left|A'\left((1-\tau)\frac{a+b}{2}+\tau b\right)\right|^2\right]d\tau\\ &=\int_0^1|A'((1-\tau)a+\tau b)|^2\,dt=\frac{1}{b-a}\int_0^1|A'(t)|^2\,dt\end{aligned}$$

and the inequality (2.11) is thus proved. □

We can introduce the following concept:

Definition 1. *We say that the continuous function $B:[a,b]\to\mathcal{B}(H)$ is square modulus convex on $[a,b]$ if*

$$(2.13)\qquad |B((1-t)u+tv)|^2\leq(1-t)|B(u)|^2+t|B(v)|^2$$

in the operator order of $\mathcal{B}(H)$, for all $u,v\in[a,b]$ and $t\in[0,1]$.

Let A, $B\in\mathcal{B}(H)$ and $\alpha\in[0,1]$. Then by (1.6) we get

$$\begin{aligned}
\left|(1-\alpha)A+\alpha B\right|^{2} &= \left|(1-\alpha)^{1/2}(1-\alpha)^{1/2}A+\alpha^{1/2}\alpha^{1/2}B\right|^{2}\\
&\le\left[\left((1-\alpha)^{1/2}\right)^{2}+\left(\alpha^{1/2}\right)^{2}\right]\left[\left|(1-\alpha)^{1/2}A\right|^{2}+\left|\alpha^{1/2}B\right|^{2}\right]\\
&=(1-\alpha+\alpha)\left[(1-\alpha)\left|A\right|^{2}+\alpha\left|B\right|^{2}\right]\\
&=(1-\alpha)\left|A\right|^{2}+\alpha\left|B\right|^{2}.
\end{aligned}$$

Consider the function $C:[0,1]\to\mathcal{B}(H)$, $C(t)=\left|(1-t)A+tB\right|$. Let t_1, $t_2\in[0,1]$ and $\alpha\in[0,1]$. Then

$$\begin{aligned}
\left|C\left((1-\alpha)t_1+\alpha t_2\right)\right|^{2} &= \left|(1-(1-\alpha)t_1-\alpha t_2)A+((1-\alpha)t_1+\alpha t_2)B\right|^{2}\\
&=\left|(1-\alpha)((1-t_1)A+t_1B)+\alpha((1-t_2)A+t_2B)\right|^{2}\\
&\le(1-\alpha)\left|(1-t_1)A+t_1B\right|^{2}+\alpha\left|(1-t_2)A+t_2B\right|^{2}\\
&=(1-\alpha)\left|C(t_1)\right|^{2}+\alpha\left|C(t_2)\right|^{2},
\end{aligned}$$

which shows that C is *square modulus convex* on $[0,1]$.

Assume that f is *nonnegative* on I and *operator convex*, namely

$$f((1-\alpha)A+\alpha B)\le(1-\alpha)f(A)+\alpha f(B)$$

for all $\alpha\in[0,1]$ and selfadjoint operators A, B with spectra in I.

For such function and A, B, we consider

$$D(t):=\left[f((1-t)A+tB)\right]^{1/2},t\in[0,1].$$

Then, using a similar proof as above for the modulus function, we conclude that D is *square modulus convex* on $[0,1]$.

The function $f(t)=t^{r}$ is operator convex on $(0,\infty)$ if either $1\le r\le2$ or $-1\le r\le0$ and is operator concave on $(0,\infty)$ if $0\le r\le1$. Therefore for A, $B>0$, the function

$$B(t):=((1-t)A+tB)^{r/2},\ t\in[0,1]$$

is *square modulus convex* on $[0,1]$ for $1\le r\le2$ or $-1\le r\le0$.

Proposition 1. *With the assumption of Theorem 2 and if A' is square modulus convex on (a,b), then*

$$\begin{aligned}
&\left|(b-a)A(s)-\int_a^b A(t)\,dt\right|^{2} \qquad (2.14)\\
&\le\frac{1}{2}(b-a)^{2}\left[\frac{1}{12}(b-a)^{2}+\left(s-\frac{a+b}{2}\right)^{2}\right]\left[\left|A'(a)\right|^{2}+\left|A'(b)\right|^{2}\right].
\end{aligned}$$

In particular,

$$\left|(b-a)A\left(\frac{a+b}{2}\right)-\int_a^b A(t)\,dt\right|^{2}\le\frac{1}{24}(b-a)^{4}\left[\left|A'(a)\right|^{2}+\left|A'(b)\right|^{2}\right]. \qquad (2.15)$$

Proof. We have

$$\int_a^b |A'(t)|^2 dt = (b-a)\int_0^1 |A'((1-s)a+sb)|^2 ds$$
$$\leq (b-a)\int_0^1 \left[(1-s)|A'(a)|^2 + s|A'(b)|^2\right] ds$$
$$= (b-a)\frac{|A'(a)|^2 + |A'(b)|^2}{2},$$

and by (2.1) we get (2.14). □

We also have:

Proposition 2. *With the assumption of Theorem 3 and if A' is square modulus convex on (a,b) then*

$$\left|(b-a)A(s) - \int_a^b A(t)\,dt\right|^2 \tag{2.16}$$
$$\leq \frac{1}{3}\left[(s-a)^4 + (b-s)^4\right]\left[|A'(s)|^2 + \frac{|A'(b)|^2 + |A'(a)|^2}{2}\right].$$

In particular,

$$\left|(b-a)A\left(\frac{a+b}{2}\right) - \int_a^b A(t)\,dt\right|^2 \tag{2.17}$$
$$\leq \frac{1}{24}(b-a)^2\left[\left|A'\left(\frac{a+b}{2}\right)\right|^2 + \frac{|A'(b)|^2 + |A'(a)|^2}{2}\right]$$
$$\leq \frac{1}{12}(b-a)^4\left[|A'(a)|^2 + |A'(b)|^2\right].$$

Proof. By the convexity

$$\int_0^1 |A'((1-\tau)s+\tau b)|^2 d\tau \leq \frac{1}{2}\left[|A'(s)|^2 + |A'(b)|^2\right]$$

and

$$\int_0^1 |A'((1-\tau)a+\tau s)|^2 \leq \frac{1}{2}\left[|A'(a)|^2 + |A'(s)|^2\right]$$

and by (2.7) we derive

$$\left|(b-a)A(s) - \int_a^b A(t)\,dt\right|^2$$
$$\leq \frac{1}{3}\left[(s-a)^4 + (b-s)^4\right]$$
$$\times \frac{1}{2}\left[|A'(s)|^2 + |A'(b)|^2 + |A'(a)|^2 + |A'(s)|^2\right]$$
$$= \frac{1}{3}\left[(s-a)^4 + (b-s)^4\right]\left[|A'(s)|^2 + \frac{|A'(b)|^2 + |A'(a)|^2}{2}\right],$$

which proves (2.16). □

3. Some Examples

Consider the function $A(t) = \exp(tT)$, where $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. Then $A'(t) = T\exp(tT)$, for $t \in \mathbb{R}$ and $T \in \mathcal{B}(H)$. By making use of 2.17 we get

$$\left|(b-a)\exp\left(\frac{a+b}{2}T\right) - \int_a^b \exp(tT)\,dt\right|^2 \le \frac{1}{12}(b-a)^3 \int_a^b |T\exp(tT)|^2\,dt. \tag{3.1}$$

If T is invertible, then [3]

$$\int_a^b \exp(tT)\,dt = T^{-1}[\exp(bT) - \exp(aT)]. \tag{3.2}$$

From (3.1) we derive

$$\begin{aligned} &\left|(b-a)\exp\left(\frac{a+b}{2}T\right) - T^{-1}[\exp(bT) - \exp(aT)]\right|^2 \\ &\le \frac{1}{12}(b-a)^3 \int_a^b |T\exp(tT)|^2\,dt. \end{aligned} \tag{3.3}$$

For T invertible, if we consider $B(t) = T\exp(tT)$, then $B'(t) = T^2\exp(tT)$ and

$$\int_a^b B(t)\,dt = \exp(bT) - \exp(aT).$$

By (2.17) we derive

$$\begin{aligned} &\left|(b-a)T\exp\left(\frac{a+b}{2}T\right) - \exp(bT) + \exp(aT)\right|^2 \\ &\le \frac{1}{12}(b-a)^3 \int_a^b \left|T^2\exp(tT)\right|^2\,dt. \end{aligned} \tag{3.4}$$

Since for any operator $V \in \mathcal{B}(H)$ we have $|V|^2 \le \|V\|^2$ and $\|\exp(tT)\| \le \exp(|t|\,\|T\|)$, $t \in \mathbb{R}$, $T \in \mathcal{B}(H)$, then by (3.1) we get

$$\begin{aligned} &\left|(b-a)\exp\left(\frac{a+b}{2}T\right) - \int_a^b \exp(tT)\,dt\right|^2 \\ &\le \frac{1}{12}(b-a)^3 \int_a^b \|T\exp(tT)\|^2\,dt \le \frac{1}{12}\|T\|^2(b-a)^3 \int_a^b \|\exp(tT)\|^2\,dt \\ &\le \frac{1}{12}\|T\|^2(b-a)^3 \int_a^b \exp(2\|T\|\,|t|)\,dt. \end{aligned} \tag{3.5}$$

If $0 \le a \le b$, then

$$\int_a^b \exp(2\|T\|\,|t|)\,dt = \int_a^b \exp(2\|T\|\,t)\,dt = \frac{\exp(2\|T\|\,b) - \exp(2\|T\|\,a)}{2\|T\|}$$

and by (3.5) we get

$$\begin{aligned} &\left|(b-a)\exp\left(\frac{a+b}{2}T\right) - \int_a^b \exp(tT)\,dt\right|^2 \\ &\le \frac{1}{24}(b-a)^3\|T\|[\exp(2\|T\|\,b) - \exp(2\|T\|\,a)] \end{aligned} \tag{3.6}$$

for any $T \in \mathcal{B}(H)$.

Moreover, if T is invertible, then we also have the exponential inequality

$$\left|(b-a)\exp\left(\frac{a+b}{2}T\right)-T^{-1}\left[\exp(bT)-\exp(aT)\right]\right|^{2} \le \frac{1}{24}(b-a)^{3}\|T\|\left[\exp(2\|T\|b)-\exp(2\|T\|a)\right]. \tag{3.7}$$

Consider the function $A(t)=\exp[(1-t)A](B-A)\exp(tB)$, $t\in[0,1]$. Then, integrating by parts

$$\begin{aligned}
&\int_0^1 f(t)\,dt \\
&=\int_0^1\left(\exp[(1-t)A]B\exp(tB)-\exp[(1-t)A]A\exp(tB)\right)dt \\
&=\int_0^1\exp[(1-t)A](\exp(tB))'\,dt+\int_0^1(\exp[(1-t)A])'\exp(tB)\,dt \\
&=\exp[(1-t)A]\exp(tB)\big|_0^1+A\int_0^1\exp[(1-t)A]\exp(tB)\,dt \\
&+\exp[(1-t)A]\exp(tB)\big|_0^1-\int_0^1(\exp[(1-t)A])B\exp(tB)\,dt \\
&=2(\exp B-\exp A)-\int_0^1\exp[(1-t)A](B-A)\exp(tB) \\
&=2(\exp B-\exp A)-\int_0^1 f(t)\,dt,
\end{aligned}$$

which gives the following identity of interest [4]

$$\int_0^1\exp[(1-t)A](B-A)\exp(tB)\,dt=\exp B-\exp A$$

for all $A,\ B\in\mathcal{B}(H)$.

Also

$$\begin{aligned}
A'(t)&=-A\exp[(1-t)A](B-A)\exp(tB) \\
&\quad+\exp[(1-t)A](B-A)B\exp(tB)\,dt \\
&=\exp[(1-t)A](B-A)B\exp(tB)\,d \\
&\quad-\exp[(1-t)A]A(B-A)\exp(tB) \\
&=\exp[(1-t)A]\left[(B-A)B-A(B-A)\right]\exp(tB) \\
&=\exp[(1-t)A]\left(B^2-2AB+A^2\right)\exp(tB).
\end{aligned}$$

By utilising (2.4) we get

$$\left|\exp\left(\frac{1}{2}A\right)(B-A)\exp\left(\frac{1}{2}B\right)-\exp B+\exp A\right|^{2} \le \frac{1}{12}\int_a^b\left|\exp[(1-t)A]\left(B^2-2AB+A^2\right)\exp(tB)\right|^{2}dt, \tag{3.8}$$

for all $A,\ B\in\mathcal{B}(H)$.

Since

$$\begin{aligned}
&\left|\exp\left[(1-t)A\right]\left(B^2-2AB+A^2\right)\exp(tB)\right|^2\\
&\leq \left\|\exp\left[(1-t)A\right]\left(B^2-2AB+A^2\right)\exp(tB)\right\|^2\\
&\leq \left\|\exp\left[(1-t)A\right]\right\|^2\left\|B^2-2AB+A^2\right\|^2\left\|\exp(tB)\right\|^2\\
&\leq \exp\left[2(1-t)\|A\|\right]\left\|B^2-2AB+A^2\right\|^2\exp\left(2t\|B\|\right)\\
&= \left\|B^2-2AB+A^2\right\|^2\exp\left[2\left[(1-t)\|A\|+t\|B\|\right]\right]\\
&= \left\|B^2-2AB+A^2\right\|^2\exp\left\{2\left[(1-t)\|A\|+t\|B\|\right]\right\},
\end{aligned}$$

hence by (3.8) we get

$$\begin{aligned}
&\left|\exp\left(\frac{1}{2}A\right)(B-A)\exp\left(\frac{1}{2}B\right)-\exp B+\exp A\right|^2 \qquad (3.9)\\
&\leq \frac{1}{12}\left\|B^2-2AB+A^2\right\|^2\int_a^b\exp\left\{2\left[(1-t)\|A\|+t\|B\|\right]\right\}dt\\
&= \frac{1}{12}\left\|B^2-2AB+A^2\right\|^2\begin{cases}\exp\left(2\|A\|\right) \text{ if } \|B\|=\|A\|,\\[2mm] \frac{\exp(2\|B\|)-\exp(2\|A\|)}{2(\|B\|-\|A\|)} \text{ if } \|B\|\neq\|A\|.\end{cases}
\end{aligned}$$

Further, let $A,\, B\in\mathcal{B}(H)$ such that $(1-t)A+tB$ is invertible for all $t\in[0,1]$. For this to happen, it is enough to assume that $A,\, B>0$ in the operator order of $\mathcal{B}(H)$. Consider the function $A(t):=((1-t)A+tB)^{-1}$, $t\in[0,1]$ and observe that

$$A'(t)=-\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1},\ t\in[0,1].$$

By utilising (2.4) we then get

$$\begin{aligned}
&\left|\left(\frac{A+B}{2}\right)^{-1}-\int_0^1\left((1-t)A+tB\right)^{-1}dt\right|^2 \qquad (3.10)\\
&\leq \frac{1}{12}\int_0^1\left|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right|^2dt
\end{aligned}$$

for $A,\, B\in\mathcal{B}(H)$ such that $(1-t)A+tB$ is invertible for all $t\in[0,1]$.

Since

$$\begin{aligned}
&\left|\left((1-t)A+tB\right)^{-1}(B-A)\left((1-t)A+tB\right)^{-1}\right|\\
&\leq \left\|\left((1-t)A+tB\right)^{-1}\right\|^2\|B-A\|,
\end{aligned}$$

hence by (3.10) we derive

$$\begin{aligned}
&\left|\left(\frac{A+B}{2}\right)^{-1}-\int_0^1\left((1-t)A+tB\right)^{-1}dt\right|^2 \qquad (3.11)\\
&\leq \frac{1}{12}\|B-A\|^2\int_0^1\left\|\left((1-t)A+tB\right)^{-1}\right\|^4dt.
\end{aligned}$$

Now, if $A \geq m > 0$ and $B \geq m > 0$, then $((1-t)A + tB)^{-1} \leq m^{-1}$ for $t \in [0,1]$ and by (3.11) we obtain the simpler inequality

$$\left\| \left(\frac{A+B}{2} \right)^{-1} - \int_0^1 ((1-t)A + tB)^{-1} dt \right\|^2 \leq \frac{1}{12} \frac{\|B-A\|^2}{m^4}.$$

Since $f(u) = u^{-1}$ is operator convex on $(0,\infty)$, then by taking the square root and using the Hermite-Hadamard operator inequality [6], we derive

$$0 \leq \int_0^1 ((1-t)A + tB)^{-1} dt - \left(\frac{A+B}{2} \right)^{-1} \leq \frac{\sqrt{3}}{6} \frac{\|B-A\|}{m^2}, \tag{3.12}$$

provided that $A \geq m > 0$ and $B \geq m > 0$.

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[1]Mathematics, College of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.

E-mail address: sever.dragomir@vu.edu.au

URL: http://rgmia.org/dragomir

[2]DST-NRF Centre of Excellence in the Mathematical, and Statistical Sciences, School of Computer Science, & Applied Mathematics, University of the Witwatersrand,, Private Bag 3, Johannesburg 2050, South Africa