STEKLOFF AND ALMANSI ABSOLUTE VALUE INTEGRAL INEQUALITIES FOR FUNCTIONS OF OPERATORS IN HILBERT SPACES

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Abstract. Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. In this paper we show among others that, if $w : [a,b] \to (0,\infty)$ is continuous on $[a,b]$ such that $\int_a^b w(s)\,ds = 1$ and $A \in C^1([a,b], \mathcal{B}(H))$ with $\frac{A'}{\sqrt{w}} \in L_2([a,b], \mathcal{B}(H))$ and $\int_a^b w(t) A(t)\,dt = 0$, then

$$\int_a^b w(t) |A(t)|^2 \,dt \le \frac{1}{\pi^2} \int_a^b \frac{|A'(t)|^2}{w(t)} dt,$$

in the operator order of $\mathcal{B}(H)$. Applications related to the Schwarz unweighted and weighted integral inequalities are also provided.

1. Introduction

It is well known that, see for instance [4], or [7], if $u \in C^1([a,b], \mathbb{R})$, namely u is continuous on $[a,b]$ and has a derivative that is continuous on (a,b) and satisfies $u(a) = u(b) = 0$, then the following *Wirtinger type inequality* is valid

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2} \int_a^b [u'(t)]^2 \,dt \tag{1.1}$$

with the equality holding if and only if $u(t) = K \sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K \in \mathbb{R}$.

If $u \in C^1([a,b], \mathbb{R})$ satisfies the condition $u(a) = 0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2} \int_a^b [u'(t)]^2 \,dt \tag{1.2}$$

and the equality holds if and only if $u(t) = L \sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L \in \mathbb{R}$.

For some related Wirtinger type integral inequalities see [2]-[5] and [6]-[14].

In 1901, W. Stekloff, [12], proved that, if $u \in C^1([a,b], \mathbb{R})$ and $\int_a^b u(t)\,dt = 0$, then

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{\pi^2} \int_a^b [u'(x)]^2 \,dx. \tag{1.3}$$

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In addition, if $u(a)=u(b)$, then, as proved by E. Almansi in 1905, [1], the inequality (1.3) can be improved as follows

$$\int_a^b u^2(x)\,dx \le \frac{(b-a)^2}{4\pi^2}\int_a^b [u'(x)]^2\,dx. \tag{1.4}$$

We can state the following result for complex functions $h:[a,b]\to\mathbb{C}$.

Theorem 1. *If $h\in C^1([a,b],\mathbb{C})$ and $\int_a^b h(t)\,dt=0$, then*

$$\int_a^b |h(x)|^2\,dx \le \frac{(b-a)^2}{\pi^2}\int_a^b |h'(x)|^2\,dx. \tag{1.5}$$

In addition, if $h(a)=h(b)$, then

$$\int_a^b |h(x)|^2\,dx \le \frac{(b-a)^2}{4\pi^2}\int_a^b |h'(x)|^2\,dx. \tag{1.6}$$

The proof follows by (1.3) and (1.4) applied for $u=\operatorname{Re} h$ and $u=\operatorname{Im} h$ and by adding the corresponding inequalities.

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. If $\{e_\alpha\}_{\alpha\in\mathcal{U}}$ ($\mathcal{U}$ is a certain index set), is a complete orthonormal system in a Hilbert space H, then for any element $x\in H$, *Parseval's equality* holds:

$$\|x\|^2=\sum_{\alpha\in\mathcal{U}}|\langle x,e_\alpha\rangle|^2 \tag{1.7}$$

and the sum on the right-hand side is to be understood as $\sup_{\mathcal{U}_0}\sum_{\alpha\in\mathcal{U}_0}|\langle x,e_\alpha\rangle|^2$where the supremum is taken over all finite subsets $\mathcal{U}_0$ of $\mathcal{U}$.

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. For $A:[a,b]\to\mathcal{B}(H)$ strongly measurable and Bochner squared integrable, namely $A\in L_2([a,b],\mathcal{B}(H))$, we have

$$\frac{1}{b-a}\int_a^b |A(t)|^2\,dt \ge \left|\frac{1}{b-a}\int_a^b A(t)\,dt\right|^2 \tag{1.8}$$

in the operator order of $\mathcal{B}(H)$.

Indeed, since

$$\begin{aligned}0&\le |A(t)-A(s)|^2=(A(t)-A(s))^*(A(t)-A(s))\\&=|A(t)|^2-A^*(s)A(t)-A^*(t)A(s)+|A(s)|^2,\end{aligned}$$

hence

$$|A(t)|^2+|A(s)|^2\ge A^*(s)A(t)+A^*(t)A(s)$$

for all $t,s\in[a,b]$.

Integrating over $s,t\in[a,b]$, we get

$$\int_a^b\int_a^b\left[|A(t)|^2+|A(s)|^2\right]dtds \ge \int_a^b\int_a^b [A^*(s)A(t)+A^*(t)A(s)]\,dtds \tag{1.9}$$

in the operator order of $\mathcal{B}(H)$.

Observe that

$$\int_a^b \int_a^b \left[|A(t)|^2 + |A(s)|^2 \right] dtds = (b-a) \int_a^b |A(t)|^2 dt + (b-a) \int_a^b |A(s)|^2 ds$$
$$= 2(b-a) \int_a^b |A(t)|^2 dt$$

and

$$\int_a^b \int_a^b [A^*(s) A(t) + A^*(t) A(s)] dtds$$
$$= \int_a^b A^*(s) ds \int_a^b A(t) dt + \int_a^b A^*(t) dt \int_a^b A(s) ds = 2 \left(\int_a^b A(t) dt \right)^* \int_a^b A(t) dt,$$

and by (1.9) we derive (1.8).

In a similar way, if $w : [a,b] \to [0,\infty)$ with $\int_a^b w(t) dt = 1$, then

$$\int_a^b w(t) |A(t)|^2 dt \geq \left| \int_a^b w(t) A(t) dt \right|^2, \tag{1.10}$$

provided that $A \in L_{2,w}([a,b], \mathcal{B}(H)) := \left\{ A : [a,b] \to B(H),\ \int_a^b w(t) \|A(t)\|^2 dt < \infty \right\}$.

2. Main Results

We have the following inequality of Wirtinger type in the operator order of $\mathcal{B}(H)$:

Theorem 2. *Assume that $A : [a,b] \to \mathcal{B}(H)$ is of class C^1 on $[a,b]$ and $\int_a^b A(t) dt = 0$, then*

$$\int_a^b |A(t)|^2 dt \leq \frac{(b-a)^2}{\pi^2} \int_a^b |A'(t)|^2 dt. \tag{2.1}$$

In addition, if $A(a) = A(b)$, then we have the better inequality

$$\int_a^b |A(t)|^2 dt \leq \frac{(b-a)^2}{4\pi^2} \int_a^b |A'(t)|^2 dt. \tag{2.2}$$

Proof. Let $x \in H$. Assume that $\{e_\alpha\}_{\alpha \in \mathcal{U}}$ is a complete orthonormal system in the Hilbert space H. For $\alpha \in \mathcal{U}$, consider the function $h_\alpha(t) = \langle A(t) x, e_\alpha \rangle$, $t \in [a,b]$. Then h_α is of class C^1 on $[a,b]$, $h'_\alpha(t) = \langle A'(t) x, e_\alpha \rangle$ and

$$\int_a^b \langle A(t) x, e_\alpha \rangle dt = \left\langle \left(\int_a^b A(t) dt \right) x, e_\alpha \right\rangle = 0.$$

By using inequality (1.5) we get

$$\int_a^b |\langle A(t) x, e_\alpha \rangle|^2 dt \leq \frac{(b-a)^2}{\pi^2} \int_a^b |\langle A'(t) x, e_\alpha \rangle|^2 dt, \tag{2.3}$$

for all $\alpha \in \mathcal{U}$.

By summing in inequality (2.3) over $\alpha \in \mathcal{U}$, then we get

$$\sum_{\alpha \in \mathcal{U}} \int_a^b |\langle A(t) x, e_\alpha \rangle|^2 dt \leq \frac{(b-a)^2}{\pi^2} \sum_{\alpha \in \mathcal{U}} \int_a^b |\langle A'(t) x, e_\alpha \rangle|^2 dt,$$

namely

$$\int_a^b \left(\sum_{\alpha\in\mathcal{U}} |\langle A(t)x, e_\alpha\rangle|^2\right) dt \le \frac{(b-a)^2}{\pi^2} \int_a^b \left(\sum_{\alpha\in\mathcal{U}} |\langle A'(t)x, e_\alpha\rangle|^2\right) dt. \tag{2.4}$$

By Parseval's equality (1.7) we get

$$\sum_{\alpha\in\mathcal{U}} |\langle A(t)x, e_\alpha\rangle|^2 = \|A(t)x\|^2, \ t\in[a,b]$$

and

$$\sum_{\alpha\in\mathcal{U}} |\langle A'(t)x, e_\alpha\rangle|^2 = \|A'(t)x\|^2, \ t\in(a,b).$$

Therefore by (2.4) we deduce

$$\int_a^b \|A(t)x\|^2 dt \le \frac{(b-a)^2}{\pi^2}\int_a^b \|A'(t)x\|^2 dt \tag{2.5}$$

for all $x\in H$.

Now, observe that

$$\begin{aligned}\int_a^b \|A(t)x\|^2 dt &= \int_a^b \langle A(t)x, A(t)x\rangle dt = \int_a^b \langle A^*(t)A(t)x, x\rangle dt\\ &= \int_a^b \left\langle |A(t)|^2 x, x\right\rangle dt = \left\langle \left(\int_a^b |A(t)|^2 dt\right)x, x\right\rangle\end{aligned}$$

and

$$\begin{aligned}\int_a^b \|A'(t)x\|^2 dt &= \int_a^b \langle A'(t)x, A'(t)x\rangle dt = \int_a^b \left\langle (A'(t))^* A'(t)x, x\right\rangle dt\\ &= \int_a^b \left\langle |A'(t)|^2 x, x\right\rangle dt = \left\langle \left(\int_a^b |A'(t)|^2 dt\right)x, x\right\rangle\end{aligned}$$

and by (2.5) we get

$$\left\langle \left(\int_a^b |A(t)|^2 dt\right)x, x\right\rangle \le \frac{(b-a)^2}{\pi^2}\left\langle \left(\int_a^b |A'(t)|^2 dt\right)x, x\right\rangle$$

for all $x\in H$, which is equivalent to (2.1).

The inequality (2.2) follows in a similar way from (1.6). □

Recall the Löwner–Heinz inequality which says that $A\ge B\ge 0$ implies $A^\alpha \ge B^\alpha$ for all $\alpha\in[0,1]$. By taking the power $1/2$ in (2.1) we derive

$$\left(\int_a^b |A(t)|^2 dt\right)^{1/2} \le \frac{b-a}{\pi}\left(\int_a^b |A'(t)|^2 dt\right)^{1/2}, \tag{2.6}$$

provided that $A:[a,b]\to\mathcal{B}(H)$ is of class C^1 on $[a,b]$ and $\int_a^b A(t)\,dt = 0$.

In addition, if $A(a) = A(b)$, then we have the better inequality

$$\left(\int_a^b |A(t)|^2 dt\right)^{1/2} \le \frac{b-a}{2\pi}\left(\int_a^b |A'(t)|^2 dt\right)^{1/2}. \tag{2.7}$$

The following composite version also holds:

Theorem 3. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b).*

(i) *If $A\in C^1([a,b],\mathcal{B}(H))$ with $\frac{A'}{\sqrt{h'(t)}}\in L_2([a,b],\mathcal{B}(H))$ and $\int_a^b h'(t)A(t)\,dt=0$, then*

$$\int_a^b h'(t)|A(t)|^2\,dt\le\frac{[h(b)-h(a)]^2}{\pi^2}\int_a^b\frac{|A'(t)|^2}{h'(t)}dt. \tag{2.8}$$

(ii) *In addition, if $A(a)=A(b)$, then*

$$\int_a^b h'(t)|A(t)|^2\,dt\le\frac{[h(b)-h(a)]^2}{4\pi^2}\int_a^b\frac{|A'(t)|^2}{h'(t)}dt. \tag{2.9}$$

Proof. (i) We write the inequality (2.1) for the function $A\circ h^{-1}$ on the interval $[h(a),h(b)]$ for which $\int_{h(a)}^{h(b)}A\circ h^{-1}(z)\,dx=0$ to get

$$\int_{h(a)}^{h(b)}\left|\left(A\circ h^{-1}\right)(z)\right|^2dz\le\frac{(h(b)-h(a))^2}{\pi^2}\int_{h(a)}^{h(b)}\left|\left(A\circ h^{-1}\right)'(z)\right|^2dz. \tag{2.10}$$

If $A:[c,d]\to H$ is strongly differentiable on (c,d), then $A\circ h^{-1}:(h(c),h(d))\to H$ is strongly differentiable on $(h(c),h(d))$ and using the chain rule and the derivative of inverse functions we have

$$\left(A\circ h^{-1}\right)'(z)=\left(A'\circ h^{-1}\right)(z)\left(h^{-1}\right)'(z)=\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)} \tag{2.11}$$

for every $z\in(h(c),h(d))$.

Using the inequality (2.10) we then get

$$\int_{h(a)}^{h(b)}\left|\left(A\circ h^{-1}\right)(z)\right|^2dz\le\frac{(h(b)-h(a))^2}{\pi^2}\int_{h(a)}^{h(b)}\left|\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right|^2dz. \tag{2.12}$$

Observe also that, by the change of variable $t=h^{-1}(z)$, $z\in[h(a),h(b)]$, we have $z=h(t)$ that gives $dz=h'(t)\,dt$,

$$\int_{h(a)}^{h(b)}A\circ h^{-1}(z)\,dx=\int_a^b A(t)h'(t)\,dt=0$$

and

$$\int_{h(a)}^{h(b)}\left|\left(A\circ h^{-1}\right)(z)\right|^2dz=\int_a^b|A(t)|^2h'(t)\,dt. \tag{2.13}$$

We also have

$$\int_{h(a)}^{h(b)}\left|\frac{\left(A'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right|^2dz=\int_a^b\left|\frac{A'(t)}{h'(t)}\right|^2h'(t)\,dt=\int_a^b\frac{|A'(t)|^2}{h'(t)}dt.$$

By making use of (2.12) we get (2.8).

(ii) The inequality (2.9) follows by (2.2) in a similar way. □

If $w:[a,b]\to\mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W:[a,b]\to[0,\infty)$, $W(x):=\int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a,b). We have $W'(x)=w(x)$ for any $x\in(a,b)$.

Corollary 1. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds=1$ *and* $A\in C^1([a,b],\mathcal{B}(H))$.

(i) *If* $\frac{A'}{\sqrt{w}}\in L_2([a,b],\mathcal{B}(H))$ *and* $\int_a^b w(t)A(t)\,dt=0$, *then*

$$\int_a^b w(t)\left|A(t)\right|^2 dt\le\frac{1}{\pi^2}\int_a^b\frac{\left|A'(t)\right|^2}{w(t)}dt. \tag{2.14}$$

(ii) *In addition, if* $A(a)=A(b)$, *then we have the better inequality*

$$\int_a^b w(t)\left|A(t)\right|^2 dt\le\frac{1}{4\pi^2}\int_a^b\frac{\left|A'(t)\right|^2}{w(t)}dt. \tag{2.15}$$

3. Applications

We have the following reverse of the Cauchy-Bunyakowsky-Schwarz inequality (1.10).

Theorem 4. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds=1$ *and* $B\in C^1([a,b],\mathcal{B}(H))$. *If* $\frac{B'}{\sqrt{w}}\in L_2([a,b],\mathcal{B}(H))$, *then*

$$0\le\int_a^b w(t)\left|B(t)\right|^2 dt-\left|\int_a^b w(t)B(t)\,dt\right|^2\le\frac{1}{\pi^2}\int_a^b\frac{\left|B'(t)\right|^2}{w(t)}dt. \tag{3.1}$$

In addition, if $B(a)=B(b)$, *then we have the better inequality*

$$0\le\int_a^b w(t)\left|B(t)\right|^2 dt-\left|\int_a^b w(t)B(t)\,dt\right|^2\le\frac{1}{4\pi^2}\int_a^b\frac{\left|B'(t)\right|^2}{w(t)}dt. \tag{3.2}$$

Proof. Let $A(t):=B(t)-\int_a^b w(s)B(s)\,ds$, $t\in[a,b]$. Then

$$\begin{aligned}\int_a^b w(t)A(t)\,dt&=\int_a^b w(t)\left(B(t)-\int_a^b w(s)B(s)\,ds\right)dt\\&=\int_a^b w(t)B(t)\,dt-\int_a^b w(t)\,dt\int_a^b w(s)B(s)\,ds=0.\end{aligned}$$

By utilising (2.14) we get

$$\int_a^b w(t)\left|B(t)-\int_a^b w(s)B(s)\,ds\right|^2 dt\le\frac{1}{\pi^2}\int_a^b\frac{\left|B'(t)\right|^2}{w(t)}dt. \tag{3.3}$$

Observe that

$$\int_a^b w(t)\left|B(t)-\int_a^b w(s)B(s)\,ds\right|^2 dt$$
$$=\int_a^b w(t)\left(B(t)-\int_a^b w(s)B(s)\,ds\right)^*\left(B(t)-\int_a^b w(s)B(s)\,ds\right)dt$$
$$=\int_a^b w(t)\left(B^*(t)-\int_a^b w(s)B(s)\,ds\right)^*\left(B(t)-\int_a^b w(s)B(s)\,ds\right)dt$$
$$=\int_a^b w(t)\left[|B(t)|^2-\left(\int_a^b w(s)B(s)\,ds\right)^* B(t)\right.$$
$$\left.-B^*(t)\int_a^b w(s)B(s)\,ds+\left(\int_a^b w(s)B(s)\,ds\right)^*\left(\int_a^b w(s)B(s)\,ds\right)\right]dt$$
$$=\int_a^b w(t)|B(t)|^2\,dt-\left(\int_a^b w(s)B(s)\,ds\right)^*\int_a^b w(t)B(t)\,dt$$
$$-\left(\int_a^b w(t)B(t)\,dt\right)^*\int_a^b w(s)B(s)\,ds$$
$$+\left(\int_a^b w(s)B(s)\,ds\right)^*\left(\int_a^b w(s)B(s)\,ds\right)$$
$$=\int_a^b w(t)|B(t)|^2\,dt-\left|\int_a^b w(s)B(s)\,ds\right|^2$$

and by (3.3) we derive (3.1).

The inequality (3.2) follows by (2.15). □

We can state the following reverse of inequality (1.8) as well:

Corollary 2. *If $B\in C^1([a,b],\mathcal{B}(H))$ with $B'\in L_2([a,b],\mathcal{B}(H))$, then*

$$0\le(b-a)\int_a^b|B(t)|^2\,dt-\left|\int_a^b B(t)\,dt\right|^2\le\frac{(b-a)^2}{\pi^2}\int_a^b|B'(t)|^2\,dt. \tag{3.4}$$

In addition, if $B(a)=B(b)$, then we have the better inequality

$$0\le(b-a)\int_a^b|B(t)|^2\,dt-\left|\int_a^b B(t)\,dt\right|^2\le\frac{(b-a)^2}{4\pi^2}\int_a^b|B'(t)|^2\,dt. \tag{3.5}$$

For a function f defined on the interval $[a,b]$ we consider the *symmetrical transform*

$$\check{f}(t):=\frac{1}{2}\left[f(t)+f(a+b-t)\right],\ t\in[a,b]$$

and the *anti-symmetrical transform*

$$\tilde{f}(t):=\frac{1}{2}\left[f(t)-f(a+b-t)\right],\ t\in[a,b].$$

Corollary 3. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$, $B\in C^1([a,b],\mathcal{B}(H))$ and there exists a constant $K>0$ such that $|B'(t)-B'(s)|\le K|t-s|$ for all $t,s\in[a,b]$. Then*

$$0\le\int_a^b w(t)\left|\check{B}(t)\right|^2dt-\left|\int_a^b \check{w}(t)B(t)\,dt\right|^2\le\frac{K^2}{4\pi^2}\int_a^b\frac{\left(t-\frac{a+b}{2}\right)^2}{w(t)}dt. \tag{3.6}$$

Moreover, if B is symmetrical on $[a,b]$, namely $B(t)=B(a+b-t)$ for all $t\in[a,b]$, then

$$0\le\int_a^b w(t)|B(t)|^2dt-\left|\int_a^b w(t)B(t)\,dt\right|^2\le\frac{K^2}{4\pi^2}\int_a^b\frac{\left(t-\frac{a+b}{2}\right)^2}{w(t)}dt. \tag{3.7}$$

Proof. Observe that $\check{B}(a)=\check{B}(b)$ and $\left(\check{B}(t)\right)'=\widetilde{(B')}(t)=\frac{1}{2}\left[B'(t)-B'(a+b-t)\right]$, $t\in[a,b]$. If we write the inequality (3.2) for $\check{B}$ we get

$$0\le\int_a^b w(t)\left|\check{B}(t)\right|^2dt-\left|\int_a^b w(t)\check{B}(t)\,dt\right|^2\le\frac{1}{4\pi^2}\int_a^b\frac{\left|\widetilde{(B')}(t)\right|^2}{w(t)}dt. \tag{3.8}$$

Observe that

$$\begin{aligned}\int_a^b w(t)\check{B}(t)\,dt&=\frac{1}{2}\int_a^b w(t)\left[B(t)+B(a+b-t)\right]dt\\&=\frac{1}{2}\int_a^b w(t)B(t)\,dt+\frac{1}{2}\int_a^b w(t)B(a+b-t)\,dt\\&=\frac{1}{2}\int_a^b w(t)B(t)\,dt+\frac{1}{2}\int_a^b w(a+b-t)B(t)\,dt\\&=\int_a^b \check{w}(t)B(t)\,dt\end{aligned}$$

and

$$\left|\widetilde{(B')}(t)\right|^2=\left|\frac{1}{2}\left[B'(t)-B'(a+b-t)\right]\right|^2\le K^2\left(t-\frac{a+b}{2}\right)^2,\ t\in[a,b]$$

for all $[a,b]$.

By utilising (3.8) we derive (3.6). □

Remark 1. *If $w\equiv 1/(b-a)$, then by (3.6) we get*

$$0\le(b-a)\int_a^b\left|\check{B}(t)\right|^2dt-\left|\int_a^b B(t)\,dt\right|^2\le\frac{K^2(b-a)^4}{48\pi^2}. \tag{3.9}$$

Moreover, if B is symmetrical on $[a,b]$, then

$$0\le(b-a)\int_a^b|B(t)|^2dt-\left|\int_a^b B(t)\,dt\right|^2\le\frac{K^2(b-a)^4}{48\pi^2}. \tag{3.10}$$

Consider the functional

$$D_w(\alpha,B):=\int_a^b w(t)\alpha(t)B(t)\,dt-\int_a^b w(t)\alpha(t)\,dt\int_a^b w(t)B(t)\,dt,$$

where $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$ and the other functions are such that the Bochner integrals exist.

We have for $\alpha:[a,b]\to\mathbb{C}$ and $A:[a,b]\to\mathcal{B}(H)$,

$$0\le\left|\overline{\alpha(t)}A(s)-\overline{\alpha(s)}A(t)\right|^2=|\alpha(t)|\,|A(s)|^2-\alpha(s)\overline{\alpha(t)}A^*(t)A(s)$$
$$-\alpha(t)\overline{\alpha(s)}A^*(s)A(t)+|\alpha(s)|^2|A(t)|^2,$$

which gives that

$$|\alpha(t)|^2|A(s)|^2+|\alpha(s)|^2|A(t)|^2\ge\alpha(s)\overline{\alpha(t)}A^*(t)A(s)+\alpha(t)\overline{\alpha(s)}A^*(s)A(t)$$

for all $s,t\in[a,b]$.

Now, multiply this with $w(s)w(t)\ge0$ to get

$$w(t)|\alpha(t)|^2w(s)|A(s)|^2+w(s)|\alpha(s)|^2w(t)|A(t)|^2$$
$$\ge w(t)\overline{\alpha(t)}A^*(t)w(s)\alpha(s)A(s)+w(s)\overline{\alpha(s)}A^*(s)w(t)\alpha(t)A(t)$$

for all $s,t\in[a,b]$.

Integrating over t and s on $[a,b]$, then we get

$$\int_a^b w(t)|\alpha(t)|^2dt\int_a^b|A(s)|^2ds+\int_a^b|\alpha(s)|^2ds\int_a^b w(t)|A(t)|^2dt$$
$$\ge\int_a^b w(t)\overline{\alpha(t)}A^*(t)\,dt\int_a^b\alpha(s)A(s)\,ds$$
$$+\int_a^b w(s)\overline{\alpha(s)}A^*(s)\,ds\int_a^b\alpha(t)A(t)\,dt$$
$$=2\left|\int_a^b w(s)\alpha(s)A(s)\,ds\right|^2,$$

which proves that

$$\int_a^b w(t)|\alpha(t)|^2dt\int_a^b w(t)|A(t)|^2dt\ge\left|\int_a^b w(t)\alpha(t)A(t)\,dt\right|^2,\tag{3.11}$$

provided that $\alpha\in L_{2,w}([a,b],\mathbb{C})$ and $A\in L_{2,w}([a,b],\mathcal{B}(H))$.

Theorem 5. *Assume that* $w:[a,b]\to(0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds=1$, $\alpha\in L_{2,w}([a,b],\mathbb{C})$ *and* $B\in L_{2,w}([a,b],\mathcal{B}(H))$. *Then*

$$|D_w(\alpha,B)|^2\le\left(\int_a^b w(t)|\alpha(t)|^2dt-\left|\int_a^b w(s)\alpha(s)\,ds\right|^2\right)\tag{3.12}$$
$$\times\left(\int_a^b w(t)|B(t)|^2dt-\left|\int_a^b w(s)B(s)\,ds\right|^2\right)$$

in the operator order of $\mathcal{B}(H)$.

Proof. We use the Sonin type identity

$$D_w(\alpha,B)=\int_a^b w(t)\left(\alpha(t)-\int_a^b w(s)\alpha(s)\,ds\right)\left(B(t)-\int_a^b w(s)B(s)\,ds\right)dt,$$

which can be easily proved by calculating the Bochner integral in the right side.

By using now the weighted Schwarz inequality (3.11), we derive

$$\left|\int_a^b w(t)\left(\alpha(t)-\int_a^b w(s)\alpha(s)\,ds\right)\left(B(t)-\int_a^b w(s)B(s)\,ds\right)dt\right|^2 \leq \int_a^b w(t)\left|\alpha(t)-\int_a^b w(s)\alpha(s)\,ds\right|^2 dt \times \int_a^b w(t)\left|B(t)-\int_a^b w(s)B(s)\,ds\right|^2 dt, \tag{3.13}$$

in the operator order of $\mathcal{B}(H)$.

Since

$$\int_a^b w(t)\left|\alpha(t)-\int_a^b w(s)\alpha(s)\,ds\right|^2 dt = \int_a^b w(t)|\alpha(t)|^2 dt - \left|\int_a^b w(s)\alpha(s)\,ds\right|^2$$

and

$$\int_a^b w(t)\left|B(t)-\int_a^b w(s)B(s)\,ds\right|^2 dt = \int_a^b w(t)|B(t)|^2 dt - \left|\int_a^b w(s)B(s)\,ds\right|^2,$$

hence by (3.13) we get (3.12). □

Corollary 4. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds = 1$, $\alpha\in L_{2,w}([a,b],\mathbb{C})$ and $B\in C^1([a,b],\mathcal{B}(H))$ with $\frac{B'}{\sqrt{w}}\in L_2([a,b],\mathcal{B}(H))$. Then*

$$|D_w(\alpha,B)|^2 \leq \frac{1}{\pi^2}\left(\int_a^b w(t)|\alpha(t)|^2 dt - \left|\int_a^b w(s)\alpha(s)\,ds\right|^2\right) \times \int_a^b \frac{|B'(t)|^2}{w(t)}dt. \tag{3.14}$$

In addition, if $B(a)=B(b)$, then we have the better inequality

$$|D_w(\alpha,B)|^2 \leq \frac{1}{4\pi^2}\left(\int_a^b w(t)|\alpha(t)|^2 dt - \left|\int_a^b w(s)\alpha(s)\,ds\right|^2\right) \times \int_a^b \frac{|B'(t)|^2}{w(t)}dt. \tag{3.15}$$

Remark 2. *If $\alpha\in C^1([a,b],\mathbb{C})$ with $\frac{\alpha'}{\sqrt{w}}\in L_2([a,b],\mathbb{C})$ and $B\in C^1([a,b],\mathcal{B}(H))$ with $\frac{B'}{\sqrt{w}}\in L_2([a,b],\mathcal{B}(H))$, then we get*

$$|D_w(\alpha,B)|^2 \leq \frac{1}{\pi^4}\int_a^b \frac{|\alpha'(t)|^2}{w(t)}dt \int_a^b \frac{|B'(t)|^2}{w(t)}dt. \tag{3.16}$$

If we take the square root in (3.16) we derive the Grüss' type inequality

$$|D_w(\alpha, B)| \le \frac{1}{\pi^2}\left(\int_a^b \frac{|\alpha'(t)|^2}{w(t)}dt\right)^{1/2}\left(\int_a^b \frac{|B'(t)|^2}{w(t)}dt\right)^{1/2} \tag{3.17}$$

in the operator order of $\mathcal{B}(H)$.

The unweighted and scalar version of this inequality for real valued functions was obtained in 1973 by Lupaş, [8].

If $\alpha \in C^1([a,b],\mathbb{C})$ *with* $\frac{\alpha'}{\sqrt{w}} \in L_2([a,b],\mathbb{C})$ *and* $B \in C^1([a,b],\mathcal{B}(H))$ *with* $\frac{B'}{\sqrt{w}} \in L_2([a,b],\mathcal{B}(H))$ *and either* $\alpha(a) = \alpha(b)$ *or* $B(a) = B(b)$, *then*

$$|D_w(\alpha, B)|^2 \le \frac{1}{4\pi^4}\int_a^b \frac{|\alpha'(t)|^2}{w(t)}dt \int_a^b \frac{|B'(t)|^2}{w(t)}dt. \tag{3.18}$$

Finally, if $\alpha(a) = \alpha(b)$ *and* $B(a) = B(b)$, *then*

$$|D_w(\alpha, B)|^2 \le \frac{1}{16\pi^4}\int_a^b \frac{|\alpha'(t)|^2}{w(t)}dt \int_a^b \frac{|B'(t)|^2}{w(t)}dt. \tag{3.19}$$

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