GENERALIZED TRAPEZOID TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. In this paper we show among others that, if the function $A:[a,b]\to B(H)$ is continuous on $[a,b]$ and strongly differentiable on (a,b) with $A'\in L_2([a,b],\mathcal{B}(H))$, then

$$\left|(b-s)A(b)+(s-a)A(a)-\int_a^b A(t)\,dt\right|^2$$
$$\le (b-a)\left[\frac{1}{12}(b-a)^2+\left(s-\frac{a+b}{2}\right)^2\right]\int_a^b \left|A'(t)\right|^2 dt$$

for $s\in(a,b)$, in the operator order of $\mathcal{B}(H)$. Some examples for the operator exponential and inverse functions are also provided.

1. Introduction

In 1999, Cerone and Dragomir proved the following *generalized trapezoid* type inequality for p-norm [5].

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. If $f'\in L_p[a,b]$, then we have the inequality*

$$\left|(b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt\right| \le \frac{1}{(q+1)^{1/q}}\left[(x-a)^{q+1}+(b-x)^{q+1}\right]^{1/q}\|f'\|_{[a,b],p}, \tag{1.1}$$

for all $x\in[a,b]$, where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ and $\|\cdot\|_{[a,b],p}$ is the p-Lebesgue norm on $L_p[a,b]$, i.e., we recall it

$$\|g\|_{[a,b],p}:=\left(\int_a^b |g(t)|^p\,dt\right)^{1/p}.$$

From (1.1) we get the following *trapezoid inequality*

$$\left|(b-a)\frac{f(a)+f(b)}{2}-\int_a^b f(t)\,dt\right|\le\frac{1}{2(q+1)^{1/q}}(b-a)^{1+1/q}\|f'\|_{[a,b],p}, \tag{1.2}$$

and $\frac{1}{2}$ is a best possible constant.

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For $p=q=2$ we derive the $L_2[a,b]$-inequality

$$\left|(b-x)f(b)+(x-a)f(a)-\int_a^b f(t)\,dt\right|^2 \le \frac{1}{3}\left[(x-a)^3+(b-x)^3\right]\|f'\|_{[a,b],2}^2, \tag{1.3}$$

for all $x\in[a,b]$ and the trapezoid inequality

$$\left|(b-a)\frac{f(a)+f(b)}{2}-\int_a^b f(t)\,dt\right|^2 \le \frac{1}{12}(b-a)^3\|f'\|_{[a,b]}^2. \tag{1.4}$$

For a survey on scalar trapezoid inequality, see [5]. For recent papers on this inequality see also [1]-[4] and [6]-[15].

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A+B|\le|A|+|B|$, so the classical arguments using this inequality can not be used. In order to obtain the corresponding version for the operator modulus we need the following preparations.

In this paper we show among others that, if the function $A:[a,b]\to B(H)$ is continuous on $[a,b]$ and strongly differentiable on (a,b) with $A'\in L_2([a,b],\mathcal{B}(H))$, then

$$\left|(b-s)A(b)+(s-a)A(a)-\int_a^b A(t)\,dt\right|^2 \le (b-a)\left[\frac{1}{12}(b-a)^2+\left(s-\frac{a+b}{2}\right)^2\right]\int_a^b |A'(t)|^2\,dt$$

for $s\in(a,b)$, in the operator order of $\mathcal{B}(H)$. Some examples for the operator exponential and inverse functions are also provided.

2. Main Results

We use the following Cauchy-Bunyakowsky-Schwarz inequality:

Lemma 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds=1$. If $\alpha\in L_{2,w}([a,b],\mathbb{C})$ and*

$$A\in L_{2,w}([a,b],\mathcal{B}(H)):=\left\{A:[a,b]\to B(H),\ \int_a^b w(t)\|A(t)\|^2\,dt<\infty\right\},$$

then

$$\int_a^b w(t)|\alpha(t)|^2\,dt\int_a^b w(t)|A(t)|^2\,dt \ge \left|\int_a^b w(t)\alpha(t)A(t)\,dt\right|^2. \tag{2.1}$$

Proof. We have for $\alpha:[a,b]\to\mathbb{C}$ and $A:[a,b]\to\mathcal{B}(H)$,

$$0\le\left|\overline{\alpha(t)}A(s)-\overline{\alpha(s)}A(t)\right|^2=|\alpha(t)||A(s)|^2-\alpha(s)\overline{\alpha(t)}A^*(t)A(s)$$
$$-\alpha(t)\overline{\alpha(s)}A^*(s)A(t)+|\alpha(s)|^2|A(t)|^2,$$

which gives that

$$|\alpha(t)|^2 |A(s)|^2 + |\alpha(s)|^2 |A(t)|^2 \geq \alpha(s)\overline{\alpha(t)}A^*(t)A(s) + \alpha(t)\overline{\alpha(s)}A^*(s)A(t)$$

for all $s, t \in [a,b]$.

Now, multiply this with $w(s)w(t) \geq 0$ to get

$$\begin{aligned} & w(t)|\alpha(t)|^2 w(s)|A(s)|^2 + w(s)|\alpha(s)|^2 w(t)|A(t)|^2 \\ & \geq w(t)\overline{\alpha(t)}A^*(t)w(s)\alpha(s)A(s) + w(s)\overline{\alpha(s)}A^*(s)w(t)\alpha(t)A(t) \end{aligned}$$

for all $s, t \in [a,b]$.

Integrating over t and s on $[a,b]$, then we get

$$\begin{aligned} & \int_a^b w(t)|\alpha(t)|^2 dt \int_a^b |A(s)|^2 ds + \int_a^b |\alpha(s)|^2 ds \int_a^b w(t)|A(t)|^2 dt \\ & \geq \int_a^b w(t)\overline{\alpha(t)}A^*(t)\,dt \int_a^b \alpha(s)A(s)\,ds \\ & + \int_a^b w(s)\overline{\alpha(s)}A^*(s)\,ds \int_a^b \alpha(t)A(t)\,dt \\ & = 2\left|\int_a^b w(s)\alpha(s)A(s)\,ds\right|^2, \end{aligned}$$

and the inequality (2.1) is obtained. □

In a similar way we can prove the following discrete inequality

$$\sum_{k=1}^{n} w_k |z_k|^2 \sum_{k=1}^{n} w_k |A_k|^2 \geq \left|\sum_{k=1}^{n} w_k z_k A_k\right|^2, \tag{2.2}$$

where $z_k \in \mathbb{C}$, $A_k \in \mathcal{B}(H)$, $w_k \geq 0$ for $k \in \{1,...,n\}$ and $\sum_{k=1}^{n} w_k = 1$.

We have the following trapezoid type inequality for the modulus of operators:

Theorem 2. *Assume that for $s \in (a,b)$, the function $A : [a,b] \to B(H)$ is continuous on $[a,b]$ and strongly differentiable on (a,b) with $A' \in L_2([a,b], \mathcal{B}(H))$. Then*

$$\begin{aligned} & \left|(b-s)A(b) + (s-a)A(a) - \int_a^b A(t)\,dt\right|^2 \\ & \leq (b-a)\left[\frac{1}{12}(b-a)^2 + \left(s - \frac{a+b}{2}\right)^2\right] \int_a^b |A'(t)|^2 dt. \end{aligned} \tag{2.3}$$

In particular, we have the trapezoid inequality

$$\left|(b-a)\frac{A(b)+A(a)}{2} - \int_a^b A(t)\,dt\right|^2 \leq \frac{1}{12}(b-a)^3 \int_a^b |A'(t)|^2 dt. \tag{2.4}$$

The constant $\frac{1}{12}$ is best possible.

Proof. Using integration by parts for Bochner's integral, we have, see also for the scalar case,

$$\int_a^b (t-s) A'(t)\, dt = (t-s) A(t)\big|_a^b - \int_a^b A(t)\, ds = (b-s) A(b) + (s-a) A(a) - \int_a^b A(t)\, ds \tag{2.5}$$

for all $s \in [a,b]$.

If we take the modulus and use the inequality (2.1) we get

$$\begin{aligned} &\left|(b-s) A(b) + (s-a) A(a) - \int_a^b A(t)\, ds\right| \\ &\leq \left|\int_a^b (t-s) A'(t)\, dt\right|^2 \leq \int_a^b (t-s)^2\, dt \int_a^b |A'(t)|^2\, dt \\ &= \frac{1}{3}\left[(b-s)^3 + (s-a)^3\right] \int_a^b |A'(t)|^2\, dt. \end{aligned} \tag{2.6}$$

Observe that, by simple calculations we get

$$\frac{1}{3}\left[(s-a)^3 + (b-s)^3\right] = (b-a)\left[\frac{1}{12}(b-a)^2 + \left(s - \frac{a+b}{2}\right)^2\right].$$

By making use of (2.6), we derive (2.3).

The inequality (2.4) follows by (2.3) for $s = \frac{a+b}{2}$.

Now, consider the function

$$A_0(t) = \frac{1}{2}\left(t - \frac{a+b}{2}\right)^2, \ t \in [a,b].$$

Then

$$A_0(b) = A_0(a) = \frac{1}{8}(b-a)^2, \quad \int_a^b A_0(t)\, dt = \frac{1}{24}(b-a)^3$$

and

$$(b-a)\frac{A_0(b) + A_0(a)}{2} - \int_a^b A_0(t)\, dt = \frac{1}{12}(b-a)^3.$$

Also

$$\int_a^b |A'(t)|^2\, dt = \int_a^b \left(t - \frac{a+b}{2}\right)^2 dt = \frac{1}{12}(b-a)^3.$$

Therefore

$$\left|(b-a)\frac{A_0(b) + A_0(a)}{2} - \int_a^b A_0(t)\, dt\right|^2 = \frac{1}{144}(b-a)^6$$

and

$$\frac{1}{12}(b-a)^3 \int_a^b |A_0'(t)|^2\, dt = \frac{1}{144}(b-a)^6,$$

which proves the sharpness of the inequality (2.4). □

Remark 1. *Recall the Löwner–Heinz inequality which says that $A \geq B \geq 0$ implies $A^{\alpha} \geq B^{\alpha}$ for all $\alpha \in [0,1]$. By taking the power $1/2$ in (2.3) we derive*

$$\left|(b-s)A(b)+(s-a)A(a)-\int_a^b A(t)\,dt\right| \tag{2.7}$$
$$\leq (b-a)^{1/2}\left[\frac{1}{12}(b-a)^2+\left(s-\frac{a+b}{2}\right)^2\right]^{1/2}\left(\int_a^b \left|A'(t)\right|^2 dt\right)^{1/2},$$

while from (2.4) we get

$$\left|(b-a)\frac{A(b)+A(a)}{2}-\int_a^b A(t)\,dt\right| \leq \frac{\sqrt{3}}{6}(b-a)^{3/2}\left(\int_a^b \left|A'(t)\right|^2 dt\right)^{1/2}. \tag{2.8}$$

The constant $\frac{\sqrt{3}}{6}$ is best possible in (2.8).

Theorem 3. *Assume that the function $A:[a,b]\to B(H)$ is continuous on $[a,b]$ and strongly differentiable on (a,b) with $A' \in L_2([a,b],\mathcal{B}(H))$. Then*

$$\left|(b-s)A(b)+(s-a)A(a)-\int_a^b A(t)\,dt\right|^2 \tag{2.9}$$
$$\leq \left[(b-s)^4+(s-a)^4\right]\left[\left|\int_0^1 \tau A'((1-\tau)s+\tau b)\,d\tau\right|^2\right.$$
$$\left.+\left|\int_0^1 (1-\tau)A'((1-\tau)a+\tau s)\,d\tau\right|^2\right]$$
$$\leq \frac{1}{3}\left[(b-s)^4+(s-a)^4\right]$$
$$\times\int_0^1 \left[\left|A'((1-\tau)a+\tau s)\right|^2+\left|A'((1-\tau)s+\tau b)\right|^2\right]d\tau$$

for $s \in [a,b]$.

Proof. We use the following change of variable $t=(1-\tau)c+\tau d$, $dt=(d-c)\,d\tau$ and

$$\int_c^d A(t)\,dt=(d-c)\int_0^1 A((1-\tau)c+\tau d)\,d\tau.$$

Observe that

$$\int_a^b (t-s)A'(t)\,dt=\int_a^s (t-s)A'(t)\,dt+\int_s^b (t-s)A'(t)\,dt.$$

Therefore

$$\int_a^s (t-s)A'(t)\,dt=(s-a)\int_0^1 ((1-\tau)a+\tau s-s)A'((1-\tau)a+\tau s)\,d\tau$$
$$=-(s-a)^2\int_0^1 (1-\tau)A'((1-\tau)a+\tau s)\,d\tau$$

and

$$\int_s^b (t-s)\, A'(t)\, dt = (b-s)\int_0^1 ((1-\tau)s+\tau b - s)\, A'((1-\tau)s+\tau b)\, d\tau$$
$$= (b-s)^2 \int_0^1 \tau A'((1-\tau)s+\tau b)\, d\tau.$$

By (2.5) we get the following identity of interest

$$(b-s)\, A(b) + (s-a)\, A(a) - \int_a^b A(t)\, ds \tag{2.10}$$
$$= (b-s)^2 \int_0^1 \tau A'((1-\tau)s+\tau b)\, d\tau$$
$$- (s-a)^2 \int_0^1 (1-\tau)\, A'((1-\tau)a+\tau s)\, d\tau$$

for all $s \in [a,b]$.

If we take the modulus and use the elementary inequality which follows by (2.2)

$$\left(|z_1|^2+|z_2|^2\right)\left(|A_1|^2+|A_2|^2\right) \geq |z_1A_1+z_2A_2|^2,$$

then we get

$$\left|(b-s)\, A(b) + (s-a)\, A(a) - \int_a^b A(t)\, ds\right|^2 \tag{2.11}$$
$$= \left|(b-s)^2 \int_0^1 \tau A'((1-\tau)s+\tau b)\, d\tau\right.$$
$$\left. - (s-a)^2 \int_0^1 (1-\tau)\, A'((1-\tau)a+\tau s)\, d\tau\right|^2$$
$$\leq \left[(b-s)^4 + (s-a)^4\right]\left[\left|\int_0^1 \tau A'((1-\tau)s+\tau b)\, d\tau\right|^2\right.$$
$$\left. + \left|\int_0^1 (1-\tau)\, A'((1-\tau)a+\tau s)\, d\tau\right|^2\right]$$

which proves the first inequality in (2.9).

By (2.1) we have

$$\left|\int_0^1 \tau A'((1-\tau)s+\tau b)\, d\tau\right|^2 \leq \int_0^1 \tau^2 d\tau \int_0^1 |A'((1-\tau)s+\tau b)|^2\, d\tau$$
$$= \frac{1}{3}\int_0^1 |A'((1-\tau)s+\tau b)|^2\, d\tau$$

and

$$\left|\int_0^1 (1-\tau)\, A'((1-\tau)a+\tau s)\, d\tau\right|^2 \leq \int_0^1 (1-\tau)^2\, d\tau \int_0^1 |A'((1-\tau)a+\tau s)|^2\, d\tau$$
$$= \frac{1}{3}\int_0^1 |A'((1-\tau)a+\tau s)|^2\, d\tau,$$

which proves the last part of (2.9). □

The following representation result holds.

Lemma 2. *Let* $B : [a, b] \to \mathcal{B}(H)$ *be a Bochner integrable on* $[a, b]$. *Then for any* $\lambda \in [0, 1]$ *we have the representation*

$$\int_0^1 B[(1-t)a + tb]\,dt = (1-\lambda)\int_0^1 B[(1-t)((1-\lambda)a + \lambda b) + tb]\,dt + \lambda \int_0^1 B[(1-t)a + t((1-\lambda)a + \lambda b)]\,dt. \tag{2.12}$$

In particular,

$$\int_0^1 B[(1-t)a + tb]\,dt = \frac{1}{2}\int_0^1 \left(B\left[(1-t)\frac{a+b}{2} + tb\right] + B\left[(1-t)a + t\frac{a+b}{2}\right]\right) dt. \tag{2.13}$$

Proof. For $\lambda = 0$ and $\lambda = 1$ the equality (2.12) is obvious.

Let $\lambda \in (0, 1)$. Observe that

$$\int_0^1 B[(1-t)(\lambda b + (1-\lambda)a) + tb]\,dt = \int_0^1 B[((1-t)\lambda + t)b + (1-t)(1-\lambda)a]\,dt$$

and

$$\int_0^1 B[t(\lambda b + (1-\lambda)a) + (1-t)a]\,dt = \int_0^1 B[t\lambda b + (1-\lambda t)a]\,dt.$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1 - u = (1-t)(1-\lambda)$ and $du = (1-\lambda)\,dt$. Then

$$\int_0^1 B[((1-t)\lambda + t)b + (1-t)(1-\lambda)a]\,dt = \frac{1}{1-\lambda}\int_\lambda^1 B[ub + (1-u)a]\,du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 B[t\lambda b + (1-\lambda t)a]\,dt = \frac{1}{\lambda}\int_0^\lambda B[ub + (1-u)a]\,du.$$

Therefore

$$\begin{aligned}
&(1-\lambda)\int_0^1 B[(1-t)(\lambda b + (1-\lambda)a) + tb]\,dt \\
&+ \lambda \int_0^1 B[t(\lambda b + (1-\lambda)a) + (1-t)a]\,dt \\
&= \int_\lambda^1 B[ub + (1-u)a]\,du + \int_0^\lambda B[ub + (1-u)a]\,du \\
&= \int_0^1 B[ub + (1-u)a]\,du
\end{aligned}$$

and the identity (2.12) is proved. □

Corollary 1. *Assume that the function $A:[a,b]\to\mathcal{B}(H)$ is continuous on $[a,b]$ and strongly differentiable on (a,b) with $A'\in L_2([a,b],\mathcal{B}(H))$. Then*

$$\begin{aligned}(2.14)\qquad &\left|(b-a)\frac{A(b)+A(a)}{2}-\int_a^b A(t)\,dt\right|^2\\ &\le\frac{1}{8}(b-a)^4\left[\left|\int_0^1\tau A'\left((1-\tau)\frac{a+b}{2}+\tau b\right)d\tau\right|^2\right.\\ &\left.+\left|\int_0^1(1-\tau)A'\left((1-\tau)a+\tau\frac{a+b}{2}\right)d\tau\right|^2\right]\\ &\le\frac{1}{12}(b-a)^3\int_0^1|A'(t)|^2\,dt.\end{aligned}$$

The constants $\frac{1}{8}$ and $\frac{1}{12}$ are best possible in (2.14).

Proof. From (2.9) we have for $s=\frac{a+b}{2}$ that

$$\begin{aligned}&\left|(b-a)\frac{A(b)+A(a)}{2}-\int_a^b A(t)\,dt\right|^2\\ &\le\frac{1}{8}(b-a)^4\left[\left|\int_0^1\tau A'\left((1-\tau)\frac{a+b}{2}+\tau b\right)d\tau\right|^2\right.\\ &\left.+\left|\int_0^1(1-\tau)A'\left((1-\tau)a+\tau\frac{a+b}{2}\right)d\tau\right|^2\right]\\ &\le\frac{1}{24}(b-a)^4\\ &\times\int_0^1\left[\left|A'\left((1-\tau)a+\tau\frac{a+b}{2}\right)\right|^2+\left|A'\left((1-\tau)\frac{a+b}{2}+\tau b\right)\right|^2\right]d\tau.\end{aligned}$$

By (2.13) we also have

$$\begin{aligned}&\frac{1}{2}\int_0^1\left[\left|A'\left((1-\tau)a+\tau\frac{a+b}{2}\right)\right|^2+\left|A'\left((1-\tau)\frac{a+b}{2}+\tau b\right)\right|^2\right]d\tau\\ &=\int_0^1|A'((1-\tau)a+\tau b)|^2\,dt=\frac{1}{b-a}\int_0^1|A'(t)|^2\,dt\end{aligned}$$

and the inequality (2.14) is thus proved. □

We can introduce the following concept:

Definition 1. *We say that the continuous function $B:[a,b]\to\mathcal{B}(H)$ is square modulus convex on $[a,b]$ if*

$$(2.15)\qquad |B((1-t)u+tv)|^2\le(1-t)|B(u)|^2+t|B(v)|^2$$

in the operator order of $\mathcal{B}(H)$, for all $u,\ v\in[a,b]$ and $t\in[0,1]$.

Let $A,\ B\in\mathcal{B}(H)$ and $\alpha\in[0,1]$. Then by (2.2) we get

$$\begin{aligned}\left|(1-\alpha)A+\alpha B\right|^{2}&=\left|(1-\alpha)^{1/2}(1-\alpha)^{1/2}A+\alpha^{1/2}\alpha^{1/2}B\right|^{2}\\&\leq\left[\left((1-\alpha)^{1/2}\right)^{2}+\left(\alpha^{1/2}\right)^{2}\right]\left[\left|(1-\alpha)^{1/2}A\right|^{2}+\left|\alpha^{1/2}B\right|^{2}\right]\\&=(1-\alpha+\alpha)\left[(1-\alpha)|A|^{2}+\alpha|B|^{2}\right]\\&=(1-\alpha)|A|^{2}+\alpha|B|^{2}.\end{aligned}$$

Consider the function $C:[0,1]\to\mathcal{B}(H)$, $C(t)=|(1-t)A+tB|$. Let $t_1,\ t_2\in[0,1]$ and $\alpha\in[0,1]$. Then

$$\begin{aligned}\left|C((1-\alpha)t_1+\alpha t_2)\right|^{2}&=\left|(1-(1-\alpha)t_1-\alpha t_2)A+((1-\alpha)t_1+\alpha t_2)B\right|^{2}\\&=\left|(1-\alpha)((1-t_1)A+t_1B)+\alpha((1-t_2)A+t_2B)\right|^{2}\\&\leq(1-\alpha)\left|(1-t_1)A+t_1B\right|^{2}+\alpha\left|(1-t_2)A+t_2B\right|^{2}\\&=(1-\alpha)|C(t_1)|^{2}+\alpha|C(t_2)|^{2},\end{aligned}$$

which shows that C is *square modulus convex* on $[0,1]$.

Assume that f is *nonnegative* on I and *operator convex*, namely

$$f((1-\alpha)A+\alpha B)\leq(1-\alpha)f(A)+\alpha f(B)$$

for all $\alpha\in[0,1]$ and selfadjoint operators $A,\ B$ with spectra in I.

For such function and $A,\ B$, we consider

$$D(t):=\left[f((1-t)A+tB)\right]^{1/2},t\in[0,1].$$

Then, using a similar proof as above for the modulus function, we conclude that D is *square modulus convex* on $[0,1]$.

The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\leq r\leq 2$ or $-1\leq r\leq 0$ and is operator concave on $(0,\infty)$ if $0\leq r\leq 1$. Therefore for $A,\ B>0$, the function

$$B(t):=((1-t)A+tB)^{r/2},\ t\in[0,1]$$

is *square modulus convex* on $[0,1]$ for $1\leq r\leq 2$ or $-1\leq r\leq 0$.

Proposition 1. *With the assumption of Theorem 2 and if A' is square modulus convex on (a,b), then*

$$\begin{aligned}&\left|(b-s)A(b)+(s-a)A(a)-\int_a^b A(t)\,dt\right|^{2}\\&\leq\frac{1}{2}(b-a)^{2}\left[\frac{1}{12}(b-a)^{2}+\left(s-\frac{a+b}{2}\right)^{2}\right]\left[|A'(a)|^{2}+|A'(b)|^{2}\right].\end{aligned}\tag{2.16}$$

In particular,

$$\left|(b-a)\frac{A(b)+A(a)}{2}-\int_a^b A(t)\,dt\right|^{2}\leq\frac{1}{24}(b-a)^{4}\left[|A'(a)|^{2}+|A'(b)|^{2}\right].\tag{2.17}$$

Proof. We have, by the square modulus convexity of A', that

$$\begin{aligned}\int_a^b |A'(t)|^2 dt &= (b-a)\int_0^1 |A'((1-s)a+sb)|^2 ds \\ &\le (b-a)\int_0^1 \left[(1-s)|A'(a)|^2 + s|A'(b)|^2\right] ds \\ &= (b-a)\frac{|A'(a)|^2+|A'(b)|^2}{2},\end{aligned}$$

and by (2.3) we get (2.16). □

We also have:

Proposition 2. *With the assumption of Theorem 3 and if A' is square modulus convex on (a,b) then*

$$\begin{aligned}&\left|(b-s)A(b)+(s-a)A(a)-\int_a^b A(t)\,dt\right|^2 \\ &\le \frac{1}{3}\left[(s-a)^4+(b-s)^4\right]\left[|A'(s)|^2+\frac{|A'(b)|^2+|A'(a)|^2}{2}\right].\end{aligned} \tag{2.18}$$

In particular,

$$\begin{aligned}&\left|(b-a)\frac{A(b)+A(a)}{2}-\int_a^b A(t)\,dt\right|^2 \\ &\le \frac{1}{24}(b-a)^2\left[\left|A'\left(\frac{a+b}{2}\right)\right|^2+\frac{|A'(b)|^2+|A'(a)|^2}{2}\right] \\ &\le \frac{1}{12}(b-a)^4\left[|A'(a)|^2+|A'(b)|^2\right].\end{aligned} \tag{2.19}$$

Proof. We have, by the square modulus convexity of A', that

$$\begin{aligned}&\int_0^1 \left[|A'((1-\tau)a+\tau s)|^2+|A'((1-\tau)s+\tau b)|^2\right] d\tau \\ &\le \int_0^1 \left[(1-\tau)|A'(a)|^2+\tau|A'(s)|^2+(1-\tau)|A'(s)|^2+\tau|A'(b)|^2\right] dt \\ &= |A'(s)|^2+\frac{|A'(b)|^2+|A'(a)|^2}{2}\end{aligned}$$

and by (2.9) we derive (2.18). □

3. Some Examples

Consider the function $A(t)=\exp(tT)$, where $t\in\mathbb{R}$ and $T\in\mathcal{B}(H)$. Then $A'(t)=T\exp(tT)$, for $t\in\mathbb{R}$ and $T\in\mathcal{B}(H)$. By making use of 2.19 we get

$$\begin{aligned}&\left|(b-a)\frac{\exp(aT)+\exp(bT)}{2}-\int_a^b \exp(tT)\,dt\right|^2 \\ &\le \frac{1}{12}(b-a)^3\int_a^b |T\exp(tT)|^2 dt.\end{aligned} \tag{3.1}$$

If T is invertible, then [2]

$$\int_a^b \exp(tT)\,dt = T^{-1}\left[\exp(bT) - \exp(aT)\right]. \tag{3.2}$$

From (3.1) we derive

$$\left|(b-a)\frac{\exp(aT)+\exp(bT)}{2} - T^{-1}\left[\exp(bT)-\exp(aT)\right]\right|^2 \leq \frac{1}{12}(b-a)^3\int_a^b |T\exp(tT)|^2\,dt. \tag{3.3}$$

For T invertible, if we consider $B(t) = T\exp(tT)$, then $B'(t) = T^2\exp(tT)$ and

$$\int_a^b B(t)\,dt = \exp(bT) - \exp(aT).$$

By (2.19) we derive

$$\left|(b-a)T\frac{\exp(aT)+\exp(bT)}{2} - \exp(bT) + \exp(aT)\right|^2 \leq \frac{1}{12}(b-a)^3\int_a^b \left|T^2\exp(tT)\right|^2 dt. \tag{3.4}$$

Since for any operator $V \in \mathcal{B}(H)$ we have $|V|^2 \leq \|V\|^2$ and $\|\exp(tT)\| \leq \exp(|t|\,\|T\|)$, $t \in \mathbb{R}$, $T \in \mathcal{B}(H)$, then by (3.1) we get

$$\begin{aligned}\left|(b-a)\frac{\exp(aT)+\exp(bT)}{2} - \int_a^b \exp(tT)\,dt\right|^2 &\leq \frac{1}{12}(b-a)^3\int_a^b \|T\exp(tT)\|^2\,dt \leq \frac{1}{12}\|T\|^2(b-a)^3\int_a^b \|\exp(tT)\|^2\,dt \\ &\leq \frac{1}{12}\|T\|^2(b-a)^3\int_a^b \exp(2\|T\|\,|t|)\,dt.\end{aligned} \tag{3.5}$$

If $0 \leq a \leq b$, then

$$\int_a^b \exp(2\|T\|\,|t|)\,dt = \int_a^b \exp(2\|T\|\,t)\,dt = \frac{\exp(2\|T\|\,b) - \exp(2\|T\|\,a)}{2\|T\|}$$

and by (3.5) we get

$$\left|(b-a)\frac{\exp(aT)+\exp(bT)}{2} - \int_a^b \exp(tT)\,dt\right|^2 \leq \frac{1}{24}(b-a)^3\|T\|\left[\exp(2\|T\|\,b) - \exp(2\|T\|\,a)\right] \tag{3.6}$$

for any $T \in \mathcal{B}(H)$.

Moreover, if T is invertible, then we also have the exponential inequality

$$\left|(b-a)\frac{\exp(aT)+\exp(bT)}{2} - T^{-1}\left[\exp(bT)-\exp(aT)\right]\right|^2 \leq \frac{1}{24}(b-a)^3\|T\|\left[\exp(2\|T\|\,b) - \exp(2\|T\|\,a)\right]. \tag{3.7}$$

Consider the function $A(t) = \exp[(1-t)A](B-A)\exp(tB)$, $t \in [0,1]$. Then, integrating by parts

$$
\begin{aligned}
&\int_0^1 f(t)\,dt \\
&= \int_0^1 (\exp[(1-t)A]B\exp(tB) - \exp[(1-t)A]A\exp(tB))\,dt \\
&= \int_0^1 \exp[(1-t)A](\exp(tB))'\,dt + \int_0^1 (\exp[(1-t)A])'\exp(tB)\,dt \\
&= \exp[(1-t)A]\exp(tB)|_0^1 + A\int_0^1 \exp[(1-t)A]\exp(tB)\,dt \\
&+ \exp[(1-t)A]\exp(tB)|_0^1 - \int_0^1 (\exp[(1-t)A])B\exp(tB)\,dt \\
&= 2(\exp B - \exp A) - \int_0^1 \exp[(1-t)A](B-A)\exp(tB) \\
&= 2(\exp B - \exp A) - \int_0^1 f(t)\,dt,
\end{aligned}
$$

which gives the following identity of interest [4]

$$
\int_0^1 \exp[(1-t)A](B-A)\exp(tB)\,dt = \exp B - \exp A
$$

for all A, $B \in \mathcal{B}(H)$.

Also

$$
\begin{aligned}
A'(t) &= -A\exp[(1-t)A](B-A)\exp(tB) \\
&\quad + \exp[(1-t)A](B-A)B\exp(tB)\,dt \\
&= \exp[(1-t)A](B-A)B\exp(tB)\,d \\
&\quad - \exp[(1-t)A]A(B-A)\exp(tB) \\
&= \exp[(1-t)A][(B-A)B - A(B-A)]\exp(tB) \\
&= \exp[(1-t)A]\left(B^2 - 2AB + A^2\right)\exp(tB).
\end{aligned}
$$

By utilising (2.4) we get

$$
\begin{aligned}
&\left|\frac{\exp(A)(B-A) + (B-A)\exp(B)}{2} - \exp B + \exp A\right|^2 \\
&\le \frac{1}{12}\int_a^b \left|\exp[(1-t)A]\left(B^2 - 2AB + A^2\right)\exp(tB)\right|^2 dt,
\end{aligned} \tag{3.8}
$$

for all A, $B \in \mathcal{B}(H)$.

Since

$$\begin{aligned}
&\left|\exp\left[(1-t)A\right]\left(B^2-2AB+A^2\right)\exp(tB)\right|^2\\
&\leq \left\|\exp\left[(1-t)A\right]\left(B^2-2AB+A^2\right)\exp(tB)\right\|^2\\
&\leq \left\|\exp\left[(1-t)A\right]\right\|^2\left\|B^2-2AB+A^2\right\|^2\left\|\exp(tB)\right\|^2\\
&\leq \exp\left[2(1-t)\|A\|\right]\left\|B^2-2AB+A^2\right\|^2\exp(2t\|B\|)\\
&= \left\|B^2-2AB+A^2\right\|^2\exp\left[2\left[(1-t)\|A\|+t\|B\|\right]\right]\\
&= \left\|B^2-2AB+A^2\right\|^2\exp\left\{2\left[(1-t)\|A\|+t\|B\|\right]\right\},
\end{aligned}$$

hence by (3.8) we get

$$\begin{aligned}
&\left|\frac{\exp(A)(B-A)+(B-A)\exp(B)}{2}-\exp B+\exp A\right|^2 \qquad (3.9)\\
&\leq \frac{1}{12}\left\|B^2-2AB+A^2\right\|^2\int_a^b \exp\left\{2\left[(1-t)\|A\|+t\|B\|\right]\right\}dt\\
&= \frac{1}{12}\left\|B^2-2AB+A^2\right\|^2\begin{cases}\exp(2\|A\|) \text{ if } \|B\|=\|A\|,\\[2mm] \frac{\exp(2\|B\|)-\exp(2\|A\|)}{2(\|B\|-\|A\|)} \text{ if } \|B\|\neq\|A\|.\end{cases}
\end{aligned}$$

Further, let $A,\ B\in\mathcal{B}(H)$ such that $(1-t)A+tB$ is invertible for all $t\in[0,1]$. For this to happen, it is enough to assume that $A,\ B>0$ in the operator order of $\mathcal{B}(H)$. Consider the function $A(t):=((1-t)A+tB)^{-1}$, $t\in[0,1]$ and observe that

$$A'(t)=-((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1},\ t\in[0,1].$$

By utilising (2.4) we then get

$$\begin{aligned}
&\left|\frac{A^{-1}+B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}dt\right|^2 \qquad (3.10)\\
&\leq \frac{1}{12}\int_0^1\left|((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1}\right|^2dt
\end{aligned}$$

for $A,\ B\in\mathcal{B}(H)$ such that $(1-t)A+tB$ is invertible for all $t\in[0,1]$.

Since

$$\begin{aligned}
&\left|((1-t)A+tB)^{-1}(B-A)((1-t)A+tB)^{-1}\right|\\
&\leq \left\|((1-t)A+tB)^{-1}\right\|^2\|B-A\|,
\end{aligned}$$

hence by (3.10) we derive

$$\begin{aligned}
&\left|\frac{A^{-1}+B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}dt\right|^2 \qquad (3.11)\\
&\leq \frac{1}{12}\|B-A\|^2\int_0^1\left\|((1-t)A+tB)^{-1}\right\|^4dt.
\end{aligned}$$

Now, if $A\geq m>0$ and $B\geq m>0$, then $((1-t)A+tB)^{-1}\leq m^{-1}$ for $t\in[0,1]$ and by (3.11) we obtain the simpler inequality

$$\left|\frac{A^{-1}+B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}dt\right|^2\leq\frac{1}{12}\frac{\|B-A\|^2}{m^4}.$$

Since $f(u)=u^{-1}$ is operator convex on $(0,\infty)$, then by taking the square root and using the Hermite-Hadamard operator inequality [8], we derive

$$0\leq\frac{A^{-1}+B^{-1}}{2}-\int_0^1((1-t)A+tB)^{-1}dt\leq\frac{\sqrt{3}}{6}\frac{\|B-A\|}{m^2}, \tag{3.12}$$

provided that $A\geq m>0$ and $B\geq m>0$.

References

[1] M. W. Alomari, New upper and lower bounds for the trapezoid inequality of absolutely continuous functions and applications. *Konuralp J. Math.* **7** (2019), no. 2, 319–323.

[2] N. S. Barnett, C. Buşe, P. Cerone and S. S. Dragomir, Ostrowski's inequality for vector-valued functions and applications, *Computers and Mathematics with Applications* **44** (2002), 559-572.

[3] C. Buşe, S. S. Dragomir, J. Roumeliotis and A. Sofo, Generalised trapezoid type inequalities for vector-valued functions and applications. Inequalities, 2001 (Timişoara). *Math. Inequal. Appl.* **5** (2002), no. 3, 435–450

[4] C. Buşe, S. S. Dragomir and A. Sofo, Ostrowski's inequality for vector-valued functions of bounded semivariation and applications, *New Zealand J. Math.* **31** (2002), 137-152.

[5] P. Cerone and S. S. Dragomir, Trapezoidal-type rules from an inequalities point of view. *Handbook of analytic-computational methods in applied mathematics*, 65–134, Chapman & Hall/CRC, Boca Raton, FL, 2000. Preprint *RGMIA Res. Rep. Coll.* **2** (1999), No. 6, Art. 8, 47 pp. [Online https://rgmia.org/papers/v2n6/TrapRulesIneqRep.pdf].

[6] P. Cerone, S. S. Dragomir and Eder Kikianty, Ostrowski and trapezoid type inequalities related to Pompeiu's mean value theorem with complex exponential weight. *J. Math. Inequal.* **11** (2017), no. 4, 947–964.

[7] S. S. Dragomir, Improvements of Ostrowski and generalised trapezoid inequality in terms of the upper and lower bounds of the first derivative. *Tamkang J. Math.* **34** (2003), no. 3, 213–222.

[8] S. S. Dragomir, Hermite-Hadamard's type inequalities for operator convex functions. *Appl. Math. Comput.* **218** (2011), no. 3, 766–772.

[9] S. S. Dragomir, Vector and operator trapezoidal type inequalities for continuous functions of selfadjoint operators in Hilbert spaces. *Electron. J. Linear Algebra* **22** (2011), 161–178.

[10] S. S. Dragomir, *Operator inequalities of Ostrowski and trapezoidal type*. SpringerBriefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[11] S. S. Dragomir and A. Sofo, Trapezoidal type inequalities for n-time differentiable functions. *J. Appl. Math. Comput.* **28** (2008), no. 1-2, 367–379.

[12] A. Kashuri and R. Liko, Generalized trapezoidal type integral inequalities and their applications. *J. Anal.* **28** (2020), no. 4, 1023–1043

[13] E. R. Nwaeze, Generalized weighted trapezoid and Grüss type inequalities on time scales. *Aust. J. Math. Anal. Appl.* **14** (2017), no. 1, Art. 4, 13 pp.

[14] M. Rostamian Delavar, Sharp trapezoid and mid-point type inequalities on closed balls in $\mathbb{R}^3$. *J. Inequal. Appl.* 2020, Paper No. 114, 17 pp.

[15] K. L.Tseng and S. H. Hwang, Some extended trapezoid-type inequalities and applications. *Hacet. J. Math. Stat.* **45** (2016), no. 3, 827–850.

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