HERMITE-HADAMARD TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Let $(H; \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on H. We say that the continuous function $B : [a,b] \to \mathcal{B}(H)$ is square modulus convex (concave) on $[a,b]$ if

$$|B((1-t)u + tv)|^2 \leq (\geq)(1-t)|B(u)|^2 + t|B(v)|^2$$

in the operator order of $\mathcal{B}(H)$, for all $u, v \in [a,b]$ and $t \in [0,1]$. In this paper, we show among others that,

$$\begin{aligned}
&\left|B\left(\frac{u+v}{2}\right)\right|^2 \\
&\leq (\geq)(1-\lambda)\left|B\left[\frac{(1-\lambda)u + (1+\lambda)v}{2}\right]\right|^2 + \lambda\left|B\left[\frac{(2-\lambda)u + \lambda v}{2}\right]\right|^2 \\
&\leq (\geq)\int_0^1 |B[(1-t)u + tv]|^2 dt \\
&\leq (\geq)\frac{1}{2}\left\{(1-\lambda)|B(v)|^2 + \lambda|B(u)|^2 + |B[(1-\lambda)u + \lambda v]|^2\right\} \\
&\leq \frac{1}{2}(\geq)\left[|B(u)|^2 + |B(v)|^2\right]
\end{aligned}$$

for all $u, v \in [a,b]$ and $\lambda \in [0,1]$. Applications for power functions are provided as well.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $I \subseteq \mathbb{R}$ be an interval. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be *convex* in the classical sense if it satisfies the following inequality

$$f(tx + (1-t)t) \leq tf(x) + (1-t)f(y)$$

for all $x, y \in I$ and $t \in [0,1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and $a, b \in I$ with $a < b$. Then the inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds if f is convex, and is known in the literature as *Hermite-Hadamard inequality*, after the name of C. Hermite and J. Hadamard (see [14]). The inequalities in (1.1) hold in reversed direction if f is a concave function.

A vast literature related to (1.1) have been produced by a large number of mathematicians [8] since it is considered to be one of the most famous inequality for

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convex functions due to its usefulness and many applications in various branches of Pure and Applied Mathematics, such as Numerical Analysis [3], Information Theory [2], Operator Theory [6] and others.

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x, y \in X,\ x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty,\ t \in [0,1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x,y) : [0,1] \to \mathbb{R},\ g(x,y)(t) := f[(1-t)x + ty],\ t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [4, p. 2], [5, p. 2])

$$f\left(\frac{x+y}{2}\right) \leq \int_0^1 f[(1-t)x + ty]dt \leq \frac{f(x) + f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x, y) : [0, 1] \to \mathbb{R}$.

Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \leq p < \infty$) is a convex function, then for any $x, y \in X$ we have the following norm inequality from (1.2) (see [13, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p \leq \int_0^1 \|(1-t)x + ty\|^p dt \leq \frac{\|x\|^p + \|y\|^p}{2}. \tag{1.3}$$

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A + B| \leq |A| + |B|$, so the classical arguments using this inequality can not be used.

The following Cauchy-Bunyakowsky-Schwarz inequality holds

$$\sum_{k=1}^n w_k |z_k|^2 \sum_{k=1}^n w_k |A_k|^2 \geq \left|\sum_{k=1}^n w_k z_k A_k\right|^2, \tag{1.4}$$

where $z_k \in \mathbb{C}$, $A_k \in \mathcal{B}(H)$, $w_k \geq 0$ for $k \in \{1, ..., n\}$ and $\sum_{k=1}^n w_k = 1$.

A general version of this inequality is proved below in Lemma 1.

Definition 1. *We say that the continuous function $B : [a, b] \to \mathcal{B}(H)$ is square modulus convex (concave) on $[a, b]$ if*

$$|B((1-t)u + tv)|^2 \leq (\geq)\, (1-t)|B(u)|^2 + t|B(v)|^2 \tag{1.5}$$

in the operator order of $\mathcal{B}(H)$, for all $u, v \in [a, b]$ and $t \in [0, 1]$.

Let $A, B \in \mathcal{B}(H)$ and $\alpha \in [0, 1]$. Then by (1.4), we get

$$\begin{aligned}
|(1-\alpha)A + \alpha B|^2 &= \left|(1-\alpha)^{1/2}(1-\alpha)^{1/2}A + \alpha^{1/2}\alpha^{1/2}B\right|^2 \\
&\leq \left[\left((1-\alpha)^{1/2}\right)^2 + \left(\alpha^{1/2}\right)^2\right]\left[\left|(1-\alpha)^{1/2}A\right|^2 + \left|\alpha^{1/2}B\right|^2\right] \\
&= (1-\alpha+\alpha)\left[(1-\alpha)|A|^2 + \alpha|B|^2\right] \\
&= (1-\alpha)|A|^2 + \alpha|B|^2.
\end{aligned}$$

Consider the function $C:[0,1]\to\mathcal{B}(H)$, $C(t)=|(1-t)A+tB|$. Let $t_1,t_2\in[0,1]$ and $\alpha\in[0,1]$. Then

$$\begin{aligned}|C((1-\alpha)t_1+\alpha t_2)|^2 &= |(1-(1-\alpha)t_1-\alpha t_2)A+((1-\alpha)t_1+\alpha t_2)B|^2\\ &= |(1-\alpha)((1-t_1)A+t_1B)+\alpha((1-t_2)A+t_2B)|^2\\ &\le (1-\alpha)|(1-t_1)A+t_1B|^2+\alpha|(1-t_2)A+t_2B|^2\\ &= (1-\alpha)|C(t_1)|^2+\alpha|C(t_2)|^2,\end{aligned}$$

which shows that C is *square modulus convex* on $[0,1]$.

In this paper, we show among others that, if $B:[a,b]\to\mathcal{B}(H)$ is square modulus convex (concave) on $[a,b]$, then

$$\begin{aligned}&\left|B\left(\frac{u+v}{2}\right)\right|^2\\ &\le(\ge)(1-\lambda)\left|B\left[\frac{(1-\lambda)u+(1+\lambda)v}{2}\right]\right|^2+\lambda\left|B\left[\frac{(2-\lambda)u+\lambda v}{2}\right]\right|^2\\ &\le(\ge)\int_0^1|B[(1-t)u+tv]|^2dt\\ &\le(\ge)\frac{1}{2}\left\{(1-\lambda)|B(v)|^2+\lambda|B(u)|^2+|B[(1-\lambda)u+\lambda v]|^2\right\}\\ &\le\frac{1}{2}(\ge)\left[|B(u)|^2+|B(v)|^2\right]\end{aligned}$$

for all $u,v\in[a,b]$ and $\lambda\in[0,1]$. Applications for power functions are provided as well.

For recent Hermite-Hadamard type operator inequalities, see [7], [9]-[12] and [16]-[17].

2. General Inequalities

Let $(\Omega,\mathcal{A},\mu)$ be a measurable space consisting of a set Ω, a σ-algebra $\mathcal{A}$ of parts of Ω and a countably additive and positive measure μ on $\mathcal{A}$ with values in $\mathbb{R}\cup\{\infty\}$.

For a μ-measurable function $w:\Omega\to\mathbb{R}$, with $w(x)\ge0$ for μ-a.e. $x\in\Omega$, consider the Lebesgue space $L_w^2(\Omega):=\{f:\Omega\to\mathbb{C},\ f$ is μ-measurable and $\int_\Omega w(x)|f(x)|^2d\mu(x)<\infty\}$. Assume $\int_\Omega w(x)d\mu(x)=1$. In order to simplify the notation for the integrals, we do not write the variable, namely, instead of $\int_\Omega w(x)d\mu(x)$ we simply write $\int_\Omega wd\mu$.

We have the following Cauchy-Bunyakowsky-Schwarz inequality:

Lemma 1. *If* $\alpha\in L_w^2(\Omega)$ *and*

$$A\in L_{2,w}(\Omega,\mathcal{B}(H)):=\left\{A:\Omega\to B(H),\ \int_\Omega w(x)\|A(x)\|^2d\mu(x)<\infty\right\},$$

then

$$\begin{aligned}\int_\Omega w(x)|\alpha(x)|^2d\mu(x)\int_\Omega w(x)|A(x)|^2d\mu(x)\\ \ge\left|\int_\Omega w(x)\alpha(x)A(x)d\mu(x)\right|^2,\end{aligned}\tag{2.1}$$

in the operator order of $\mathcal{B}(H)$.

Proof. We have for $\alpha \in L_w^2(\Omega)$ and $A \in L_{2,w}(\Omega, \mathcal{B}(H))$,

$$0 \leq \left|\overline{\alpha(x)}A(y) - \overline{\alpha(y)}A(x)\right|^2 = |\alpha(x)|\,|A(y)|^2 - \alpha(y)\overline{\alpha(x)}A^*(x)A(y) - \alpha(x)\overline{\alpha(y)}A^*(y)A(x) + |\alpha(y)|^2|A(x)|^2,$$

which gives that

$$|\alpha(x)|^2|A(y)|^2 + |\alpha(y)|^2|A(x)|^2 \geq \alpha(y)\overline{\alpha(x)}A^*(x)A(y) + \alpha(x)\overline{\alpha(y)}A^*(y)A(x)$$

for all $y,\ x \in \Omega$.

Now, multiply this with $w(y)w(x) \geq 0$ to get

$$w(x)|\alpha(x)|^2 w(y)|A(y)|^2 + w(y)|\alpha(y)|^2 w(x)|A(x)|^2 \geq w(x)\overline{\alpha(x)}A^*(x)w(y)\alpha(y)A(y) + w(y)\overline{\alpha(y)}A^*(y)w(x)\alpha(x)A(x)$$

for all $y,\ x \in \Omega$.

Integrating over x and y on Ω, then we get

$$\begin{aligned}&\int_a^b w(x)|\alpha(x)|^2 d\mu(x)\int_a^b |A(y)|^2 d\mu(y)\\&+\int_a^b |\alpha(y)|^2 d\mu(y)\int_a^b w(x)|A(x)|^2 d\mu(x)\\&\geq \int_a^b w(x)\overline{\alpha(x)}A^*(x)d\mu(x)\int_a^b \alpha(y)A(y)d\mu(y)\\&+\int_a^b w(y)\overline{\alpha(y)}A^*(y)d\mu(y)\int_a^b \alpha(x)A(x)d\mu(x)\\&= 2\left|\int_a^b w(y)\alpha(y)A(y)d\mu(y)\right|^2,\end{aligned}$$

and the inequality (2.1) is obtained. □

Assume that f is *nonnegative* on I and *operator convex*, namely

$$f((1-\alpha)A + \alpha B) \leq (1-\alpha)f(A) + \alpha f(B)$$

for all $\alpha \in [0,1]$ and selfadjoint operators A, B with spectra in I.

For such function and A, B, we consider

$$D(t) := [f((1-t)A + tB)]^{1/2}, t \in [0,1].$$

Then, using a similar proof as above for the modulus function, we conclude that D is *square modulus convex* on $[0,1]$.

The function $f(t) = t^r$ is operator convex on $(0,\infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0,\infty)$ if $0 \leq r \leq 1$. Therefore for A, $B > 0$, the function

$$B_r(t) := ((1-t)A + tB)^{r/2},\ t \in [0,1]$$

is *square modulus convex* on $[0,1]$ for $1 \leq r \leq 2$ or $-1 \leq r \leq 0$.

Let $B : [a,b] \to \mathbb{C}$ be defined by $B(t) := x(t) + y(t)i,\ t \in [a,b]$. Observe that $|B(t)|^2 = x^2(t) + y^2(t),\ t \in [a,b]$. Now, if x^2, y^2 are convex on $[a,b]$, then obviously that $x^2 + y^2$ is convex on $[a,b]$. However, if we take $x(t) = t\sin t,\ y(t) = t\cos t$,

$t \in [0, 2\pi]$, then neither x^2 nor y^2 is convex on $[0, 2\pi]$ but $x^2 + y^2$ is convex on $[0, 2\pi]$.

Proposition 1. *If the continuous function $B : [a, b] \to \mathcal{B}(H)$ is square modulus concave on $[a, b]$ then for $p \in (0, 1)$, $|B(\cdot)|^p$ is also square modulus concave on $[a, b]$.*

Proof. By the operator monotonicity and operator concavity of the function $h(t) = t^p$ for $p \in (0, 1)$, we have

$$\begin{aligned} |B((1-t)u + tv)|^{2p} &\geq \left((1-t)|B(u)|^2 + t|B(v)|^2\right)^p \\ &\geq (1-t)|B(u)|^{2p} + t|B(v)|^{2p} \end{aligned}$$

for $t \in [0, 1]$, which shows that $|B(\cdot)|^p$ is also square modulus concave on $[a, b]$. □

For A, $B > 0$, the function

$$B_q(t) := ((1-t)A + tB)^{q/2}, \ t \in [0, 1]$$

is *square modulus concave* on $[0, 1]$ for $q \in (0, 1)$.

Indeed, we have for t_1, $t_2 \in [0, 1]$ and $\alpha \in [0, 1]$ that

$$\begin{aligned} |B_q((1-\alpha)t_1 + \alpha t_2)|^2 &= ((1-(1-\alpha)t_1 - +\alpha t_2)A + ((1-\alpha)t_1 + \alpha t_2)B)^q \\ &= [(1-\alpha)((1-t_1)A + t_1 B) + \alpha((1-t_2)A + t_2 B)]^q \\ &\geq (1-\alpha)((1-t_1)A + t_1 B)^r + \alpha((1-t_2)A + t_2 B)^q \\ &= (1-\alpha)|B_q(t_1)|^2 + \alpha |B_q(t_2)|^2, \end{aligned}$$

which shows that B_q is *square modulus concave* on $[0, 1]$.

We have the following Hermite-Hadamard type inequalities:

Theorem 1. *Assume that the continuous function $B : [a, b] \to \mathcal{B}(H)$ is square modulus convex (concave) on $[a, b]$. Then for all u, $v \in [a, b]$ we have*

$$\begin{aligned} \left|B\left(\frac{u+v}{2}\right)\right|^2 &\leq (\geq) \int_0^1 |B((1-t)u + tv)|^2 \, dt \\ &\leq (\geq) \frac{1}{2}\left[|B(u)|^2 + |B(v)|^2\right]. \end{aligned} \tag{2.2}$$

Proof. The proof goes along the classical one, namely, if we integrate the inequality (1.5) over $t \in [0, 1]$, then we get

$$\begin{aligned} \int_0^1 |B((1-t)u + tv)|^2 \, dt &\leq \int_0^1 \left[(1-t)|B(u)|^2 + t|B(v)|^2\right] dt \\ &= \frac{|B(u)|^2 + |B(v)|^2}{2}, \end{aligned}$$

which proves the right side of (2.2).

From (1.5) we have for all x, $y \in [a, b]$

$$\left|B\left(\frac{x+y}{2}\right)\right|^2 \leq \frac{|B(x)|^2 + |B(y)|^2}{2}. \tag{2.3}$$

If we take in (2.3) $x = (1-t)u + tv$ and $y = tu + (1-t)v$, then we get

$$\left|B\left(\frac{u+v}{2}\right)\right|^2 \leq \frac{1}{2}\left[|B((1-t)u + tv)|^2 + |B(tu + (1-t)v)|^2\right] \tag{2.4}$$

for all $t \in [0,1]$.

If we take the integral in (2.4), then we get

$$\begin{aligned}\left|B\left(\frac{u+v}{2}\right)\right|^2 &\le \frac{1}{2}\left[\int_0^1 |B((1-t)u+tv)|^2 dt + \int_0^1 |B(tu+(1-t)v)|^2 dt\right] \\ &= \frac{1}{2}\left[\int_0^1 |B((1-t)u+tv)|^2 dt + \int_0^1 |B((1-t)u+tv)|^2 dt\right] \\ &= \int_0^1 |B((1-t)u+tv)|^2 dt,\end{aligned}$$

which proves the first inequality in (2.2). □

Remark 1. *For $u \neq v$, u, $v \in [a,b]$, by changing the variable $s = (1-t)u + tv$, $t \in [0,1]$, we get $ds = (v-u)\,dt$ and*

$$\int_0^1 |B((1-t)u+tv)|^2 dt = \frac{1}{v-u}\int_u^v |B(s)|^2 ds,$$

and by (2.2)

$$\left|B\left(\frac{u+v}{2}\right)\right|^2 \le (\ge) \frac{1}{v-u}\int_u^v |B(s)|^2 ds \le (\ge) \frac{1}{2}\left[|B(u)|^2 + |B(v)|^2\right], \tag{2.5}$$

for $u \neq v$, u, $v \in [a,b]$.

In fact, the following refinement of (2.2) holds:

Theorem 2. *Assume that the continuous function $B : [a,b] \to \mathcal{B}(H)$ is square modulus convex (concave) on $[a,b]$. Then for all u, $v \in [a,b]$ and $\lambda \in [0,1]$, we have*

$$\begin{aligned}&\left|B\left(\frac{u+v}{2}\right)\right|^2 \\ &\le (\ge)(1-\lambda)\left|B\left[\frac{(1-\lambda)u+(1+\lambda)v}{2}\right]\right|^2 + \lambda\left|B\left[\frac{(2-\lambda)u+\lambda v}{2}\right]\right|^2 \\ &\le (\ge)\int_0^1 |B[(1-t)u+tv]|^2 dt \\ &\le (\ge)\frac{1}{2}\left\{(1-\lambda)|B(v)|^2 + \lambda|B(u)|^2 + |B[(1-\lambda)u+\lambda v]|^2\right\} \\ &\le \frac{1}{2}(\ge)\left[|B(u)|^2 + |B(v)|^2\right].\end{aligned} \tag{2.6}$$

In particular,

$$\begin{aligned}\left|B\left(\frac{u+v}{2}\right)\right|^2 &\le (\ge)\frac{1}{2}\left\{\left|B\left(\frac{u+3v}{4}\right)\right|^2 + \left|B\left(\frac{3u+v}{2}\right)\right|^2\right\} \\ &\le (\ge)\int_0^1 |B[(1-t)u+tv]|^2 dt \\ &\le (\ge)\frac{1}{2}\left\{\frac{|B(v)|^2+|B(u)|^2}{2} + \left|B\left(\frac{u+v}{2}\right)\right|^2\right\} \\ &\le \frac{1}{2}(\ge)\left[|B(u)|^2 + |B(v)|^2\right].\end{aligned} \tag{2.7}$$

Proof. Let $\lambda \in (0,1)$. Observe that

$$\int_0^1 |B[(1-t)(\lambda v+(1-\lambda)u)+tv]|^2\,dt$$
$$=\int_0^1 |B[((1-t)\lambda+t)v+(1-t)(1-\lambda)u]|^2\,dt$$

and

$$\int_0^1 |B[t(\lambda v+(1-\lambda)u)+(1-t)u]|^2\,dt=\int_0^1 |B[t\lambda v+(1-\lambda t)u]|^2\,dt.$$

If we make the change of variable $\tau := (1-t)\lambda+t$ then we have $1-\tau = (1-t)(1-\lambda)$ and $d\tau = (1-\lambda)\,dt$. Then

$$\int_0^1 |B[((1-t)\lambda+t)v+(1-t)(1-\lambda)u]|^2\,dt$$
$$=\frac{1}{1-\lambda}\int_\lambda^1 |B[\tau v+(1-\tau)u]|^2\,d\tau.$$

If we make the change of variable $\tau := \lambda t$ then we have $d\tau = \lambda dt$ and

$$\int_0^1 |B[t\lambda v+(1-\lambda t)u]|^2\,dt=\frac{1}{\lambda}\int_0^\lambda |B[\tau v+(1-\tau)u]|^2\,d\tau.$$

Therefore

$$(1-\lambda)\int_0^1 |B[(1-t)(\lambda v+(1-\lambda)u)+tv]|^2\,dt$$
$$+\lambda\int_0^1 |B[t(\lambda v+(1-\lambda)u)+(1-t)u]|^2\,dt$$
$$=\int_\lambda^1 |B[\tau v+(1-\tau)u]\,d\tau|^2+\int_0^\lambda |B[\tau v+(1-\tau)u]|^2\,d\tau$$
$$=\int_0^1 |B[\tau v+(1-\tau)u]|^2\,d\tau$$

and then for any $\lambda \in [0,1]$ we have the representation

$$\int_0^1 |B[(1-t)u+tv]|^2\,dt \tag{2.8}$$
$$=(1-\lambda)\int_0^1 |B[(1-t)((1-\lambda)u+\lambda v)+tv]|^2\,dt$$
$$+\lambda\int_0^1 |B[(1-t)u+t((1-\lambda)u+\lambda v)]|^2\,dt.$$

Using (2.2) we get

$$\left|B\left[\frac{(1-\lambda)u+(1+\lambda)v}{2}\right]\right|^2 \le \int_0^1 |B[(1-t)((1-\lambda)u+\lambda v)+tv]|^2\,dt \tag{2.9}$$
$$\le \frac{1}{2}\left[|B[(1-\lambda)u+\lambda v]|^2+|B(v)|^2\right]$$

and

$$\left|B\left[\frac{(2-\lambda)u+\lambda v}{2}\right]\right|^2 \leq \int_0^1 |B[(1-t)u+t((1-\lambda)u+\lambda v)]|^2\,dt \leq \frac{1}{2}\left[|B(u)|^2+|B[(1-\lambda)u+\lambda v]|^2\right]. \tag{2.10}$$

Then by (2.8)-(2.10) we get

$$\begin{aligned}
&(1-\lambda)\left|B\left[\frac{(1-\lambda)u+(1+\lambda)v}{2}\right]\right|^2+\lambda\left|B\left[\frac{(2-\lambda)u+\lambda v}{2}\right]\right|^2\\
&\leq \int_0^1 |B[(1-t)u+tv]|^2\,dt\\
&\leq \frac{1}{2}(1-\lambda)\left[|B[(1-\lambda)u+\lambda v]|^2+|B(v)|^2\right]\\
&+\frac{1}{2}\lambda\left[|B(u)|^2+|B[(1-\lambda)u+\lambda v]|^2\right]\\
&=\frac{1}{2}(1-\lambda)|B(v)|^2+\frac{1}{2}\lambda|B(u)|^2+\frac{1}{2}|B[(1-\lambda)u+\lambda v]|^2.
\end{aligned}$$

By the square modulus convexity of $B:[u,v]\to\mathcal{B}(H)$, we get

$$\begin{aligned}
&(1-\lambda)\left|B\left[\frac{(1-\lambda)u+(1+\lambda)v}{2}\right]\right|^2+\lambda\left|B\left[\frac{(2-\lambda)u+\lambda v}{2}\right]\right|^2\\
&\geq \left|B\left[(1-\lambda)\frac{(1-\lambda)u+(1+\lambda)v}{2}+\lambda\frac{(2-\lambda)u+\lambda v}{2}\right]\right|^2\\
&=\left|B\left[\frac{(1-\lambda)^2u+\left(1-\lambda^2\right)v}{2}+\frac{(2-\lambda)\lambda u+\lambda^2 v}{2}\right]\right|^2\\
&=\left|B\left(\frac{u+v}{2}\right)\right|^2
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}(1-\lambda)|B(v)|^2+\frac{1}{2}\lambda|B(u)|^2+\frac{1}{2}|B[(1-\lambda)u+\lambda v]|^2\\
&\leq \frac{1}{2}(1-\lambda)|B(v)|^2+\frac{1}{2}\lambda|B(u)|^2+\frac{1}{2}\left[(1-\lambda)|B(u)|^2+\lambda|B(v)|^2\right]\\
&=\frac{1}{2}\left[|B(u)|^2+|B(v)|^2\right],
\end{aligned}$$

which proves the desired result (2.6). □

Remark 2. *A scalar version of inequality (2.6) was obtained by Barnett, Cerone and Dragomir in* [2].

3. Gradient Inequalities

Following Roberts and Varberg [15, p. 5], we recall that if $f:I\to\mathbb{R}$ is a convex function, then for any $s_0\in\mathring{I}$ (the interior of the interval I) the limits

$$f'_-(s_0):=\lim_{s\to s_0-}\frac{f(s)-f(s_0)}{s-s_0} \text{ and } f'_+(s_0):=\lim_{s\to s_0+}\frac{f(s)-f(s_0)}{s-s_0}$$

exists and $f'_{-}(s_0) \le f'_{+}(s_0)$. The functions f'_{-} and f'_{+} are monotonic nondecreasing on $\mathring{I}$ and this property can be extended to the whole interval I (see [15, p. 7]).

From the monotonicity of the lateral derivatives f'_{-} and f'_{+} we also have *the gradient inequality*

$$f'_{-}(s)(s-\tau) \ge f(s) - f(\tau) \ge f'_{+}(\tau)(s-\tau)$$

for any $s,\ \tau \in \mathring{I}$.

If $I = [a,b]$, then at the end points we also have the inequalities

$$f(s) - f(a) \ge f'_{+}(a)(s-a)$$

for any $s \in (a,b]$ and

$$f(\tau) - f(b) \ge f'_{-}(b)(\tau - b)$$

for any $\tau \in [a,b)$.

For the operator $T \in \mathcal{B}(H)$ we define the selfadjoint operator

$$\operatorname{Re}(T) = \frac{1}{2}(T^* + T).$$

Assume that function $B : [a,b] \to \mathcal{B}(H)$ is continuous on $[a,b]$. The function $B : [a,b] \to \mathcal{B}(H)$ is square modulus convex on $[a,b]$ if and only if for all $x \in H \setminus \{0\}$ the auxiliary function $\varphi_{B,x} : [a,b] \to [0,\infty)$, $\varphi_{B,x}(u) = \|B(u)x\|^2$ is convex on $[a,b]$.

Indeed, condition (1.5) is equivalent to

$$\left\langle |B((1-t)u+tv)|^2 x, x\right\rangle \le (1-t)\left\langle |B(u)|^2 x, x\right\rangle + t\left\langle |B(v)|^2 x, x\right\rangle,$$

namely

$$\begin{aligned}&\left\langle [B((1-t)u+tv)]^* B((1-t)u+tv)x, x\right\rangle \\ &\le (1-t)\left\langle [B(u)]^* B(u)x, x\right\rangle + t\left\langle [B(v)]^* B(v)x, x\right\rangle,\end{aligned}$$

or

$$\|B((1-t)u+tv)x\|^2 \le (1-t)\|B(u)x\|^2 + t\|B(v)x\|^2$$

for all $t \in [0,1]$ and $u,\ v \in [a,b]$.

We also have

$$\begin{aligned}\varphi'_{\pm B,x}(u) &= (\langle B(u)x, B(u)x\rangle)'_{\pm} = \left\langle B'_{\pm}(u)x, B(u)x\right\rangle + \left\langle B(u)x, B'_{\pm}(u)x\right\rangle \\ &= \left\langle (B(u))^* B'_{\pm}(u)x, x\right\rangle + \left\langle \left(B'_{\pm}(u)\right)^* B(u)x, x\right\rangle \\ &= \left\langle (B(u))^* B'_{\pm}(u)x, x\right\rangle + \left\langle \left((B(u))^* B'_{\pm}(u)\right)^* x, x\right\rangle \\ &= \left\langle 2\operatorname{Re}\left((B(u))^* B'_{\pm}(u)\right)x, x\right\rangle\end{aligned}$$

and

$$\begin{aligned}\varphi'_{+B,x}(a) &= 2\left\langle \operatorname{Re}\left((B(a))^* B'_{+}(a)\right)x, x\right\rangle, \\ \varphi'_{-B,x}(b) &= 2\left\langle \operatorname{Re}\left((B(b))^* B'_{-}(b)\right)x, x\right\rangle\end{aligned}$$

for all $x \in H \setminus \{0\}$.

We have for $t\in(a,b)$ and small $h\neq 0$ such that $t+h\in(a,b)$,

$$\left\langle \frac{|B(t+h)|^2-|B(t)|^2}{h}x,x\right\rangle = \frac{1}{h}\left[\left\langle |B(t+h)|^2 x,x\right\rangle-\left\langle |B(t)|^2 x,x\right\rangle\right] = \frac{1}{h}\left[\|B(t+h)x\|^2-\|B(t)x\|^2\right]$$

for all $x\in H\setminus\{0\}$.

By taking the lateral limits, we get

$$\lim_{h\to\pm 0}\left\langle \frac{|B(t+h)|^2-|B(t)|^2}{h}x,x\right\rangle = \lim_{h\to\pm 0}\frac{1}{h}\left[\|B(t+h)x\|^2-\|B(t)x\|^2\right] = \left\langle 2\operatorname{Re}\left((B(t))^* B'_{\pm}(t)\right)x,x\right\rangle$$

for all $x\in H\setminus\{0\}$.

Therefore for the function $\varphi(t):=|B(t)|^2$, $t\in(a,b)$ we have

$$\varphi'_{\pm}(t)=2\operatorname{Re}\left((B(t))^* B'_{\pm}(t)\right)$$

and

$$\varphi'_{+}(a)=2\operatorname{Re}\left((B(a))^* B_{+}(a)\right),\ \varphi'_{-}(b)=2\operatorname{Re}\left((B(b))^* B'_{-}(b)\right).$$

We also have *the operator gradient inequality*

$$2(s-\tau)\operatorname{Re}\left((B(s))^* B'_{-}(s)\right)\geq |B(s)|^2-|B(\tau)|^2 \geq 2(s-\tau)\operatorname{Re}\left((B(\tau))^* B'_{+}(\tau)\right) \tag{3.1}$$

for any s, $\tau\in(a,b)$.

Moreover, at the end points of the interval we have the operator inequalities

$$|B(s)|^2-|B(a)|^2\geq 2(s-a)\operatorname{Re}\left((B(a))^* B'_{+}(a)\right) \tag{3.2}$$

for any $s\in(a,b]$ and

$$2(b-\tau)\operatorname{Re}\left((B(b))^* B'_{-}(b)\right)\geq |B(b)|^2-|B(\tau)|^2 \tag{3.3}$$

for any $\tau\in[a,b)$.

Finally, we notice that for $a\leq t_1<t_2\leq b$, we have

$$\operatorname{Re}\left((B(a))^* B'_{+}(a)\right)\leq \operatorname{Re}\left((B(t_1))^* B'_{\pm}(t_1)\right) \leq \operatorname{Re}\left((B(t_2))^* B'_{\pm}(t_2)\right)\leq \operatorname{Re}\left((B(b))^* B'_{-}(b)\right) \tag{3.4}$$

in the operator order of $\mathcal{B}(H)$.

Recall the following Ostrowski's type scalar inequality [4]:

Lemma 2. *Let $h:[\alpha,\beta]\subset\mathbb{R}\to\mathbb{R}$ be a convex function on $[\alpha,\beta]$. Then for any $\gamma\in(\alpha,\beta)$ one has the inequality*

$$\frac{1}{2}\left[(\beta-\gamma)^2 h'_{+}(\gamma)-(\gamma-\alpha)^2 h'_{-}(\gamma)\right] \leq \int_\alpha^\beta h(t)\,dt-(\beta-\alpha)h(\gamma) \leq \frac{1}{2}\left[(\beta-\gamma)^2 h'_{-}(\beta)-(\gamma-\alpha)^2 h'_{+}(\alpha)\right]. \tag{3.5}$$

The constant $\frac{1}{2}$ *is sharp in both inequalities.*
The second inequality also holds for $\gamma = \alpha$ *or* $\gamma = \beta$*. In particular, we have for* $\gamma = \frac{\alpha+\beta}{2}$ *that*

$$0 \le \frac{1}{8}(\beta-\alpha)^2\left[h'_+(\gamma) - h'_-(\gamma)\right] \le \int_\alpha^\beta h(t)\,dt - (\beta-\alpha)\,h\left(\frac{\alpha+\beta}{2}\right) \le \frac{1}{8}(\beta-\alpha)^2\left[h'_-(\beta) - h'_+(\alpha)\right], \tag{3.6}$$

with $\frac{1}{8}$ *best possible in both inequalities.*

We have the following result:

Theorem 3. *Assume that function* $B : [a,b] \to \mathcal{B}(H)$ *is continuous on* $[a,b]$ *and square modulus convex on* $[a,b]$*. Then for* $s \in (a,b)$ *we have*

$$\begin{aligned} &\operatorname{Re}\left\{(B(s))^*\left[(b-s)^2 B'_+(s) - (s-a)^2 B'_-(s)\right]\right\} \\ &\le \int_a^b |B(t)|^2\,dt - (b-a)\,|B(s)|^2 \\ &\le (b-s)^2 \operatorname{Re}\left((B(b))^* B'_-(b)\right) - (s-a)^2 \operatorname{Re}\left((B(a))^* B'_-(a)\right) \end{aligned} \tag{3.7}$$

In particular, we have

$$\begin{aligned} 0 &\le \frac{1}{4}(b-a)^2 \operatorname{Re}\left\{\left(B\left(\frac{a+b}{2}\right)\right)^*\left[B'_+\left(\frac{a+b}{2}\right) - B'_-\left(\frac{a+b}{2}\right)\right]\right\} \\ &\le \int_a^b |B(t)|^2\,dt - (b-a)\left|B\left(\frac{a+b}{2}\right)\right|^2 \\ &\le \frac{1}{4}(b-a)^2\left[\operatorname{Re}\left((B(b))^* B'_-(b)\right) - \operatorname{Re}\left((B(a))^* B'_-(a)\right)\right]. \end{aligned} \tag{3.8}$$

Proof. Let $x \in H \setminus \{0\}$ and apply Lemma 2 for the function $h = \varphi_{B,x}$ on the interval $[a,b]$ and for $\gamma = s \in (a,b)$ to get

$$\begin{aligned} &\frac{1}{2}\left[(b-s)^2\left\langle 2\operatorname{Re}\left((B(s))^* B'_+(s)\right)x, x\right\rangle - (s-a)^2\left\langle 2\operatorname{Re}\left((B(s))^* B'_-(s)\right)x, x\right\rangle\right] \\ &\le \int_a^b \|B(t)\,x\|^2\,dt - (b-a)\,\|B(s)\,x\|^2 \\ &\le \frac{1}{2}\left[(b-s)^2\left\langle 2\operatorname{Re}\left((B(s))^* B'_-(b)\right)x, x\right\rangle - (s-a)^2\left\langle 2\operatorname{Re}\left((B(a))^* B'_-(a)\right)x, x\right\rangle\right], \end{aligned}$$

namely

$$\begin{aligned} &\left\langle\left[(b-s)^2 \operatorname{Re}\left((B(s))^* B'_+(s)\right) - (s-a)^2 \operatorname{Re}\left((B(s))^* B'_-(s)\right)\right]x, x\right\rangle \\ &\le \left\langle\left[\left(\int_a^b |B(t)|^2\,dt\right) - (b-a)\,|B(s)|^2\right]x, x\right\rangle \\ &\le \left\langle\left[(b-s)^2 \operatorname{Re}\left((B(b))^* B'_-(b)\right) - (s-a)^2 \operatorname{Re}\left((B(a))^* B'_-(a)\right)\right]x, x\right\rangle, \end{aligned}$$

which is equivalent to (3.7). □

Recall the following trapezoid type inequality [5]:

Lemma 3. *Let $h : [\alpha, \beta] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on $[\alpha, \beta]$. Then for any $\gamma \in (\alpha, \beta)$ one has the inequality*

$$\begin{aligned} &\frac{1}{2}\left[(\beta-\gamma)^2 h'_+(\gamma) - (\gamma-\alpha)^2 h'_-(\gamma)\right] \\ &\le (\gamma-\alpha) h(\alpha) + (\beta-\gamma) h(\beta) - \int_\alpha^\beta h(t)\,dt \\ &\le \frac{1}{2}\left[(\beta-\gamma)^2 h'_-(\beta) - (\gamma-\alpha)^2 h'_+(\alpha)\right]. \end{aligned} \tag{3.9}$$

The constant $\frac{1}{2}$ is sharp in both inequalities.
The second inequality also holds for $\gamma = \alpha$ or $\gamma = \beta$. In particular, we have for $\gamma = \frac{\alpha+\beta}{2}$ that

$$\begin{aligned} 0 &\le \frac{1}{8}(\beta-\alpha)^2\left[h'_+(\gamma) - h'_-(\gamma)\right] \le (\beta-\alpha)\frac{h(\alpha)+h(\beta)}{2} - \int_\alpha^\beta h(t)\,dt \\ &\le \frac{1}{8}(\beta-\alpha)^2\left[h'_-(\beta) - h'_+(\alpha)\right], \end{aligned} \tag{3.10}$$

with $\frac{1}{8}$ best possible in both inequalities.

We can state the following result as well:

Theorem 4. *Assume that function $B : [a, b] \to \mathcal{B}(H)$ is continuous on $[a, b]$ and square modulus convex on $[a, b]$. Then for $s \in (a, b)$ we have*

$$\begin{aligned} &\operatorname{Re}\left\{(B(s))^*\left[(b-s)^2 B'_+(s) - (s-a)^2 B'_-(s)\right]\right\} \\ &\le (b-s)|B(b)|^2 + (s-a)|B(a)|^2 - \int_a^b |B(t)|^2\,dt \\ &\le (b-s)^2 \operatorname{Re}\left((B(b))^* B'_-(b)\right) - (s-a)^2 \operatorname{Re}\left((B(a))^* B'_-(a)\right). \end{aligned} \tag{3.11}$$

In particular,

$$\begin{aligned} 0 &\le \frac{1}{4}(b-a)^2 \operatorname{Re}\left\{\left(B\left(\frac{a+b}{2}\right)\right)^*\left[B'_+\left(\frac{a+b}{2}\right) - B'_-\left(\frac{a+b}{2}\right)\right]\right\} \\ &\le (b-a)\frac{|B(a)|^2 + |B(b)|^2}{2} - \int_a^b |B(t)|^2\,dt \\ &\le \frac{1}{4}(b-a)^2\left[\operatorname{Re}\left((B(b))^* B'_-(b)\right) - \operatorname{Re}\left((B(a))^* B'_-(a)\right)\right]. \end{aligned} \tag{3.12}$$

4. Example for Power Function

We have the following integral representation for the power function when $t > 0$, $r \in (0, 1]$, see for instance [1, p. 145]

$$t^{r-1} = \frac{\sin(r\pi)}{\pi}\int_0^\infty \frac{\lambda^{r-1}}{\lambda+t}\,d\lambda. \tag{4.1}$$

Motivated by these representation, we introduced in , for a continuous and positive function $w(\lambda)$, $\lambda > 0$, the following *integral transform*

$$\mathcal{D}(w,\mu)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda+t}\,d\mu(\lambda),\ t > 0, \tag{4.2}$$

where μ is a positive measure on $(0, \infty)$ and the integral (4.2) exists for all $t > 0$.

For μ the Lebesgue usual measure, we put

$$\mathcal{D}(w)(t) := \int_0^\infty \frac{w(\lambda)}{\lambda + t} d\lambda,\ t > 0. \tag{4.3}$$

If we take μ to be the usual Lebesgue measure and the kernel $w_r(\lambda) = \lambda^{r-1}$, $r \in (0, 1]$, then

$$t^{r-1} = \frac{\sin(r\pi)}{\pi} \mathcal{D}(w_r)(t),\ t > 0. \tag{4.4}$$

For a continuous and positive function $w(\lambda)$, $\lambda > 0$ and a positive measure μ on $(0, \infty)$, we can define the following mapping, which we call *monotonic integral transform,* by

$$\mathcal{M}(w, \mu)(t) := t\mathcal{D}(w, \mu)(t),\ t > 0. \tag{4.5}$$

For all $T > 0$ we have, by the continuous functional calculus for selfadjoint operators, that

$$\mathcal{M}(w, \mu)(T) = T\mathcal{D}(w, \mu)(T) = \int_0^\infty w(\lambda) \left[1 - \lambda (T + \lambda)^{-1}\right] d\mu(\lambda). \tag{4.6}$$

This gives the representation

$$T^r = \frac{\sin(r\pi)}{\pi} \mathcal{M}(w_r, \mu)(T), \tag{4.7}$$

for $T > 0$.

We have the following representation of the Fréchet derivative $D(\mathcal{M}(w, \mu))$:

Lemma 4. *For all $A > 0$,*

$$D(\mathcal{M}(w, \mu))(A)(V) = \int_0^\infty \lambda w(\lambda) (\lambda + A)^{-1} V (\lambda + A)^{-1} d\mu(\lambda) \tag{4.8}$$

for all $V \in S(H)$, the class of all selfadjoint operators on H.

Proof. The proof follows directly from the fact that the Fréchet derivative of the map $\operatorname{Inv}(A) = A^{-1}$ is

$$D(\operatorname{Inv})(A)(V) = -A^{-1} V A^{-1}$$

for all $A > 0$ and $V \in S(H)$. □

For a continuous function f on $(0, \infty)$ and A, $B > 0$ we consider the auxiliary function $f_{A,B} : [0, 1] \to \mathbb{R}$ defined by

$$f_{A,B}(t) := f((1 - t) A + tB),\ t \in [0, 1].$$

We have the following representations of the derivatives:

Lemma 5. *Assume that the operator function generated by f is Fréchet differentiable in each $A > 0$, then for $B > 0$ we have that $f_{A,B}$ is differentiable on $[0, 1]$,*

$$\frac{df_{A,B}(t)}{dt} = D(f)((1 - t) A + tB)(B - A) \tag{4.9}$$

for $t \in [0, 1]$, where in 0 and 1 the derivatives are the right and left derivatives.

Proof. We prove only for the interior points $t\in(0,1)$. Let h be in a small interval around 0 such that $t+h\in(0,1)$. Then for $h\neq 0$,

$$\begin{aligned}&\frac{f_{A,B}(t+h)-f(t)}{h}\\&=\frac{f((1-(t+h))A+(t+h)B)-f((1-t)A+tB)}{h}\\&=\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\end{aligned}$$

and by taking the limit over $h\to 0$, we get

$$\begin{aligned}\frac{df_{A,B}(t)}{dt}&=\lim_{h\to 0}\frac{f_{A,B}(t+h)-f(t)}{h}\\&=\lim_{h\to 0}\left[\frac{f((1-t)A+tB+h(B-A))-f((1-t)A+tB)}{h}\right]\\&=D(f)((1-t)A+tB)(B-A),\end{aligned}$$

which proves (4.9). □

For the transform $\mathcal{M}(w,\mu)(t)$ defined above, we consider the auxiliary function

$$\mathcal{M}(w,\mu)_{A,B}(t):=\mathcal{M}(w,\mu)((1-t)A+tB)$$

where $A,\ B>0$ and $t\in[0,1]$.

Corollary 1. *For all $A,\ B>0$ and $t\in[0,1]$,*

$$\begin{aligned}\frac{d\mathcal{M}(w,\mu)_{A,B}(t)}{dt}&=D(\mathcal{M}(w,\mu))((1-t)A+tB)(B-A)\\&=\int_0^\infty\lambda w(\lambda)(\lambda+(1-t)A+tB)^{-1}(B-A)\\&\times(\lambda+(1-t)A+tB)^{-1}d\mu(\lambda).\end{aligned}\tag{4.10}$$

By utilising (4.7) and (4.10) we derive for $A,\ B>0$ and $r\in(0,1)$ that

$$\begin{aligned}&\ell'_{r,A,B}(t)\\&=\frac{\sin(r\pi)}{\pi}\\&\times\int_0^\infty\lambda^r(\lambda+(1-t)A+tB)^{-1}(B-A)(\lambda+(1-t)A+tB)^{-1}d\lambda,\end{aligned}$$

where $\ell_{r,A,B}(t):=((1-t)A+tB)^r$. This equality holds for $t\in[0,1]$, where in 0 and 1 the derivatives are the right and left derivatives, namely

$$\ell'_{r,A,B+}(0)=\frac{\sin(r\pi)}{\pi}\int_0^\infty\lambda^r(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1}d\lambda$$

and

$$\ell'_{r,A,B-}(1)=\frac{\sin(r\pi)}{\pi}\int_0^\infty\lambda^r(\lambda+B)^{-1}(B-A)(\lambda+B)^{-1}d\lambda.$$

Since the function

$$\ell_{r/2,A,B}(t)=((1-t)A+tB)^{r/2}$$

is *square modulus concave* on $[0,1]$, then we get the following reverse Hermite-Hadamard inequalities

$$0 \le \left(\frac{A+B}{2}\right)^r - \int_0^1 ((1-t)A+tB)^r \, dt \le \frac{1}{8}\frac{\sin(r\pi)}{\pi} \times \int_0^\infty \lambda^r \left[(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} - (\lambda+B)^{-1}(B-A)(\lambda+B)^{-1}\right] d\lambda \tag{4.11}$$

and

$$0 \le \int_0^1 ((1-t)A+tB)^r \, dt - \frac{A^r+B^r}{2} \le \frac{1}{8}\frac{\sin(r\pi)}{\pi} \times \int_0^\infty \lambda^r \left[(\lambda+A)^{-1}(B-A)(\lambda+A)^{-1} - (\lambda+B)^{-1}(B-A)(\lambda+B)^{-1}\right] d\lambda \tag{4.12}$$

for $A, B > 0$ and $r \in (0,1)$.

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