SOME INEQUALITIES RELATED TO VRÂNCEANU'S EXTENSION OF OPIAL'S RESULT

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Abstract. In this paper we establish some refinements and generalizations of Vrânceanu's extension in Hilbert spaces of Opial's inequality. Examples for two positive weights are also given.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u'\in L_2[a,b]$.*

(i) *If $u(a)=u(b)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt \le \frac{1}{4}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.1}$$

with equality if and only if

$$u(t)=\begin{cases} c(t-a) & \text{if } a\le t\le \frac{a+b}{2},\\ c(b-t) & \text{if } \frac{a+b}{2}<t\le b,\end{cases}$$

where c is an arbitrary constant;

(ii) *If $u(a)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt \le \frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ for some constant c.

The inequality (1.1) was obtained by Olech in [7] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [8].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [3], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [4]-[6] and [9]. For some recent result related to Opial's inequality see [1], [2], [10] and [11].

In 1975, G. G. Vrânceanu extended Opial's inequality (1.2) for functions with values in Hilbert spaces $(H;\langle\cdot,\cdot\rangle)$ as follows:

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Theorem 2. *Assume that the function $f:[a,b]\to H$ has a continuous derivative and $f(a)=0$, then*

$$\int_a^b |\langle f(t), f'(t)\rangle|\,dt \le \frac{1}{2}(b-a)\int_a^b \|f'(t)\|^2\,dt. \tag{1.3}$$

In this paper we establish some refinements and generalizations of Vrânceanu's extension in Hilbert spaces of Opial's inequality. Examples for two positive weights are also given.

2. Main Results

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. We have the following refinement and generalization for vector valued functions of the Opial inequalities:

Theorem 3. *Assume that $f:[a,b]\to H$ is absolutely continuous on $[a,b]$ and $f'\in L_2([a,b],H)$.*

(i) *If either $f(a)=0$ or $f(b)=0$, then*

$$\begin{aligned}\int_a^b |\langle f'(t), f(t)\rangle|\,dt &\le \left(\int_a^b (t-a)\|f'(t)\|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)\|f'(t)\|^2\,dt\right)^{1/2}\\ &\le \frac{1}{2}(b-a)\int_a^b \|f'(t)\|^2\,dt.\end{aligned} \tag{2.1}$$

(ii) *If $f(a)=f(b)=0$, then*

$$\begin{aligned}\int_a^b |\langle f'(t), f(t)\rangle|\,dt &\le \left[\int_a^b K(t)\|f'(t)\|^2\,dt\right]^{1/2}\left[\int_a^b \left|\frac{a+b}{2}-t\right|\|f'(t)\|^2\,dt\right]^{1/2}\\ &\le \frac{1}{4}(b-a)\int_a^b \|f'(t)\|^2\,dt,\end{aligned} \tag{2.2}$$

where

$$K(t) := \begin{cases} t-a & \text{if } a\le t\le \frac{a+b}{2},\\ b-t & \text{if } \frac{a+b}{2}<t\le b.\end{cases}$$

Proof. (i). Since $f(a)=0$, then $f(t)=\int_a^t f'(s)\,ds$ for $t\in[a,b]$. We have by Schwarz inequality for the inner product,

$$\begin{aligned}\int_a^b |\langle f'(t), f(t)\rangle|\,dt &\le \int_a^b \|f'(t)\|\,\|f(t)\|\,dt = \int_a^b (t-a)^{1/2}\|f'(t)\|(t-a)^{-1/2}\|f(t)\|\,dt\\ &= \int_a^b (t-a)^{1/2}\|f'(t)\|(t-a)^{-1/2}\left\|\int_a^t f'(s)\,ds\right\|dt =: A.\end{aligned} \tag{2.3}$$

Using Cauchy-Bunyakovsky-Schwarz (CBS) integral inequality, we have

$$
\begin{aligned}
(2.4)\qquad A &\le \left(\int_a^b \left[(t-a)^{1/2}\left\|f'(t)\right\|\right]^2 dt\right)^{1/2} \\
&\quad\times\left(\int_a^b \left[(t-a)^{-1/2}\left\|\int_a^t f'(s)\,ds\right\|\right]^2 dt\right)^{1/2} \\
&= \left(\int_a^b (t-a)\left\|f'(t)\right\|^2 dt\right)^{1/2}\left(\int_a^b (t-a)^{-1}\left\|\int_a^t f'(s)\,ds\right\|^2 dt\right)^{1/2} \\
&=: B.
\end{aligned}
$$

By (CBS) integral inequality we also have

$$
(t-a)^{-1}\left\|\int_a^t f'(s)\,ds\right\|^2 \le \int_a^t \left\|f'(s)\right\|^2 ds,
$$

which gives

$$
(2.5)\qquad B \le \left(\int_a^b (t-a)\left\|f'(t)\right\|^2 dt\right)^{1/2}\left(\int_a^b \left(\int_a^t \left\|f'(s)\right\|^2 ds\right) dt\right)^{1/2}.
$$

Using integration by parts, we have

$$
\begin{aligned}
\int_a^b \left(\int_a^t \left\|f'(s)\right\|^2 ds\right) dt &= b\int_a^b \left\|f'(s)\right\|^2 ds - \int_a^b t\left\|f'(t)\right\|^2 dt \\
&= \int_a^b (b-t)\left\|f'(t)\right\|^2
\end{aligned}
$$

and by (2.4) we get the first inequality in (2.1).

The last part follows by the elementary inequality

$$
(2.6)\qquad \sqrt{\alpha\beta} \le \frac{1}{2}(\alpha+\beta),\ \ \alpha,\beta \ge 0.
$$

The case $f(b) = 0$ can be proved in a similar way and the details are omitted.

(ii). If we write the inequality (2.1) on the interval $\left[a, \frac{a+b}{2}\right]$, we have

$$
\begin{aligned}
(2.7)\qquad &\int_a^{\frac{a+b}{2}} \left|\left\langle f'(t), f(t)\right\rangle\right| dt \\
&\le \left(\int_a^{\frac{a+b}{2}} (t-a)\left\|f'(t)\right\|^2 dt\right)^{1/2}\left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right)\left\|f'(t)\right\|^2 dt\right)^{1/2}
\end{aligned}
$$

and if we write the inequality (2.1) on the interval $\left[\frac{a+b}{2}, b\right]$, we have

$$
\begin{aligned}
(2.8)\qquad &\int_{\frac{a+b}{2}}^b \left|\left\langle f'(t), f(t)\right\rangle\right| dt \\
&\le \left(\int_{\frac{a+b}{2}}^b (b-t)\left\|f'(t)\right\|^2 dt\right)^{1/2}\left(\int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right)\left\|f'(t)\right\|^2 dt\right)^{1/2}.
\end{aligned}
$$

If we add the inequalities (2.7) and (2.8) we get

$$\begin{aligned}
&\int_a^b |\langle f'(t), f(t)\rangle| \, dt \\
&\leq \left(\int_a^{\frac{a+b}{2}} (t-a) \|f'(t)\|^2 \, dt\right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) \|f'(t)\|^2 \, dt\right)^{1/2} \\
&+ \left(\int_{\frac{a+b}{2}}^b (b-t) \|f'(t)\|^2 \, dt\right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right) \|f'(t)\|^2 \, dt\right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leq \left[\int_a^{\frac{a+b}{2}} (t-a) \|f'(t)\|^2 \, dt + \int_{\frac{a+b}{2}}^b (b-t) \|f'(t)\|^2 \, dt\right]^{1/2} \\
&\times \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) \|f'(t)\|^2 \, dt + \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right) \|f'(t)\|^2 \, dt\right]^{1/2} \\
&= \left[\int_a^b K(t) \|f'(t)\|^2 \, dt\right]^{1/2} \left[\int_a^b \left|\frac{a+b}{2} - t\right| \|f'(t)\|^2 \, dt\right]^{1/2},
\end{aligned}$$

where for the last inequality we used the elementary (CBS) inequality

$$\alpha\beta + \gamma\delta \leq \left(\alpha^2 + \gamma^2\right)^{1/2} \left(\beta^2 + \delta^2\right)^{1/2}, \ \alpha, \ \beta, \ \gamma, \ \delta \geq 0.$$

The last part follows by (2.6), namely

$$\begin{aligned}
&\left[\int_a^b K(t) \|f'(t)\|^2 \, dt\right]^{1/2} \left[\int_a^b \left|\frac{a+b}{2} - t\right| \|f'(t)\|^2 \, dt\right]^{1/2} \\
&\leq \frac{1}{2} \left[\int_a^b K(t) \|f'(t)\|^2 \, dt + \int_a^b \left|\frac{a+b}{2} - t\right| \|f'(t)\|^2 \, dt\right] \\
&= \frac{1}{2} \int_a^b \left[K(t) + \left|\frac{a+b}{2} - t\right|\right] \|f'(t)\|^2 \, dt = \frac{1}{4} \int_a^b \|f'(t)\|^2 \, dt,
\end{aligned}$$

since

$$K(t) + \left|\frac{a+b}{2} - t\right| = \frac{1}{2}(b-a) \text{ for } t \in [a,b].$$

□

Remark 1. *The inequality (2.2) can also be written as*

$$\begin{aligned}\int_a^b |\langle f'(t), f(t)\rangle| \, dt &\le \left[\frac{1}{2}(b-a)\int_a^b \|f'(t)\|^2 \, dt - \int_a^b \left|\frac{a+b}{2} - t\right| \|f'(t)\|^2 \, dt\right]^{1/2} \\ &\times \left[\int_a^b \left|\frac{a+b}{2} - t\right| \|f'(t)\|^2 \, dt\right]^{1/2} \\ &\le \frac{1}{4}(b-a)\int_a^b \|f'(t)\|^2 \, dt. \end{aligned} \tag{2.9}$$

We also have the following composite inequality:

Theorem 4. *Let $h : [a,b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b). Assume that $f : [a,b] \subset \mathbb{R} \to H$ is an absolutely continuous vector valued function on the interval $[a,b]$ and such that $\frac{f'}{[h']^{1/2}} \in L_2([a,b], H)$.*

(i) *If $f(a) = 0$ or $f(b) = 0$, then*

$$\begin{aligned}\int_a^b |\langle f'(t), f(t)\rangle| \, dt &\le \left(\int_a^b [h(t) - h(a)] \frac{\|f'(t)\|^2}{h'(t)} dt\right)^{1/2} \left(\int_a^b [h(b) - h(t)] \frac{\|f'(t)\|^2}{h'(t)} dt\right)^{1/2} \\ &\le \frac{1}{2}[h(b) - h(a)] \int_a^b \frac{\|f'(t)\|^2}{h'(t)} dt. \end{aligned} \tag{2.10}$$

(ii) *If $f(a) = f(b) = 0$, then*

$$\begin{aligned}\int_a^b |\langle f'(t), f(t)\rangle| \, dt \le& \left[\frac{1}{2}[h(b) - h(a)] \int_a^b \frac{\|f'(t)\|^2}{h'(t)} dt \right. \\ &\left. - \int_a^b \left|\frac{h(a) + h(b)}{2} - h(t)\right| \frac{\|f'(t)\|^2}{h'(t)} dt\right]^{1/2} \\ &\times \left[\int_a^b \left|\frac{h(a) + h(b)}{2} - h(t)\right| \frac{\|f'(t)\|^2}{h'(t)} dt\right]^{1/2} \\ &\le \frac{1}{4}[h(b) - h(a)] \int_a^b \frac{\|f'(t)\|^2}{h'(t)} dt. \end{aligned} \tag{2.11}$$

Proof. (i). Consider the function $u := f \circ h^{-1} : [h(a), h(b)] \to \mathbb{R}$. The function u is absolutely continuous on $[h(a), h(b)]$, $u(h(a)) = f \circ h^{-1}(h(a)) = f(a) = 0$ or $u(h(b)) = f \circ h^{-1}(h(b)) = f(b) = 0$.

Using the chain rule and the derivative of inverse functions we have

$$\left(f \circ h^{-1}\right)'(z) = \left(f' \circ h^{-1}\right)(z) \left(h^{-1}\right)'(z) = \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \tag{2.12}$$

for almost every (a.e.) $z \in [h(a), h(b)]$.

If we apply the inequality (2.1) for the function $u = f \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$\begin{aligned}
(2.13)\qquad & \int_{h(a)}^{h(b)} \left| \left\langle f \circ h^{-1}(z), \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\rangle \right| dz \\
& \leq \left(\int_{h(a)}^{h(b)} (z - h(a)) \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz \right)^{1/2} \\
& \times \left(\int_{h(a)}^{h(b)} (h(b) - z) \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz \right)^{1/2} \\
& \leq \frac{1}{2} [h(b) - h(a)] \int_{h(a)}^{h(b)} \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz.
\end{aligned}$$

If we make the change of variable $t = h^{-1}(z)$, $z \in [h(a), h(b)]$, then $z = h(t)$, $dz = h'(t)\,dt$,

$$\begin{aligned}
\int_{h(a)}^{h(b)} \left| \left\langle f \circ h^{-1}(z), \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\rangle \right| dz &= \int_a^b \left| \left\langle f(t), \frac{f'(t)}{h'(t)} \right\rangle \right| h'(t)\,dt \\
&= \int_a^b \left| \left\langle f(t), f'(t) \right\rangle \right| dt,
\end{aligned}$$

$$\begin{aligned}
\int_{h(a)}^{h(b)} (z - h(a)) \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz &= \int_a^b [h(t) - h(a)] \left\| \frac{f'(t)}{h'(t)} \right\|^2 h'(t)\,dt \\
&= \int_a^b [h(t) - h(a)] \frac{\left\| f'(t) \right\|^2}{h'(t)} dt
\end{aligned}$$

$$\begin{aligned}
\int_{h(a)}^{h(b)} (h(b) - z) \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz &= \int_{h(a)}^{h(b)} [h(b) - h(t)] \left\| \frac{f'(t)}{h'(t)} \right\|^2 h'(t)\,dt \\
&= \int_a^b [h(b) - h(t)] \frac{\left\| f'(t) \right\|^2}{h'(t)} dt
\end{aligned}$$

and

$$\int_{h(a)}^{h(b)} \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz = \int_a^b \left\| \frac{f'(t)}{h'(t)} \right\|^2 h'(t)\,dt = \int_a^b \frac{\left\| f'(t) \right\|^2}{h'(t)} dt.$$

By utilising (2.13), we then get the desired inequality (2.10).

(ii). By using the inequality (2.2) for the function $u = f \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$\int_{h(a)}^{h(b)} \left| \left\langle f \circ h^{-1}(z), \frac{(f' \circ h^{-1})(z)}{(h' \circ h^{-1})(z)} \right\rangle \right| dz \tag{2.14}$$
$$\leq \left[\int_{h(a)}^{h(b)} \left(\frac{1}{2}(h(b) - h(a)) - \left| \frac{h(a)+h(b)}{2} - z \right| \right) \right.$$
$$\left. \times \left\| \frac{(f' \circ h^{-1})(z)}{(h' \circ h^{-1})(z)} \right\|^2 dz \right]^{1/2}$$
$$\times \left[\int_a^b \left| \frac{h(a)+h(b)}{2} - z \right| \left\| \frac{(f' \circ h^{-1})(z)}{(h' \circ h^{-1})(z)} \right\|^2 dz \right]^{1/2}$$
$$\leq \frac{1}{4}[h(b) - h(a)] \int_{h(a)}^{h(b)} \left\| \frac{(f' \circ h^{-1})(z)}{(h' \circ h^{-1})(z)} \right\|^2 dz.$$

If we make the change of variable $t = h^{-1}(z)$, $z \in [h(a), h(b)]$, then by (2.14) we get the desired result (2.11). □

If $w : [a, b] \to \mathbb{R}$ is continuous and positive on the interval $[a, b]$, then the function $W : [a, b] \to [0, \infty)$, $W(x) := \int_a^x w(s)\,ds$ is strictly increasing and differentiable on (a, b). We have $W'(x) = w(x)$ for any $x \in (a, b)$.

Corollary 1. *Assume that $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$ and that $f : [a, b] \subset \mathbb{R} \to H$ is an absolutely continuous vector valued function on the interval $[a, b]$ and such that $\frac{f'}{w^{1/2}} \in L_2([a, b], H)$.*

(i) *If $f(a) = 0$ or $f(b) = 0$, then*

$$\int_a^b |\langle f'(t), f(t) \rangle| \, dt \leq \left(\int_a^b \left(\int_a^t w(s)\,ds \right) \frac{\|f'(t)\|^2}{w(t)} dt \right)^{1/2} \tag{2.15}$$
$$\times \left(\int_a^b \left(\int_t^b w(s)\,ds \right) \frac{\|f'(t)\|^2}{w(t)} dt \right)^{1/2}$$
$$\leq \frac{1}{2} \int_a^b w(s)\,ds \int_a^b \frac{\|f'(t)\|^2}{w(t)} dt.$$

(ii) *If* $f(a) = f(b) = 0,$ *then*

$$\int_a^b |\langle f'(t), f(t)\rangle| \, dt \leq \frac{1}{2}\left[\int_a^b w(s)\,ds \int_a^b \frac{\|f'(t)\|^2}{w(t)}dt - \int_a^b \left|\int_t^b w(s)\,ds - \int_a^t w(s)\,ds\right| \frac{\|f'(t)\|^2}{w(t)}dt\right]^{1/2} \times \left[\int_a^b \left|\int_t^b w(s)\,ds - \int_a^t w(s)\,ds\right| \frac{\|f'(t)\|^2}{w(t)}dt\right]^{1/2} \leq \frac{1}{4}\int_a^b w(s)\,ds \int_a^b \frac{\|f'(t)\|^2}{w(t)}dt. \tag{2.16}$$

3. Some examples

Consider the function $h(t) = \ln t,\ t \in [a,b] \subset (0,\infty)$. Assume that $f : [a,b] \subset \mathbb{R} \to H$ is an absolutely continuous vector valued function on the interval $[a,b]$ and such that $\ell^{1/2} f' \in L_2([a,b], H)$, where $\ell(t) = t$.

If $f(a) = 0$ or $f(b) = 0$, then

$$\int_a^b |\langle f'(t), f(t)\rangle| \, dt \leq \left(\int_a^b t \ln\left(\frac{t}{a}\right) |f'(t)|^2 \, dt\right)^{1/2} \left(\int_a^b t \ln\left(\frac{b}{t}\right) |f'(t)|^2 \, dt\right)^{1/2} \leq \frac{1}{2}\ln\left(\frac{b}{a}\right)\int_a^b t\,|f'(t)|^2 \, dt. \tag{3.1}$$

If $f(a) = f(b) = 0$, then

$$\int_a^b |\langle f'(t), f(t)\rangle| \, dt \leq \left[\ln\left(\sqrt{\frac{b}{a}}\right)\int_a^b t\,\|f'(t)\|^2 \, dt - \int_a^b \left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right| t\,\|f'(t)\|^2 \, dt\right]^{1/2} \times \left[\int_a^b \left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right| t\,\|f'(t)\|^2 \, dt\right]^{1/2} \leq \frac{1}{4}\ln\left(\frac{b}{a}\right)\int_a^b t\,\|f'(t)\|^2 \, dt. \tag{3.2}$$

Consider the function $h(t) = \frac{t^2}{2},\ t \in [a,b] \subset (0,\infty)$. Assume that $f : [a,b] \subset \mathbb{R} \to H$ is an absolutely continuous vector valued function on the interval $[a,b]$ and such that $\frac{f'}{\ell^{1/2}} \in L_2([a,b], H)$, where $\ell(t) = t$.

If $f(a)=0$ or $f(b)=0$, then

$$\int_a^b |\langle f'(t), f(t)\rangle| \, dt \tag{3.3}$$
$$\leq \frac{1}{2}\left(\int_a^b \left(t^2-a^2\right)\frac{|f'(t)|^2}{t}dt\right)^{1/2}\left(\int_a^b \left(b^2-t^2\right)\frac{|f'(t)|^2}{t}dt\right)^{1/2}$$
$$\leq \frac{1}{4}\left(b^2-a^2\right)\int_a^b \frac{|f'(t)|^2}{t}dt.$$

If $f(a)=f(b)=0$, then

$$\int_a^b |\langle f'(t), f(t)\rangle| \, dt \tag{3.4}$$
$$\leq \frac{1}{2}\left[\frac{b^2-a^2}{2}\int_a^b \frac{\|f'(t)\|^2}{t}dt - \int_a^b \left|\frac{a^2+b^2}{2}-t^2\right|\frac{\|f'(t)\|^2}{t}dt\right]^{1/2}$$
$$\times\left[\int_a^b \left|\frac{a^2+b^2}{2}-t^2\right|\frac{\|f'(t)\|^2}{t}dt\right]^{1/2} \leq \frac{1}{8}\left(b^2-a^2\right)\int_a^b \frac{\|f'(t)\|^2}{t}dt.$$

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