DOUBLE INTEGRALS HERMITE-HADAMARD TYPE INEQUALITIES FOR THE OPERATOR MODULUS IN HILBERT SPACES

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Abstract. Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. Denote by $\mathcal{B}(H)$ the Banach C^{*}-algebra of bounded linear operators on H. For $A\in\mathcal{B}(H)$ we define the modulus of A by $|A|:=(A^{*}A)^{1/2}$. We say that the continuous function $B:[a,b]\to\mathcal{B}(H)$ is square modulus convex on $[a,b]$ if

$$|B((1-t)u+tv)|^{2}\leq(1-t)|B(u)|^{2}+t|B(v)|^{2}$$

in the operator order of $\mathcal{B}(H)$, for all $u,v\in[a,b]$ and $t\in[0,1]$. In this paper, we show among others that,

$$\begin{aligned}
0 &\leq 2\min\{v,1-v\}\\
&\times\left[\frac{1}{b-a}\int_{a}^{b}|B(s)|^{2}ds-\frac{1}{(b-a)^{2}}\int_{a}^{b}\int_{a}^{b}\left|B\left(\frac{s+\tau}{2}\right)\right|^{2}dsd\tau\right]\\
&\leq\frac{1}{b-a}\int_{a}^{b}|B(s)|^{2}ds-\frac{1}{(b-a)^{2}}\int_{a}^{b}\int_{a}^{b}|B((1-\nu)s+\nu\tau)|^{2}dsd\tau\\
&\leq 2\max\{v,1-v\}\\
&\times\left[\frac{1}{b-a}\int_{a}^{b}|B(s)|^{2}ds-\frac{1}{(b-a)^{2}}\int_{a}^{b}\int_{a}^{b}\left|B\left(\frac{s+\tau}{2}\right)\right|^{2}dsd\tau\right].
\end{aligned}$$

for all $\nu\in[0,1]$. Applications for exponential function are provided as well.

1. Introduction

Let $\mathbb{R}$ be the set of real numbers and $I\subseteq\mathbb{R}$ be an interval. A function $f:I\subseteq\mathbb{R}\to\mathbb{R}$ is said to be *convex* in the classical sense if it satisfies the following inequality

$$f(tx+(1-t)t)\leq tf(x)+(1-t)f(y)$$

for all $x,y\in I$ and $t\in[0,1]$.

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ and $a,b\in I$ with $a<b$. Then the inequality

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_{a}^{b}f(x)dx\leq\frac{f(a)+f(b)}{2} \tag{1.1}$$

holds if f is convex, and is known in the literature as *Hermite-Hadamard inequality*, after the name of C. Hermite and J. Hadamard (see [16]). The inequalities in (1.1) hold in reversed direction if f is a concave function.

A vast literature related to (1.1) have been produced by a large number of mathematicians [10] since it is considered to be one of the most famous inequality

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for convex functions due to its usefulness and many applications in various branches of Pure and Applied Mathematics, such as Numerical Analysis [3], Information Theory [2], Operator Theory [6] and others.

Let X be a vector space over the real or complex number field $\mathbb{K}$ and $x, y \in X,\ x \neq y$. Define the segment

$$[x, y] := \{(1-t)x + ty,\ t \in [0,1]\}.$$

We consider the function $f : [x, y] \to \mathbb{R}$ and the associated function

$$g(x, y) : [0,1] \to \mathbb{R},\ g(x, y)(t) := f[(1-t)x + ty],\ t \in [0,1].$$

Note that f is convex on $[x, y]$ if and only if $g(x, y)$ is convex on $[0, 1]$.

For any convex function defined on a segment $[x, y] \subset X$, we have the *Hermite-Hadamard integral inequality* (see [4, p. 2], [5, p. 2])

$$f\left(\frac{x+y}{2}\right) \le \int_0^1 f[(1-t)x + ty]dt \le \frac{f(x) + f(y)}{2}, \tag{1.2}$$

which can be derived from the classical Hermite-Hadamard inequality (1.1) for the convex function $g(x, y) : [0, 1] \to \mathbb{R}$.

Since $f(x) = \|x\|^p$ ($x \in X$ and $1 \le p < \infty$) is a convex function, then for any $x, y \in X$ we have the following norm inequality from (1.2) (see [15, p. 106])

$$\left\|\frac{x+y}{2}\right\|^p \le \int_0^1 \|(1-t)x + ty\|^p dt \le \frac{\|x\|^p + \|y\|^p}{2}. \tag{1.3}$$

Denote by $\mathcal{B}(H)$ the Banach C^*-algebra of bounded linear operators on Hilbert space H. For $A \in \mathcal{B}(H)$ we define the modulus of A by $|A| := (A^*A)^{1/2}$. It is well known that the modulus of operators does not satisfy, in general, the triangle inequality $|A + B| \le |A| + |B|$, so the classical arguments using this inequality can not be used.

We use the following Cauchy-Bunyakowsky-Schwarz discrete inequality:

$$\sum_{k=1}^n w_k |z_k|^2 \sum_{k=1}^n w_k |A_k|^2 \ge \left|\sum_{k=1}^n w_k z_k A_k\right|^2, \tag{1.4}$$

where $z_k \in \mathbb{C}$, $A_k \in \mathcal{B}(H)$, $w_k \ge 0$ for $k \in \{1, ..., n\}$ and $\sum_{k=1}^n w_k = 1$.

Definition 1. *We say that the continuous function $B : [a, b] \to \mathcal{B}(H)$ is square modulus convex (concave) on $[a, b]$ if*

$$|B((1-t)u + tv)|^2 \le (\ge)\, (1-t)|B(u)|^2 + t|B(v)|^2 \tag{1.5}$$

in the operator order of $\mathcal{B}(H)$, for all u, $v \in [a, b]$ and $t \in [0, 1]$.

Let A, $B \in \mathcal{B}(H)$ and $\alpha \in [0, 1]$. Then by (1.4) we get

$$\begin{aligned}
|(1-\alpha)A + \alpha B|^2 &= \left|(1-\alpha)^{1/2}(1-\alpha)^{1/2}A + \alpha^{1/2}\alpha^{1/2}B\right|^2 \\
&\le \left[\left((1-\alpha)^{1/2}\right)^2 + \left(\alpha^{1/2}\right)^2\right]\left[\left|(1-\alpha)^{1/2}A\right|^2 + \left|\alpha^{1/2}B\right|^2\right] \\
&= (1-\alpha+\alpha)\left[(1-\alpha)|A|^2 + \alpha|B|^2\right] \\
&= (1-\alpha)|A|^2 + \alpha|B|^2.
\end{aligned}$$

Consider the function $C:[0,1]\to\mathcal{B}(H)$, $C(t)=|(1-t)A+tB|$. Let $t_1, t_2\in[0,1]$ and $\alpha\in[0,1]$. Then

$$\begin{aligned}|C((1-\alpha)t_1+\alpha t_2)|^2 &= |(1-(1-\alpha)t_1-\alpha t_2)A+((1-\alpha)t_1+\alpha t_2)B|^2\\ &= |(1-\alpha)((1-t_1)A+t_1B)+\alpha((1-t_2)A+t_2B)|^2\\ &\le (1-\alpha)|(1-t_1)A+t_1B|^2+\alpha|(1-t_2)A+t_2B|^2\\ &= (1-\alpha)|C(t_1)|^2+\alpha|C(t_2)|^2,\end{aligned}$$

which shows that C is *square modulus convex* on $[0,1]$.

Assume that f is *nonnegative* on I and *operator convex*, namely

$$f((1-\alpha)A+\alpha B)\le(1-\alpha)f(A)+\alpha f(B)$$

for all $\alpha\in[0,1]$ and selfadjoint operators A, B with spectra in I.

For such function and A, B, we consider

$$D(t):=[f((1-t)A+tB)]^{1/2},t\in[0,1].$$

Then, using a similar proof as above for the modulus function, we conclude that D is *square modulus convex* on $[0,1]$.

The function $f(t)=t^r$ is operator convex on $(0,\infty)$ if either $1\le r\le 2$ or $-1\le r\le 0$ and is operator concave on $(0,\infty)$ if $0\le r\le 1$. Therefore for A, $B>0$, the function

$$B_r(t):=((1-t)A+tB)^{r/2},\ t\in[0,1]$$

is *square modulus convex* on $[0,1]$ for $1\le r\le 2$ or $-1\le r\le 0$.

Let $B:[a,b]\to\mathbb{C}$ be defined by $B(t):=x(t)+y(t)i$, $t\in[a,b]$. Observe that $|B(t)|^2=x^2(t)+y^2(t)$, $t\in[a,b]$. Now, if x^2, y^2 are convex on $[a,b]$, then obviously that x^2+y^2 is convex on $[a,b]$. However, if we take $x(t)=t\sin t$, $y(t)=t\cos t$, $t\in[0,2\pi]$, then neither x^2 nor y^2 is convex on $[0,2\pi]$ but x^2+y^2 is convex on $[0,2\pi]$.

Proposition 1. *If the continuous function $B:[a,b]\to\mathcal{B}(H)$ is square modulus concave on $[a,b]$ then for $p\in(0,1)$, $|B(\cdot)|^p$ is also square modulus concave on $[a,b]$.*

Proof. By the operator monotonicity and operator concavity of the function $h(t)=t^p$ for $p\in(0,1)$, we have

$$\begin{aligned}|B((1-t)u+tv)|^{2p} &\ge \left((1-t)|B(u)|^2+t|B(v)|^2\right)^p\\ &\ge (1-t)|B(u)|^{2p}+t|B(v)|^{2p}\end{aligned}$$

for $t\in[0,1]$, which shows that $|B(\cdot)|^p$ is also square modulus concave on $[a,b]$. □

For A, $B>0$, the function

$$B_q(t):=((1-t)A+tB)^{q/2},\ t\in[0,1]$$

is *square modulus concave* on $[0,1]$ for $q\in(0,1)$.

Indeed, we have for t_1, $t_2\in[0,1]$ and $\alpha\in[0,1]$ that

$$\begin{aligned}|B_q((1-\alpha)t_1+\alpha t_2)|^2 &= ((1-(1-\alpha)t_1-\alpha t_2)A+((1-\alpha)t_1+\alpha t_2)B)^q\\ &= [(1-\alpha)((1-t_1)A+t_1B)+\alpha((1-t_2)A+t_2B)]^q\\ &\ge (1-\alpha)((1-t_1)A+t_1B)^r+\alpha((1-t_2)A+t_2B)^q\\ &= (1-\alpha)|B_q(t_1)|^2+\alpha|B_q(t_2)|^2,\end{aligned}$$

which shows that B_q is *square modulus concave* on $[0,1]$.

2. Preliminary Facts

Following Roberts and Varberg [17, p. 5], we recall that if $f:I\to\mathbb{R}$ is a convex function, then for any $s_0\in\mathring{I}$ (the interior of the interval I) the limits

$$f'_-(s_0):=\lim_{s\to s_0-}\frac{f(s)-f(s_0)}{s-s_0}\text{ and }f'_+(s_0):=\lim_{s\to s_0+}\frac{f(s)-f(s_0)}{s-s_0}$$

exists and $f'_-(s_0)\le f'_+(s_0)$. The functions f'_- and f'_+ are monotonic nondecreasing on $\mathring{I}$ and this property can be extended to the whole interval I (see [17, p. 7]).

From the monotonicity of the lateral derivatives f'_- and f'_+ we also have *the gradient inequality*

$$f'_-(s)(s-\tau)\ge f(s)-f(\tau)\ge f'_+(\tau)(s-\tau)$$

for any $s,\ \tau\in\mathring{I}$.

If $I=[a,b]$, then at the end points we also have the inequalities

$$f(s)-f(a)\ge f'_+(a)(s-a)$$

for any $s\in(a,b]$ and

$$f(\tau)-f(b)\ge f'_-(b)(\tau-b)$$

for any $\tau\in[a,b)$.

For the operator $T\in\mathcal{B}(H)$ we define the selfadjoint operator

$$\operatorname{Re}(T)=\frac{1}{2}(T^*+T).$$

We have the following fundamental fact:

Proposition 2. *Assume that function $B:[a,b]\to\mathcal{B}(H)$ is continuous on $[a,b]$. The function $B:[a,b]\to\mathcal{B}(H)$ is square modulus convex on $[a,b]$ if and only if for all $x\in H\setminus\{0\}$ the auxiliary function $\varphi_{B,x}:[a,b]\to[0,\infty)$, $\varphi_{B,x}(u)=\|B(u)x\|^2$ is convex on $[a,b]$.*

Proof. Indeed, condition (1.5) is equivalent to

$$\left\langle |B((1-t)u+tv)|^2x,x\right\rangle\le(1-t)\left\langle|B(u)|^2x,x\right\rangle+t\left\langle|B(v)|^2x,x\right\rangle,$$

namely

$$\begin{aligned}&\left\langle[B((1-t)u+tv)]^*B((1-t)u+tv)x,x\right\rangle\\&\le(1-t)\left\langle[B(u)]^*B(u)x,x\right\rangle+t\left\langle[B(v)]^*B(v)x,x\right\rangle,\end{aligned}$$

or

$$\|B((1-t)u+tv)x\|^2\le(1-t)\|B(u)x\|^2+t\|B(v)x\|^2$$

for all $t\in[0,1]$ and $u,\ v\in[a,b]$. □

We also have

$$\begin{aligned}\varphi'_{\pm B,x}(u)&=(\langle B(u)x,B(u)x\rangle)'_\pm=\left\langle B'_\pm(u)x,B(u)x\right\rangle+\left\langle B(u)x,B'_\pm(u)x\right\rangle\\&=\left\langle(B(u))^*B'_\pm(u)x,x\right\rangle+\left\langle\left(B'_\pm(u)\right)^*B(u)x,x\right\rangle\\&=\left\langle(B(u))^*B'_\pm(u)x,x\right\rangle+\left\langle\left((B(u))^*B'_\pm(u)\right)^*x,x\right\rangle\\&=\left\langle2\operatorname{Re}\left((B(u))^*B'_\pm(u)\right)x,x\right\rangle\end{aligned}$$

and

$$\varphi'_{+B,x}(a) = 2\left\langle \operatorname{Re}\left((B(a))^* B'_+(a)\right)x, x\right\rangle,$$
$$\varphi'_{-B,x}(b) = 2\left\langle \operatorname{Re}\left((B(b))^* B'_-(b)\right)x, x\right\rangle$$

for all $x \in H \setminus \{0\}$.

We have for $t \in (a,b)$ and small $h \neq 0$ such that $t+h \in (a,b)$,

$$\left\langle \frac{|B(t+h)|^2 - |B(t)|^2}{h} x, x \right\rangle = \frac{1}{h}\left[\left\langle |B(t+h)|^2 x, x\right\rangle - \left\langle |B(t)|^2 x, x\right\rangle\right] = \frac{1}{h}\left[\|B(t+h)x\|^2 - \|B(t)x\|^2\right]$$

for all $x \in H \setminus \{0\}$.

By taking the lateral limits, we get

$$\lim_{h\to\pm 0}\left\langle \frac{|B(t+h)|^2 - |B(t)|^2}{h} x, x \right\rangle = \lim_{h\to\pm 0}\frac{1}{h}\left[\|B(t+h)x\|^2 - \|B(t)x\|^2\right] = \left\langle 2\operatorname{Re}\left((B(t))^* B'_\pm(t)\right)x, x\right\rangle$$

for all $x \in H \setminus \{0\}$.

Therefore for the function $\varphi(t) := |B(t)|^2$, $t \in (a,b)$ we have

$$\varphi'_\pm(t) = 2\operatorname{Re}\left((B(t))^* B'_\pm(t)\right) \tag{2.1}$$

and

$$\varphi'_+(a) = 2\operatorname{Re}\left((B(a))^* B_+(a)\right),\ \varphi'_-(b) = 2\operatorname{Re}\left((B(b))^* B'_-(b)\right).$$

We also have *the operator gradient inequality*

$$2(s-\tau)\operatorname{Re}\left((B(s))^* B'_-(s)\right) \geq |B(s)|^2 - |B(\tau)|^2 \geq 2(s-\tau)\operatorname{Re}\left((B(\tau))^* B'_+(\tau)\right) \tag{2.2}$$

for any $s, \tau \in (a,b)$.

Moreover, at the end points of the interval we have the operator inequalities

$$|B(s)|^2 - |B(a)|^2 \geq 2(s-a)\operatorname{Re}\left((B(a))^* B'_+(a)\right) \tag{2.3}$$

for any $s \in (a,b]$ and

$$2(b-\tau)\operatorname{Re}\left((B(b))^* B'_-(b)\right) \geq |B(b)|^2 - |B(\tau)|^2 \tag{2.4}$$

for any $\tau \in [a,b)$.

Finally, we notice that for $a \leq t_1 < t_2 \leq b$, we have

$$\operatorname{Re}\left((B(a))^* B'_+(a)\right) \leq \operatorname{Re}\left((B(t_1))^* B'_\pm(t_1)\right) \leq \operatorname{Re}\left((B(t_2))^* B'_\pm(t_2)\right) \leq \operatorname{Re}\left((B(b))^* B'_-(b)\right) \tag{2.5}$$

in the operator order of $\mathcal{B}(H)$.

Proposition 3. *Assume that function $B : [a,b] \to \mathcal{B}(H)$ is strongly differentiable on (a,b). The function $B : [a,b] \to \mathcal{B}(H)$ is square modulus convex on $[a,b]$ if and only if*

$$\operatorname{Re}\left((B(t_1))^* B'(t_1)\right) \leq \operatorname{Re}\left((B(t_2))^* B'(t_2)\right) \tag{2.6}$$

for all $a \leq t_1 < t_2 \leq b$.

The proof follows by Proposition 2 and the fact that a scalar-valued differentiable function is convex on (a,b) if and only if its derivative is nondecreasing on (a,b).

We notice that if $\varphi(t) := |B(t)|^2$, $t \in (a,b)$, then we have from (2.1)

$$\varphi'(t) = 2\operatorname{Re}\left((B(t))^* B'(t)\right) = (B(t))^* B'(t) + (B'(t))^* B(t). \tag{2.7}$$

If $B : [a,b] \to \mathcal{B}(H)$ is strongly twice differentiable on (a,b), then

$$\begin{aligned}
&\varphi''(t) \\
&= \left((B(t))^* B'(t)\right)' + \left((B'(t))^* B(t)\right)' \\
&= \left((B'(t))^* B'(t)\right) + (B(t))^* B''(t) + \left((B''(t))^* B(t)\right) + \left((B'(t))^* B'(t)\right) \\
&= 2|B'(t)|^2 + 2\operatorname{Re}\left[(B(t))^* B''(t)\right] = 2\left\{|B'(t)|^2 + \operatorname{Re}\left[(B(t))^* B''(t)\right]\right\}
\end{aligned}$$

for all $t \in (a,b)$.

We can state then the following result as well:

Proposition 4. *Assume that function $B : [a,b] \to \mathcal{B}(H)$ is strongly twice differentiable on (a,b). The function $B : [a,b] \to \mathcal{B}(H)$ is square modulus convex on $[a,b]$ if and only if*

$$|B'(t)|^2 + \operatorname{Re}\left[(B(t))^* B''(t)\right] \geq 0$$

for all $t \in (a,b)$.

We also have the following scalar inequality of interest:

$$\begin{aligned}
0 &\leq 2\min\{t, 1-t\}\left[\frac{f(s)+f(u)}{2} - f\left(\frac{s+u}{2}\right)\right] \\
&\leq (1-t) f(s) + t f(u) - f((1-t)s + tu) \\
&\leq 2\max\{t, 1-t\}\left[\frac{f(s)+f(u)}{2} - f\left(\frac{s+u}{2}\right)\right],
\end{aligned} \tag{2.8}$$

where $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$ and $t \in [0,1]$.

The proof follows, for instance, by Corollary 1 from [7] for $n = 2$, $p_1 = 1-t$, $p_2 = t$, $t \in [0,1]$ and $x_1 = s$, $x_2 = u$.

By making use of (2.8) for the $\varphi_{B,x} : [a,b] \to [0,\infty)$, $\varphi_{B,x}(u) = \|B(u)x\|^2$, $x \in H \setminus \{0\}$, we can state the following result :

Proposition 5. *Assume that the continuous function $B : [a,b] \to \mathcal{B}(H)$ is square modulus convex on $[a,b]$, s, $\tau \in (a,b)$ and $\nu \in [0,1]$, then*

$$\begin{aligned}
0 &\leq 2\min\{v, 1-v\}\left[\frac{|B(s)|^2 + |B(\tau)|^2}{2} - \left|B\left(\frac{s+\tau}{2}\right)\right|^2\right] \\
&\leq (1-v)|B(s)|^2 + v|B(\tau)|^2 - |B((1-\nu)s + \nu\tau)|^2 \\
&\leq 2\max\{v, 1-v\}\left[\frac{|B(s)|^2 + |B(\tau)|^2}{2} - \left|B\left(\frac{s+\tau}{2}\right)\right|^2\right].
\end{aligned} \tag{2.9}$$

We have the following result for general convex functions [8]:

Lemma 1. *Let $\varphi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval $[a,b]$, s, $\tau \in (a,b)$ and $\nu \in [0,1]$. Then*

$$0 \le \nu (1-\nu)(\tau - s)\left[\varphi'_+((1-\nu)s+\nu\tau) - \varphi'_-((1-\nu)s+\nu\tau)\right] \\ \le (1-\nu)\varphi(s) + \nu\varphi(\tau) - \varphi((1-\nu)s+\nu\tau) \\ \le \nu(1-\nu)(\tau-s)\left[\varphi'_-(\tau) - \varphi'_+(s)\right]. \tag{2.10}$$

In particular, we have

$$0 \le \frac{1}{4}(\tau - s)\left[\varphi'_+\left(\frac{s+\tau}{2}\right) - \varphi'_-\left(\frac{s+\tau}{2}\right)\right] \\ \le \frac{\varphi(s)+\varphi(\tau)}{2} - \varphi\left(\frac{s+\tau}{2}\right) \\ \le \frac{1}{4}(\tau-s)\left[\varphi'_-(\tau) - \varphi'_+(s)\right]. \tag{2.11}$$

The constant $\frac{1}{4}$ is best possible in both inequalities from (2.11).

Remark 1. *If the function $\varphi : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on (a,b), then for any s, $\tau \in (a,b)$ and $\nu \in [0,1]$ we have*

$$0 \le (1-\nu)\varphi(s) + \nu\varphi(\tau) - \varphi((1-\nu)s+\nu\tau) \\ \le \nu(1-\nu)(\tau-s)\left[\varphi'(\tau) - \varphi'(s)\right]. \tag{2.12}$$

Proposition 6. *Assume that the continuous function $B : [a,b] \to \mathcal{B}(H)$ is square modulus convex on $[a,b]$, s, $\tau \in (a,b)$ and $\nu \in [0,1]$, then*

$$0 \le 2\nu(1-\nu)(\tau-s) \\ \times \left[\operatorname{Re}\left((B((1-\nu)s+\nu\tau))^* B'_+((1-\nu)s+\nu\tau)\right)\right. \\ \left. - \operatorname{Re}\left((B((1-\nu)s+\nu\tau))^* B'_-((1-\nu)s+\nu\tau)\right)\right] \\ \le (1-v)|B(s)|^2 + v|B(\tau)|^2 - |B((1-\nu)s+\nu\tau)|^2 \\ \le 2\nu(1-\nu)(\tau-s)\left[\operatorname{Re}\left((B(\tau))^* B'_-(\tau)\right) - \operatorname{Re}\left((B(s))^* B'_+(s)\right)\right]. \tag{2.13}$$

If $B : [a,b] \to \mathcal{B}(H)$ is strongly differentiable on (a,b), then

$$0 \le (1-v)|B(s)|^2 + v|B(\tau)|^2 - |B((1-\nu)s+\nu\tau)|^2 \\ \le 2\nu(1-\nu)(\tau-s)\left[\operatorname{Re}\left((B(\tau))^* B'(\tau)\right) - \operatorname{Re}\left((B(s))^* B'(s)\right)\right], \tag{2.14}$$

for s, $\tau \in (a,b)$ and $\nu \in [0,1]$.

In particular,

$$0 \le \frac{|B(s)|^2 + |B(\tau)|^2}{2} - \left|B\left(\frac{s+\tau}{2}\right)\right|^2 \\ \le \frac{1}{2}(\tau-s)\left[\operatorname{Re}\left((B(\tau))^* B'(\tau)\right) - \operatorname{Re}\left((B(s))^* B'(s)\right)\right], \tag{2.15}$$

for s, $\tau \in (a,b)$.

Proof. Let $x \in H \setminus \{0\}$, $s, \tau \in (a,b)$ and $\nu \in [0,1]$, then by the inequality (2.10) for the function $\varphi_{B,x} : [a,b] \to [0,\infty)$, $\varphi_{B,x}(u) = \|B(u)x\|^2$ we get

$$\begin{aligned}
&2\nu(1-\nu)(\tau-s)\\
&\times\left[\left\langle \operatorname{Re}\left((B((1-\nu)s+\nu\tau))^* B'_+((1-\nu)s+\nu\tau)\right)x,x\right\rangle\right.\\
&\left.-\left\langle \operatorname{Re}\left((B((1-\nu)s+\nu\tau))^* B'_-((1-\nu)s+\nu\tau)\right)x,x\right\rangle\right]\\
&\le (1-\nu)\left\langle |B(s)|^2 x,x\right\rangle + \nu\left\langle |B(\tau)|^2 x,x\right\rangle - \left\langle |B((1-\nu)s+\nu\tau)|^2 x,x\right\rangle\\
&\le 2\nu(1-\nu)(\tau-s)\\
&\times\left[\left\langle \operatorname{Re}\left((B(\tau))^* B'_-(\tau)\right)x,x\right\rangle - \left\langle \operatorname{Re}\left((B(s))^* B'_+(s)\right)x,x\right\rangle\right],
\end{aligned}$$

which is equivalent to (2.13). □

We have the following result for the second derivative holds, see for instance :

Lemma 2. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on the interval (a,b). If there exists the constants d, D such that*

$$d \le f''(t) \le D \text{ for any } t \in (s,\tau), \tag{2.16}$$

then

$$\begin{aligned}
\frac{1}{2}\nu(1-\nu)d(\tau-s)^2 &\le (1-\nu)f(s)+\nu f(\tau) - f((1-\nu)s+\nu\tau)\\
&\le \frac{1}{2}\nu(1-\nu)D(\tau-s)^2
\end{aligned} \tag{2.17}$$

for any $s, \tau \in (s,\tau)$ and $\nu \in [0,1]$.

In particular, we have

$$\frac{1}{8}(\tau-s)^2 d \le \frac{f(s)+f(\tau)}{2} - f\left(\frac{s+\tau}{2}\right) \le \frac{1}{8}(\tau-s)^2 D, \tag{2.18}$$

for any $s, \tau \in (s,\tau)$.

The constant $\frac{1}{8}$ is best possible in both inequalities in (2.18).

Proposition 7. *Assume that $B : [a,b] \to \mathcal{B}(H)$ is strongly twice differentiable on (a,b) and there exist C, $D \in \mathcal{B}(H)$ with C, $D \ge 0$ and*

$$(0 \le)\, C \le |B'(t)|^2 + \operatorname{Re}\left[(B(t))^* B''(t)\right] \le D \text{ for all } t \in (a,b). \tag{2.19}$$

Then for any $s, \tau \in [a,b]$ and $\nu \in [0,1]$ we have

$$\begin{aligned}
&(0 \le)\, \nu(1-\nu)(\tau-s)^2 C\\
&\le (1-v)|B(s)|^2 + v|B(\tau)|^2 - |B((1-\nu)s+\nu\tau)|^2\\
&\le \nu(1-\nu)(\tau-s)^2 D.
\end{aligned} \tag{2.20}$$

In particular,

$$\begin{aligned}
(0 \le)\, \frac{1}{4}(\tau-s)^2 C &\le \frac{|B(s)|^2 + |B(\tau)|^2}{2} - \left|B\left(\frac{s+\tau}{2}\right)\right|^2\\
&\le \frac{1}{4}(\tau-s)^2 D.
\end{aligned} \tag{2.21}$$

The proof follows by Lemma 2 for the function $\varphi_{B,x} : [a,b] \to [0,\infty)$, $\varphi_{B,x}(u) = \|B(u)x\|^2$, $x \in H \setminus \{0\}$ and by observing that

$$\varphi''_{B,x}(t) = 2\left\langle \left\{ |B'(t)|^2 + \operatorname{Re}\left[(B(t))^* B''(t)\right] \right\} x, x \right\rangle$$

for $t \in (a,b)$.

Corollary 1. *With the assumptions of Proposition 7 we have*

$$0 \le \frac{1}{6}(\tau - s)^2 C \le \frac{|B(s)|^2 + |B(\tau)|^2}{2} - \int_0^1 |B((1-\nu)s + \nu\tau)|^2 d\nu \le \frac{1}{6}(\tau - s)^2 D \tag{2.22}$$

and

$$0 \le \frac{1}{12}(\tau - s)^2 C \le \int_0^1 |B((1-\nu)s + \nu\tau)|^2 d\nu - \left|B\left(\frac{s+\tau}{2}\right)\right|^2 \le \frac{1}{12}(\tau - s)^2 D \tag{2.23}$$

for any $s,\ \tau \in [a,b]$.

Proof. The inequality (2.22) follows by (2.20) on integrating over $\nu \in [0,1]$.

From (2.21) we get

$$\begin{aligned}(0 \le)\ & \frac{1}{4}((1-\nu)s + \nu\tau - (1-\nu)\tau - \nu s)^2 C \\ & \le \frac{|B((1-\nu)s + \nu\tau)|^2 + |B((1-\nu)\tau + \nu s)|^2}{2} - \left|B\left(\frac{s+\tau}{2}\right)\right|^2 \\ & \le \frac{1}{4}((1-\nu)s + \nu\tau - (1-\nu)\tau - \nu s)^2 D,\end{aligned} \tag{2.24}$$

namely

$$\begin{aligned}(0 \le)\ & \frac{1}{4}(1-2\nu)^2(\tau - s)^2 C \\ & \le \frac{|B((1-\nu)s + \nu\tau)|^2 + |B((1-\nu)\tau + \nu s)|^2}{2} - \left|B\left(\frac{s+\tau}{2}\right)\right|^2 \\ & \le \frac{1}{4}(1-2\nu)^2(\tau - s)^2 D.\end{aligned} \tag{2.25}$$

Taking the integral over $\nu \in [0,1]$ and observing that

$$\int_0^1 (1-2\nu)^2 d\nu = \frac{1}{3}$$

and

$$\int_0^1 |B((1-\nu)\tau + \nu s)|^2 d\nu = \int_0^1 |B((1-\nu)s + \nu\tau)|^2 d\nu,$$

then by (2.25) we derive (2.23). □

3. Double Integral Inequalities

We have the following double integral inequalities:

Theorem 1. *Assume that the continuous function* $B:[a,b]\to\mathcal{B}(H)$ *is square modulus convex on* $[a,b]$. *Then for all* $\nu\in[0,1]$,

$$
\begin{aligned}
(3.1)\qquad & 0\le 2\min\{v,1-v\}\\
&\times\left[\frac{1}{b-a}\int_a^b|B(s)|^2ds-\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|B\left(\frac{s+\tau}{2}\right)\right|^2dsd\tau\right]\\
&\le\frac{1}{b-a}\int_a^b|B(s)|^2ds-\frac{1}{(b-a)^2}\int_a^b\int_a^b|B((1-\nu)s+\nu\tau)|^2dsd\tau\\
&\le 2\max\{v,1-v\}\\
&\times\left[\frac{1}{b-a}\int_a^b|B(s)|^2ds-\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|B\left(\frac{s+\tau}{2}\right)\right|^2dsd\tau\right].
\end{aligned}
$$

Proof. If we take the double integral in (2.9), then we get

$$
\begin{aligned}
(3.2)\qquad & 0\le 2\min\{v,1-v\}\\
&\times\left[\int_a^b\int_a^b\left(\frac{|B(s)|^2+|B(\tau)|^2}{2}\right)dsd\tau-\int_a^b\int_a^b\left|B\left(\frac{s+\tau}{2}\right)\right|^2dsd\tau\right]\\
&\le(1-v)\int_a^b\int_a^b|B(s)|^2dsd\tau+v\int_a^b\int_a^b|B(\tau)|^2dsd\tau\\
&-\int_a^b\int_a^b|B((1-\nu)s+\nu\tau)|^2dsd\tau\\
&\le 2\max\{v,1-v\}\\
&\times\left[\int_a^b\int_a^b\left(\frac{|B(s)|^2+|B(\tau)|^2}{2}\right)dsd\tau-\int_a^b\int_a^b\left|B\left(\frac{s+\tau}{2}\right)\right|^2dsd\tau\right],
\end{aligned}
$$

for all $\nu\in[0,1]$.

Observe that

$$
\int_a^b\int_a^b|B(s)|^2dsd\tau=\int_a^b\int_a^b|B(\tau)|^2dsd\tau=(b-a)\int_a^b|B(s)|^2ds
$$

and by (3.2) we get

$$
\begin{aligned}
0&\le 2\min\{v,1-v\}\left[(b-a)\int_a^b|B(s)|^2ds-\int_a^b\int_a^b\left|B\left(\frac{s+\tau}{2}\right)\right|^2dsd\tau\right]\\
&\le(b-a)\int_a^b|B(s)|^2ds-\int_a^b\int_a^b|B((1-\nu)s+\nu\tau)|^2dsd\tau\\
&\le 2\max\{v,1-v\}\left[(b-a)\int_a^b|B(s)|^2ds-\int_a^b\int_a^b\left|B\left(\frac{s+\tau}{2}\right)\right|^2dsd\tau\right],
\end{aligned}
$$

which is equivalent to (3.1). □

For functions that are strongly differentiable, we have:

Theorem 2. *Assume that* $B:[a,b]\to\mathcal{B}(H)$ *is square modulus convex on* $[a,b]$ *and strongly differentiable on* (a,b), *then for all* $\nu\in[0,1]$,

$$0\le\frac{1}{b-a}\int_a^b|B(s)|^2ds-\frac{1}{(b-a)^2}\int_a^b\int_a^b|B((1-\nu)s+\nu\tau)|^2dsd\tau \tag{3.3}$$
$$\le 4\nu(1-\nu)\frac{1}{b-a}\int_a^b\left(s-\frac{a+b}{2}\right)\operatorname{Re}\left((B(s))^*B'(s)\right)ds.$$

In particular,

$$0\le\frac{1}{b-a}\int_a^b|B(s)|^2ds-\frac{1}{(b-a)^2}\int_a^b\int_a^b\left|B\left(\frac{s+\tau}{2}\right)\right|^2dsd\tau \tag{3.4}$$
$$\le\frac{1}{b-a}\int_a^b\left(s-\frac{a+b}{2}\right)\operatorname{Re}\left((B(s))^*B'(s)\right)ds.$$

Proof. If we take the double integral in (2.14), then we get

$$0\le(1-v)\int_a^b\int_a^b|B(s)|^2dsd\tau+v\int_a^b\int_a^b|B(\tau)|^2dsd\tau \tag{3.5}$$
$$-\int_a^b\int_a^b|B((1-\nu)s+\nu\tau)|^2dsd\tau$$
$$\le 2\nu(1-\nu)$$
$$\times\int_a^b\int_a^b(\tau-s)\left[\operatorname{Re}\left((B(\tau))^*B'(\tau)\right)-\operatorname{Re}\left((B(s))^*B'(s)\right)\right]dsd\tau,$$

for all $\nu\in[0,1]$.

Observe that

$$\int_a^b\int_a^b(\tau-s)\left[\operatorname{Re}\left((B(\tau))^*B'(\tau)\right)-\operatorname{Re}\left((B(s))^*B'(s)\right)\right]dsd\tau$$
$$=\int_a^b\int_a^b\left[\tau\operatorname{Re}\left((B(\tau))^*B'(\tau)\right)+s\operatorname{Re}\left((B(s))^*B'(s)\right)\right]dsd\tau$$
$$-\int_a^b\int_a^b\left[s\operatorname{Re}\left((B(\tau))^*B'(\tau)\right)+\tau\operatorname{Re}\left((B(s))^*B'(s)\right)\right]dsd\tau$$
$$=2(b-a)\int_a^b s\operatorname{Re}\left((B(s))^*B'(s)\right)ds-2\frac{b^2-a^2}{2}\int_a^b\operatorname{Re}\left((B(s))^*B'(s)\right)ds$$
$$=2(b-a)\left[\int_a^b s\operatorname{Re}\left((B(s))^*B'(s)\right)ds-\frac{a+b}{2}\int_a^b\operatorname{Re}\left((B(s))^*B'(s)\right)ds\right]$$

and by (3.5) we get

$$(b-a)\int_a^b|B(s)|^2ds-\int_a^b\int_a^b|B((1-\nu)s+\nu\tau)|^2dsd\tau$$
$$\le 4\nu(1-\nu)(b-a)\left[\int_a^b\left(s-\frac{a+b}{2}\right)\operatorname{Re}\left((B(s))^*B'(s)\right)ds\right],$$

which is equivalent to (3.3). □

For functions that are strongly twice differentiable, we have:

Theorem 3. *Assume that $B:[a,b]\to\mathcal{B}(H)$ is strongly twice differentiable on (a,b) and there exist $C,\ D\in\mathcal{B}(H)$ with $C,\ D\geq 0$ and the condition (2.19) is satisfied, then for all $\nu\in[0,1]$,*

$$
\begin{aligned}
(3.6)\qquad 0 &\leq \frac{1}{6}\nu(1-\nu)(b-a)^2 C \\
&\leq \frac{1}{b-a}\int_a^b |B(s)|^2 ds - \frac{1}{(b-a)^2}\int_a^b\int_a^b |B((1-\nu)s+\nu\tau)|^2 dsd\tau\tau \\
&\leq \frac{1}{6}\nu(1-\nu)(b-a)^2 D.
\end{aligned}
$$

In particular,

$$
\begin{aligned}
(3.7)\qquad 0 &\leq \frac{1}{24}(b-a)^2 C \\
&\leq \frac{1}{b-a}\int_a^b |B(s)|^2 ds - \frac{1}{(b-a)^2}\int_a^b\int_a^b \left|B\left(\frac{s+\tau}{2}\right)\right|^2 dsd\tau \\
&\leq \frac{1}{24}(b-a)^2 D.
\end{aligned}
$$

Proof. By taking the double integral in (2.20), we get

$$
\begin{aligned}
(3.8)\qquad & (0\leq)\,\nu(1-\nu)\left(\int_a^b\int_a^b(\tau-s)^2 dsd\tau\right)C \\
&\leq (1-v)\int_a^b\int_a^b |B(s)|^2 dsd\tau + v\int_a^b\int_a^b |B(\tau)|^2 dsd\tau \\
&-\int_a^b\int_a^b |B((1-\nu)s+\nu\tau)|^2 dsd\tau \\
&\leq \nu(1-\nu)\left(\int_a^b\int_a^b(\tau-s)^2 dsd\tau\right)D.
\end{aligned}
$$

Since

$$\int_a^b\int_a^b(\tau-s)^2 dsd\tau = \frac{1}{6}(b-a)^4,$$

hence by (3.8) we get (3.6). □

4. An Example for Exponential Function

Let A be a *normal operator* in $\mathcal{B}(H)$, i.e. $A^*A=AA^*$. Consider the function $B(t):=\left|e^{tA}\right|^2$, with $t\in\mathbb{R}$. Observe that

$$\varphi_A(t):=\left|e^{tA}\right|^2=\left(e^{tA}\right)^* e^{tA}=e^{tA^*}e^{tA}=e^{t(A^*+A)}=e^{2t\operatorname{Re}(A)}$$

for all $t\in\mathbb{R}$.

Observe that [1]

$$\varphi_A'(t)=2\operatorname{Re}(A)\,e^{2t\operatorname{Re}(A)},\ t\in\mathbb{R}$$

and

$$\varphi_A''(t)=4\left[\operatorname{Re}(A)\right]^2 e^{2t\operatorname{Re}(A)},\ t\in\mathbb{R}.$$

For a selfadjoint operator S, by utilising the continuous functional calculus for selfadjoint operators, we have that

$$S^2 e^{tS} \geq 0 \text{ for all } t \in \mathbb{R}.$$

By utilising Proposition 4 we conclude that the function $A(t) := \left|e^{tA}\right|$ is *square modulus convex* on any subinterval $[a, b]$ of $\mathbb{R}$.

If T is *invertible operator* in $\mathcal{B}(H)$, then [1]

$$\int_a^b \exp(tT)\, dt = T^{-1}\left[\exp(bT) - \exp(aT)\right]. \tag{4.1}$$

This implies that for $\operatorname{Re}(A)$ invertible, we have

$$\int_a^b \exp(2t \operatorname{Re}(A))\, dt = \frac{1}{2}(\operatorname{Re}(A))^{-1}\left[\exp(2b \operatorname{Re}(A)) - \exp(2a \operatorname{Re}(A))\right].$$

Also, if $0 < m \leq \operatorname{Re}(A) \leq M$, then for $t \in [a, b] \subset (0, \infty)$,

$$4m^2 e^{2ma} \leq 4m^2 e^{2mt} \leq \varphi_A''(t) = 4\left[\operatorname{Re}(A)\right]^2 e^{2t \operatorname{Re}(A)} \leq 4M^2 e^{2tM} \leq 4M^2 e^{2bM}.$$

By Corollary 2.2 we then obtain

$$\begin{aligned} 0 &\leq \frac{2}{3}(b-a)^2 m^2 e^{2ma} \\ &\leq \frac{e^{2a \operatorname{Re}(A)} + e^{2b \operatorname{Re}(A)}}{2} \\ &\quad - \frac{1}{2}(\operatorname{Re}(A))^{-1}\left[\frac{\exp(2b \operatorname{Re}(A)) - \exp(2a \operatorname{Re}(A))}{b-a}\right] \\ &\leq \frac{2}{3}(b-a)^2 M^2 e^{2bM} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} 0 &\leq \frac{1}{3}(b-a)^2 m^2 e^{2ma} \\ &\leq \frac{1}{2}(\operatorname{Re}(A))^{-1}\left[\frac{\exp(2b \operatorname{Re}(A)) - \exp(2a \operatorname{Re}(A))}{b-a}\right] - e^{(b+a)\operatorname{Re}(A)} \\ &\leq \frac{1}{3}(b-a)^2 M^2 e^{2bM}. \end{aligned} \tag{4.3}$$

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