AN EXTENSION OF BROWN-PLUM INEQUALITY TO FUNCTIONS WITH VALUES IN HILBERT SPACES

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Abstract. In this paper we establish an extension and a weighted generalization of Brown-Plum inequality to functions with values in Hilbert spaces. Examples for trapezoid type inequalities are also given.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u'\in L_2[a,b]$.*

(i) *If $u(a)=u(b)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le\frac{1}{4}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.1}$$

with equality if and only if

$$u(t)=\begin{cases} c(t-a) & \text{if } a\le t\le \frac{a+b}{2},\\ c(b-t) & \text{if } \frac{a+b}{2}<t\le b\end{cases}$$

where c is an arbitrary constant;

(ii) *If $u(a)=0$, then*

$$\int_a^b |u(t)\,u'(t)|\,dt\le\frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ for some constant c;

he inequality (1.1) was obtained by Olech in [7] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [8].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [1], which is satisfied by those u vanishing only at a.

In 2005, Brown and Plum [4] obtained the following result as well:

Theorem 2. *Assume that $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u'\in L_2[a,b]$. If $\int_a^b u(t)\,dt=0$, then the inequality (1.1) holds with equality if and only if*

$$u(t)=c\left(t-\frac{a+b}{2}\right)$$

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for any constant c.

In 1975, G. G. Vrânceanu extended Opial's inequality (1.2) for functions with values in Hilbert spaces $(H;\langle\cdot,\cdot\rangle)$ as follows:

Theorem 3. *Assume that the function $f:[a,b]\to H$ has a continuous derivative and $f(a)=0$, then*

$$\int_a^b |\langle f(t), f'(t)\rangle|\,dt \le \frac{1}{2}(b-a)\int_a^b \|f'(t)\|^2\,dt. \tag{1.3}$$

In the recent paper [5] we obtain the following refinement of (1.3):

Theorem 4. *Assume that $f:[a,b]\to H$, H a complex Hilbert space, is absolutely continuous on $[a,b]$ and $f'\in L_2([a,b],H)$.*

(i) *If either $f(a)=0$ or $f(b)=0$, then*

$$\begin{aligned}&\int_a^b |\langle f'(t), f(t)\rangle|\,dt \\ &\le \left(\int_a^b (t-a)\|f'(t)\|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)\|f'(t)\|^2\,dt\right)^{1/2} \\ &\le \frac{1}{2}(b-a)\int_a^b \|f'(t)\|^2\,dt.\end{aligned} \tag{1.4}$$

(ii) *If $f(a)=f(b)=0$, then*

$$\begin{aligned}&\int_a^b |\langle f'(t), f(t)\rangle|\,dt \\ &\le \left[\int_a^b K(t)\|f'(t)\|^2\,dt\right]^{1/2}\left[\int_a^b \left|\frac{a+b}{2}-t\right|\|f'(t)\|^2\,dt\right]^{1/2} \\ &\le \frac{1}{4}(b-a)\int_a^b \|f'(t)\|^2\,dt,\end{aligned} \tag{1.5}$$

where

$$K(t):=\begin{cases} t-a \text{ if } a\le t\le \frac{a+b}{2}, \\ b-t \text{ if } \frac{a+b}{2}<t\le b.\end{cases}$$

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space. If $\{e_\alpha\}_{\alpha\in\mathcal{U}}$ ($\mathcal{U}$ is a certain index set), is a complete orthonormal system in a Hilbert space H, then for any element $x\in H$, *Parseval's equality* holds:

$$\|x\|^2=\sum_{\alpha\in\mathcal{U}}|\langle x,e_\alpha\rangle|^2 \tag{1.6}$$

and the sum on the right-hand side is to be understood as $\sup_{\mathcal{U}_0}\sum_{\alpha\in\mathcal{U}_0}|\langle x,e_\alpha\rangle|^2$where the supremum is taken over all finite subsets $\mathcal{U}_0$ of $\mathcal{U}$.

Assume that H is a *separable* Hilbert space and $x, y\in H$. If $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of H and if $a_n=\langle x,e_n\rangle$ and $b_n=\langle y,e_n\rangle$ are the Fourier coefficients of x and y, then

$$\langle x,y\rangle=\sum_{n=1}^{\infty}a_n\overline{b_n}, \tag{1.7}$$

the so-called *generalized Parseval equality*.

Motivated by the above results, in this paper we establish an extension and a weighted generalization of Brown-Plum inequality to functions with values in real Hilbert spaces. Examples for trapezoid type inequalities are also given.

2. Main Results

We assume in what follows that H is a *separable real Hilbert space*.

Theorem 5. *Assume that $f : [a,b] \to H$ is absolutely continuous on $[a,b]$ and $f' \in L_2([a,b],H)$. If $\int_a^b f(t)\,dt = 0$, then*

$$\int_a^b |\langle f'(t), f(t)\rangle|\,dt \le \frac{1}{4}(b-a)\int_a^b \|f'(t)\|^2\,dt. \tag{2.1}$$

The constant $\frac{1}{4}$ is best possible.

Proof. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis of H. Define $u : [a,b] \to \mathbb{R}$, $u(t) = \langle f(t), e_n\rangle$. Then

$$\int_a^b u(t)\,dt = \int_a^b \langle f(t), e_n\rangle\,dt = \left\langle \int_a^b f(t)\,dt, e_n \right\rangle = 0$$

for all $n \in \mathbb{N}$.

Also $u'(t) = (\langle f(t), e_n\rangle)' = \langle f'(t), e_n\rangle$ for all $n \in \mathbb{N}$.

Writing the inequality (1.1) for this function, we get

$$\int_a^b |\langle f(t), e_n\rangle \langle f'(t), e_n\rangle|\,dt \le \frac{1}{4}(b-a)\int_a^b |\langle f'(t), e_n\rangle|^2\,dt,$$

namely

$$\int_a^b |\langle f(t), e_n\rangle \langle e_n, f'(t)\rangle|\,dt \le \frac{1}{4}(b-a)\int_a^b |\langle f'(t), e_n\rangle|^2\,dt, \tag{2.2}$$

for all $n \in \mathbb{N}$.

Summing over n in (2.2), we get

$$\sum_{n=1}^{\infty}\int_a^b |\langle f(t), e_n\rangle \langle e_n, f'(t)\rangle|\,dt \le \frac{1}{4}(b-a)\sum_{n=1}^{\infty}\int_a^b |\langle f'(t), e_n\rangle|^2\,dt,$$

which gives

$$\int_a^b \left(\sum_{n=1}^{\infty} |\langle f(t), e_n\rangle \langle e_n, f'(t)\rangle|\right) dt \le \frac{1}{4}(b-a)\int_a^b \left(\sum_{n=1}^{\infty} |\langle f'(t), e_n\rangle|^2\right) dt. \tag{2.3}$$

By the triangle inequality we have

$$\left|\sum_{n=1}^{\infty} \langle f(t), e_n\rangle \langle e_n, f'(t)\rangle\right| \le \sum_{n=1}^{\infty} |\langle f(t), e_n\rangle \langle e_n, f'(t)\rangle|$$

and by (2.3) we get

$$\int_a^b \left|\sum_{n=1}^{\infty} \langle f(t), e_n\rangle \langle e_n, f'(t)\rangle\right| \le \frac{1}{4}(b-a)\int_a^b \left(\sum_{n=1}^{\infty} |\langle f'(t), e_n\rangle|^2\right) dt. \tag{2.4}$$

Using Parseval's identity (1.7) we have for $t\in[a,b]$ that

$$\sum_{n=1}^{\infty}\langle f(t),e_n\rangle\langle e_n,f'(t)\rangle=\langle f(t),f'(t)\rangle$$

and

$$\sum_{n=1}^{\infty}|\langle f'(t),e_n\rangle|^2=\|f'(t)\|^2$$

and by (2.4) we derive the desired result (2.1).

The sharpness of the constant follows by the scalar case. □

We have:

Theorem 6. *Let $h:[a,b]\to[h(a),h(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b). Assume that $f:[a,b]\subset\mathbb{R}\to H$ is an absolutely continuous function on the interval $[a,b]$ and such that $\frac{f'}{[h']^{1/2}}\in L_2[a,b]$. If*

$$\int_a^b f(t)\,h'(t)\,dt=0, \tag{2.5}$$

then

$$\int_a^b|\langle f(t),f'(t)\rangle|\,dt\le\frac{1}{4}[h(b)-h(a)]\int_a^b\frac{\|f'(t)\|^2}{h'(t)}dt. \tag{2.6}$$

Proof. Consider the function $u:=f\circ h^{-1}:[h(a),h(b)]\to H$. The function u is absolutely continuous on $[h(a),h(b)]$, $u(h(a))=f\circ h^{-1}(h(a))=f(a)=0$ and $u(h(b))=f\circ h^{-1}(h(b))=f(b)=0$.

Using the chain rule and the derivative of inverse functions we have

$$\left(f\circ h^{-1}\right)'(z)=\left(f'\circ h^{-1}\right)(z)\left(h^{-1}\right)'(z)=\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)} \tag{2.7}$$

for almost every (a.e.) $z\in[h(a),h(b)]$.

Also by the change of variable $t=h^{-1}(z)$, $z\in[h(a),h(b)]$, then $z=h(t)$, $dz=h'(t)\,dt$, and

$$\int_{h(a)}^{h(b)}f\circ h^{-1}(z)\,dz=\int_a^b f(t)\,h'(t)\,dt=0.$$

If we apply the inequality (2.1) for the function $u=f\circ h^{-1}$ on the interval $[h(a),h(b)]$, then we get

$$\begin{aligned}\int_{h(a)}^{h(b)}&\left|\left\langle f\circ h^{-1}(z),\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right\rangle\right|dz \\ &\le\frac{1}{4}[h(b)-h(a)]\int_{h(a)}^{h(b)}\left\|\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right\|^2dz.\end{aligned} \tag{2.8}$$

If we make the change of variable $t=h^{-1}(z)$, $z\in[h(a),h(b)]$, then

$$\begin{aligned}\int_{h(a)}^{h(b)}\left|\left\langle f\circ h^{-1}(z),\frac{\left(f'\circ h^{-1}\right)(z)}{\left(h'\circ h^{-1}\right)(z)}\right\rangle\right|dz&=\int_a^b\left|\left\langle f(t),\frac{f'(t)}{h'(t)}\right\rangle\right|h'(t)\,dt\\&=\int_a^b|\langle f(t),f'(t)\rangle|\,dt\end{aligned}$$

and

$$\int_{h(a)}^{h(b)} \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz = \int_a^b \left\| \frac{f'(t)}{h'(t)} \right\|^2 h'(t)\, dt = \int_a^b \frac{\|f'(t)\|^2}{h'(t)} dt.$$

By utilising (2.8), we then get the desired inequality (2.6). □

In what follows we consider the identity function $\ell(t) = t$.

a). If we take $h : [a,b] \subset (0,\infty) \to \mathbb{R}$, $h(t) = \ln t$ and assume that f is an absolutely continuous function with $\int_a^b \frac{f(t)}{t} dt = 0$ and $\ell^{1/2} f' \in L_2([a,b], H)$, then by (2.6) we get

$$\int_a^b |\langle f(t), f'(t)\rangle|\, dt \le \frac{1}{4} \ln\left(\frac{b}{a}\right) \int_a^b t \|f'(t)\|^2 dt. \tag{2.9}$$

b). If we take $h : [a,b] \subset \mathbb{R} \to (0,\infty)$, $h(t) = \exp t$ and assume that f is an absolutely continuous function with $\int_a^b f(t) \exp t dt = 0$ and $\exp\left(-\frac{1}{2}\ell\right) f' \in L_2([a,b], H)$, then by (2.6) we get

$$\int_a^b |\langle f(t), f'(t)\rangle|\, dt \le \frac{1}{4} (\exp b - \exp a) \int_a^b \exp(-t) \|f'(t)\|^2 dt. \tag{2.10}$$

c). If we take $h : [a,b] \subset (0,\infty) \to \mathbb{R}$, $h(t) = t^r$, $r > 0$ and assume that f is an absolutely continuous function with $\int_a^b f(t) t^{r-1} dt = 0$ and $\ell^{(1-r)/2} f' \in L_2[a,b]$, then by (2.1) we get

$$\int_a^b |\langle f(t), f'(t)\rangle|\, dt \le \frac{1}{4r} (b^r - a^r) \int_a^b \frac{\|f'(t)\|^2}{t^{r-1}} dt. \tag{2.11}$$

If $w : [a,b] \to \mathbb{R}$ is continuous and positive on the interval $[a,b]$, then the function $W : [a,b] \to [0,\infty)$, $W(x) := \int_a^x w(s)\, ds$ is strictly increasing and differentiable on (a,b). We have $W'(x) = w(x)$ for any $x \in (a,b)$.

Corollary 1. *Assume that* $w : [a,b] \to (0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\, ds = 1$ *and that* $f : [a,b] \subset \mathbb{R} \to H$ *is an absolutely continuous function on the interval* $[a,b]$ *and such that* $\frac{f'}{w^{1/2}} \in L_2([a,b], H)$. *If*

$$\int_a^b f(t) w(t)\, dt = 0, \tag{2.12}$$

then

$$\int_a^b |\langle f(t), f'(t)\rangle|\, dt \le \frac{1}{4} \int_a^b \frac{\|f'(t)\|^2}{w(t)} dt. \tag{2.13}$$

Similar results may be stated for the probability distributions that are supported on the whole axis $\mathbb{R} = (-\infty,\infty)$. Namely, if $f : \mathbb{R} \to H$ is locally absolutely continuous on $\mathbb{R}$, $w(s) > 0$ for $s \in \mathbb{R}$, $\int_{-\infty}^{\infty} w(s)\, ds = 1$ with $\frac{f}{w^{1/2}} \in L_2(\mathbb{R}, H)$ and

$$\int_{-\infty}^{\infty} f(t) w(t)\, dt = 0, \tag{2.14}$$

then

$$\int_{-\infty}^{\infty} |\langle f(t), f'(t)\rangle|\, dt \le \frac{1}{4} \int_{-\infty}^{\infty} \frac{\|f'(t)\|^2}{w(t)} dt. \tag{2.15}$$

In what follows we give an example.

The probability density of the *normal distribution* on $(-\infty,\infty)$ is

$$w_{\mu,\sigma^2}(x) := \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\ x\in\mathbb{R},$$

where μ is the *mean* or *expectation* of the distribution (and also its *median* and *mode*), σ is the *standard deviation*, and σ^2 is the *variance*.

The cumulative distribution function is

$$W_{\mu,\sigma^2}(x) = \frac{1}{2}+\frac{1}{2}\operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right),$$

where the *error function* erf is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp\left(-t^2\right)dt.$$

So, if $\frac{f}{w^{1/2}_{\mu,\sigma^2}} \in L_2(\mathbb{R},H)$ with

$$\int_{-\infty}^{\infty} f(t)\exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right)dt = 0, \tag{2.16}$$

then

$$\int_{-\infty}^{\infty}\left|\left\langle f(t), f'(t)\right\rangle\right|dt \le \frac{\sqrt{2\pi}\sigma}{4}\int_{-\infty}^{\infty}\left\|f'(t)\right\|^2\exp\left(\frac{(t-\mu)^2}{2\sigma^2}\right)dt. \tag{2.17}$$

3. Applications

We have:

Proposition 1. *Assume that $w:[a,b]\to(0,\infty)$ is continuous on $[a,b]$ with $\int_a^b w(s)\,ds = 1$ and that $h:[a,b]\subset\mathbb{R}\to H$ is an absolutely continuous function on the interval $[a,b]$ and such that $\frac{h'}{w^{1/2}}\in L_2([a,b],H)$. Then*

$$\left|\left\langle \frac{h(a)+h(b)}{2}-\int_a^b w(s)h(s)\,ds, h(b)-h(a)\right\rangle\right| \le \frac{1}{4}\int_a^b \frac{\left\|h'(t)\right\|^2}{w(t)}dt. \tag{3.1}$$

Proof. Consider the function

$$f(t) := h(t)-\int_a^b w(s)h(s)\,ds,\ t\in[a,b].$$

Then

$$\int_a^b\left(h(t)-\int_a^b w(s)h(s)\,ds\right)w(t)\,dt = 0,$$

and by the Corollary 1 we have

$$\int_a^b\left|\left\langle\left(h(t)-\int_a^b w(s)h(s)\,ds\right), h'(t)\right\rangle\right|dt \le \frac{1}{4}\int_a^b\frac{\left\|h'(t)\right\|^2}{w(t)}dt. \tag{3.2}$$

By the modulus and integral properties, we also have

$$\begin{aligned}
(3.3)\qquad & \int_a^b \left| \left\langle \left(h(t) - \int_a^b w(s) h(s) ds \right), h'(t) \right\rangle \right| dt \\
& \geq \left| \int_a^b \left\langle \left(h(t) - \int_a^b w(s) h(s) ds \right), h'(t) \right\rangle dt \right| \\
& = \left| \int_a^b \langle h(t), h'(t) \rangle dt - \left\langle \int_a^b w(s) h(s) ds, \int_a^b h'(t) dt \right\rangle \right| \\
& = \left| \frac{1}{2} \left(\|h(b)\|^2 - \|h(a)\|^2 \right) - \left\langle \int_a^b w(s) h(s) ds, h(b) - h(a) \right\rangle \right| \\
& = \left| \left\langle \frac{h(a) + h(b)}{2}, h(b) - h(a) \right\rangle - \left\langle \int_a^b w(s) h(s) ds, h(b) - h(a) \right\rangle \right| .
\end{aligned}$$

By utilising (3.2) and (3.3) we get the desired result (3.1). □

In the case of scalar functions, namely if $H = \mathbb{R}$, then we have the following trapezoid type inequality:

Corollary 2. *Assume that* $w : [a, b] \to (0, \infty)$ *is continuous on* $[a, b]$ *with* $\int_a^b w(s) ds = 1$ *and that* $h : [a, b] \subset \mathbb{R} \to \mathbb{R}$ *is an absolutely continuous function on the interval* $[a, b]$ *with* $h(b) \neq h(a)$ *and such that* $\frac{h'}{w^{1/2}} \in L_2[a, b]$. *Then*

$$\begin{aligned}
(3.4)\qquad & \left| \frac{h(a) + h(b)}{2} - \int_a^b w(s) h(s) ds \right| \\
& \leq \frac{1}{4} \frac{1}{|h(b) - h(a)|} \int_a^b \frac{[h'(t)]^2}{w(t)} dt.
\end{aligned}$$

Corollary 3. *Assume that* $h : [a, b] \subset \mathbb{R} \to \mathbb{R}$ *is an absolutely continuous function on the interval* $[a, b]$ *with* $h(b) \neq h(a)$ *and such that* $h' \in L_2[a, b]$. *Then*

$$\begin{aligned}
(3.5)\qquad & \left| \frac{h(a) + h(b)}{2} - \frac{1}{b - a} \int_a^b h(s) ds \right| \\
& \leq \frac{1}{4} \frac{b - a}{|h(b) - h(a)|} \int_a^b [h'(t)]^2 dt.
\end{aligned}$$

In 1906, Fejér [6], while studying trigonometric polynomials, obtained the following inequalities which generalize that of Hermite & Hadamard:

Theorem 7 (Fejér's Inequality). *Consider the integral* $\int_a^b h(x) w(x) dx$, *where* h *is a convex function in the interval* (a, b) *and* w *is a positive function in the same interval such that*

$$w(x) = w(a + b - x), \text{ for any } x \in [a, b]$$

i.e., $y = w(x)$ *is a symmetric curve with respect to the straight line which contains the point* $\left(\frac{1}{2}(a + b), 0\right)$ *and is normal to the* x*-axis. Under those conditions the following inequalities are valid:*

$$(3.6)\qquad h\left(\frac{a + b}{2}\right) \leq \frac{1}{\int_a^b w(x) dx} \int_a^b h(x) w(x) dx \leq \frac{h(a) + h(b)}{2}.$$

If h is concave on (a,b), then the inequalities reverse in (3.6).

If $w \equiv 1$, then (3.6) becomes the well known Hermite-Hadamard inequality

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b h(x)\,dx \leq \frac{h(a)+h(b)}{2}. \tag{3.7}$$

We have the following reverse of Fejér's inequality:

Corollary 4. *Let $h:[a,b]\to\mathbb{R}$ be a convex function with $h(b)\neq h(a)$ and $w:[a,b]\to(0,\infty)$ be continuous, symmetrical on $[a,b]$ and such that $\frac{h'}{w^{1/2}}\in L_2[a,b]$. Then*

$$\begin{aligned} 0 &\leq \frac{h(a)+h(b)}{2} - \frac{1}{\int_a^b w(s)\,ds}\int_a^b w(t)\,h(t)\,dt \\ &\leq \frac{1}{4}\frac{\int_a^b w(s)\,ds}{|h(b)-h(a)|}\int_a^b \frac{[h'(t)]^2}{w(t)}dt. \end{aligned} \tag{3.8}$$

In particular, we have the following reverse of the Hermite-Hadamard inequality

$$\begin{aligned} 0 &\leq \frac{h(a)+h(b)}{2} - \frac{1}{b-a}\int_a^b h(t)\,dt \\ &\leq \frac{1}{4}\frac{b-a}{|h(b)-h(a)|}\int_a^b [h'(t)]^2\,dt, \end{aligned} \tag{3.9}$$

provided that $h'\in L_2[a,b]$.

References

[1] P. R. Beesack, On an integral inequality of Z. Opial. *Trans. Am. Math. Soc.* **104** (1962), 470–475.

[2] D. W. Boyd, Best constants in a class of integral inequalities, *Pacific J. Math.* **30** (1969), 367-383.

[3] R. C. Brown and D. B. Hinton, Opial's inequality and oscilation of 2nd order equations, *Proc. Amer. Math. Soc.* **125** (1997), Number 4, 1123-1129.

[4] R. C. Brown and M. Plum, An Opial-type inequality with an integral boundary condition, *Proc. R. Soc. A* **461** (2005), 2635–2651. doi:10.1098/rspa.2005.1449.

[5] S. S. Dragomir, Some inequalities related to Vrânceanu's extension of Opial's result, Preprint *RGMIA Res. Rep. Coll.* **24** (2021), Art.

[6] L. Fejér, Über die Fourierreihen, II, (In Hungarian) *Math. Naturwiss, Anz. Ungar. Akad. Wiss.*, **24** (1906), 369-390.

[7] C. Olech, A simple proof of a certain result of Z. Opial. *Ann. Polon. Math.* **8** (1960), 61–63.

[8] Z. Opial, Sur une inégalité. *Ann. Polon. Math.* **8** (1960), 29–32.

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