SOME EXTENSIONS OF OPIAL'S INEQUALITIES IN BANACH SPACES

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Abstract. In this paper we establish some unweighted and weighted extensions of Opial's inequalities for functions with values in Banach spaces. Examples for *Lumer-Giles* and *lower* and *upper semi-inner products* are also given.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u'\in L_2[a,b]$.*

(i) *If $u(a)=u(b)=0$, then*

$$\int_a^b |u(t)u'(t)|\,dt\le\frac{1}{4}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.1}$$

with equality if and only if

$$u(t)=\begin{cases} c(t-a) & \text{if } a\le t\le\frac{a+b}{2},\\ c(b-t) & \text{if } \frac{a+b}{2}<t\le b,\end{cases}$$

where c is an arbitrary constant;

(ii) *If $u(a)=0$, then*

$$\int_a^b |u(t)u'(t)|\,dt\le\frac{1}{2}(b-a)\int_a^b |u'(t)|^2\,dt, \tag{1.2}$$

with equality if and only if $u(t)=c(t-a)$ for some constant c.

The inequality (1.1) was obtained by Olech in [19] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [20].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [3], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [12]-[14] and [22]. For some recent result related to Opial's inequality see [1], [2], [24] and [25].

In 1975, G. G. Vrânceanu extended Opial's inequality (1.2) for functions with values in Hilbert spaces $(H;\langle\cdot,\cdot\rangle)$ as follows:

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Theorem 2. *Assume that the function $f:[a,b]\to H$ has a continuous derivative and $f(a)=0$, then*

$$\int_a^b |\langle f(t), f'(t)\rangle|\,dt \le \frac{1}{2}(b-a)\int_a^b \|f'(t)\|^2\,dt. \tag{1.3}$$

In this paper we establish some unweighted and weighted extensions of Opial's inequalities for functions with values in Banach spaces. Examples for *Lumer-Giles* and *lower* and *upper semi-inner products* are also given.

2. Main Results

Let $(X;\|\cdot\|)$ be a complex Banach space. We have the following refinement and generalization for vector valued functions of the Opial inequalities:

Theorem 3. *Assume that $f:[a,b]\to X$ is strongly differentiable on $[a,b]$ and $f'\in L_2([a,b],X)$.*

(i) *If either $f(a)=0$ or $f(b)=0$, then*

$$\begin{aligned}\int_a^b \|f'(t)\|\,\|f(t)\|\,dt &\le \left(\int_a^b (t-a)\|f'(t)\|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)\|f'(t)\|^2\,dt\right)^{1/2}\\ &\le \frac{1}{2}(b-a)\int_a^b \|f'(t)\|^2\,dt.\end{aligned} \tag{2.1}$$

(ii) *If $f(a)=f(b)=0$, then*

$$\begin{aligned}\int_a^b \|f'(t)\|\,\|f(t)\|\,dt &\le \left[\int_a^b K(t)\|f'(t)\|^2\,dt\right]^{1/2}\left[\int_a^b \left|\frac{a+b}{2}-t\right|\|f'(t)\|^2\,dt\right]^{1/2}\\ &\le \frac{1}{4}(b-a)\int_a^b \|f'(t)\|^2\,dt,\end{aligned} \tag{2.2}$$

where

$$K(t):=\begin{cases} t-a \ \textit{if}\ a\le t\le \frac{a+b}{2},\\ b-t\ \textit{if}\ \frac{a+b}{2}<t\le b.\end{cases}$$

Proof. (i). Since $f(a)=0$, then $f(t)=\int_a^t f'(s)\,ds$ for $t\in[a,b]$. We have

$$\begin{aligned}\int_a^b \|f'(t)\|\,\|f(t)\|\,dt &= \int_a^b (t-a)^{1/2}\|f'(t)\|(t-a)^{-1/2}\|f(t)\|\,dt\\ &= \int_a^b (t-a)^{1/2}\|f'(t)\|(t-a)^{-1/2}\left\|\int_a^t f'(s)\,ds\right\|dt\\ &=: A.\end{aligned} \tag{2.3}$$

Using Cauchy-Bunyakovsky-Schwarz (CBS) integral inequality, we have

$$A \leq \left(\int_a^b \left[(t-a)^{1/2} \left\| f'(t) \right\| \right]^2 dt \right)^{1/2} \times \left(\int_a^b \left[(t-a)^{-1/2} \left\| \int_a^t f'(s)\, ds \right\| \right]^2 dt \right)^{1/2} = \left(\int_a^b (t-a) \left\| f'(t) \right\|^2 dt \right)^{1/2} \left(\int_a^b (t-a)^{-1} \left\| \int_a^t f'(s)\, ds \right\|^2 dt \right)^{1/2} =: B. \tag{2.4}$$

By (CBS) integral inequality we also have

$$(t-a)^{-1} \left\| \int_a^t f'(s)\, ds \right\|^2 \leq \int_a^t \left\| f'(s) \right\|^2 ds,$$

which gives

$$B \leq \left(\int_a^b (t-a) \left\| f'(t) \right\|^2 dt \right)^{1/2} \left(\int_a^b \left(\int_a^t \left\| f'(s) \right\|^2 ds \right) dt \right)^{1/2}. \tag{2.5}$$

Using integration by parts, we have

$$\int_a^b \left(\int_a^t \left\| f'(s) \right\|^2 ds \right) dt = b \int_a^b \left\| f'(s) \right\|^2 ds - \int_a^b t \left\| f'(t) \right\|^2 dt = \int_a^b (b-t) \left\| f'(t) \right\|^2$$

and by (2.4) we get the first inequality in (2.1).

The last part follows by the elementary inequality

$$\sqrt{\alpha\beta} \leq \frac{1}{2} (\alpha + \beta), \ \ \alpha, \beta \geq 0. \tag{2.6}$$

The case $f(b) = 0$ can be proved in a similar way and the details are omitted.

(ii). If we write the inequality (2.1) on the interval $\left[a, \frac{a+b}{2}\right]$, we have

$$\int_a^{\frac{a+b}{2}} \left\| f'(t) \right\| \left\| f(t) \right\| dt \leq \left(\int_a^{\frac{a+b}{2}} (t-a) \left\| f'(t) \right\|^2 dt \right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) \left\| f'(t) \right\|^2 dt \right)^{1/2} \tag{2.7}$$

and if we write the inequality (2.1) on the interval $\left[\frac{a+b}{2}, b\right]$, we have

$$\int_{\frac{a+b}{2}}^b \left\| f'(t) \right\| \left\| f(t) \right\| dt \leq \left(\int_{\frac{a+b}{2}}^b (b-t) \left\| f'(t) \right\|^2 dt \right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right) \left\| f'(t) \right\|^2 dt \right)^{1/2}. \tag{2.8}$$

If we add the inequalities (2.7) and (2.8) we get

$$\begin{aligned}
&\int_a^b \|f'(t)\| \|f(t)\| \, dt \\
&\leq \left(\int_a^{\frac{a+b}{2}} (t-a) \|f'(t)\|^2 \, dt \right)^{1/2} \left(\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) \|f'(t)\|^2 \, dt \right)^{1/2} \\
&+ \left(\int_{\frac{a+b}{2}}^b (b-t) \|f'(t)\|^2 \, dt \right)^{1/2} \left(\int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right) \|f'(t)\|^2 \, dt \right)^{1/2}
\end{aligned}$$

$$\begin{aligned}
&\leq \left[\int_a^{\frac{a+b}{2}} (t-a) \|f'(t)\|^2 \, dt + \int_{\frac{a+b}{2}}^b (b-t) \|f'(t)\|^2 \, dt \right]^{1/2} \\
&\times \left[\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t \right) \|f'(t)\|^2 \, dt + \int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2} \right) \|f'(t)\|^2 \, dt \right]^{1/2} \\
&= \left[\int_a^b K(t) \|f'(t)\|^2 \, dt \right]^{1/2} \left[\int_a^b \left| \frac{a+b}{2} - t \right| \|f'(t)\|^2 \, dt \right]^{1/2},
\end{aligned}$$

where for the last inequality we used the elementary (CBS) inequality

$$\alpha\beta + \gamma\delta \leq \left(\alpha^2 + \gamma^2 \right)^{1/2} \left(\beta^2 + \delta^2 \right)^{1/2}, \ \alpha, \ \beta, \ \gamma, \ \delta \geq 0.$$

The last part follows by (2.6), namely

$$\begin{aligned}
&\left[\int_a^b K(t) \|f'(t)\|^2 \, dt \right]^{1/2} \left[\int_a^b \left| \frac{a+b}{2} - t \right| \|f'(t)\|^2 \, dt \right]^{1/2} \\
&\leq \frac{1}{2} \left[\int_a^b K(t) \|f'(t)\|^2 \, dt + \int_a^b \left| \frac{a+b}{2} - t \right| \|f'(t)\|^2 \, dt \right] \\
&= \frac{1}{2} \int_a^b \left[K(t) + \left| \frac{a+b}{2} - t \right| \right] \|f'(t)\|^2 \, dt = \frac{1}{4} \int_a^b \|f'(t)\|^2 \, dt,
\end{aligned}$$

since

$$K(t) + \left| \frac{a+b}{2} - t \right| = \frac{1}{2} (b-a) \ \text{ for } t \in [a,b].$$

□

Remark 1. *The inequality (2.2) can also be written as*

$$\begin{aligned}(2.9)\qquad &\int_a^b \|f'(t)\| \|f(t)\| dt \\ &\leq \left[\frac{1}{2}(b-a)\int_a^b \|f'(t)\|^2 dt - \int_a^b \left|\frac{a+b}{2} - t\right| \|f'(t)\|^2 dt\right]^{1/2} \\ &\times \left[\int_a^b \left|\frac{a+b}{2} - t\right| \|f'(t)\|^2 dt\right]^{1/2} \\ &\leq \frac{1}{4}(b-a)\int_a^b \|f'(t)\|^2 dt.\end{aligned}$$

We also have the following composite inequality:

Theorem 4. *Let $h : [a,b] \to [h(a), h(b)]$ be a continuous strictly increasing function that is of class C^1 on (a,b). Assume that $f : [a,b] \subset \mathbb{R} \to X$ is a strongly differentiable vector valued function on the interval $[a,b]$ and such that $\frac{f'}{[h']^{1/2}} \in L_2([a,b], X)$.*

(i) *If $f(a) = 0$ or $f(b) = 0$, then*

$$\begin{aligned}(2.10)\qquad &\int_a^b \|f'(t)\| \|f(t)\| dt \\ &\leq \left(\int_a^b \frac{[h(t)-h(a)] \|f'(t)\|^2}{h'(t)} dt\right)^{1/2} \left(\int_a^b \frac{[h(b)-h(t)] \|f'(t)\|^2}{h'(t)} dt\right)^{1/2} \\ &\leq \frac{1}{2}[h(b)-h(a)] \int_a^b \frac{\|f'(t)\|^2}{h'(t)} dt.\end{aligned}$$

(ii) *If $f(a) = f(b) = 0$, then*

$$\begin{aligned}(2.11)\qquad \int_a^b \|f'(t)\| \|f(t)\| dt &\leq \left[\frac{1}{2}[h(b)-h(a)] \int_a^b \frac{\|f'(t)\|^2}{h'(t)} dt\right. \\ &\left. - \int_a^b \left|\frac{h(a)+h(b)}{2} - h(t)\right| \frac{\|f'(t)\|^2}{h'(t)} dt\right]^{1/2} \\ &\times \left[\int_a^b \left|\frac{h(a)+h(b)}{2} - h(t)\right| \frac{\|f'(t)\|^2}{h'(t)} dt\right]^{1/2} \\ &\leq \frac{1}{4}[h(b)-h(a)] \int_a^b \frac{\|f'(t)\|^2}{h'(t)} dt.\end{aligned}$$

Proof. (i). Consider the function $u := f \circ h^{-1} : [h(a), h(b)] \to \mathbb{R}$. The function u is absolutely continuous on $[h(a), h(b)]$, $u(h(a)) = f \circ h^{-1}(h(a)) = f(a) = 0$ or $u(h(b)) = f \circ h^{-1}(h(b)) = f(b) = 0$.

Using the chain rule and the derivative of inverse functions we have

$$(2.12)\qquad \left(f \circ h^{-1}\right)'(z) = \left(f' \circ h^{-1}\right)(z) \left(h^{-1}\right)'(z) = \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)}$$

for almost every (a.e.) $z \in [h(a), h(b)]$.

If we apply the inequality (2.1) for the function $u = f \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$\begin{aligned}
\int_{h(a)}^{h(b)} \left\| f \circ h^{-1}(z) \right\| \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\| dz
&\leq \left(\int_{h(a)}^{h(b)} (z - h(a)) \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz \right)^{1/2} \\
&\times \left(\int_{h(a)}^{h(b)} (h(b) - z) \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz \right)^{1/2} \\
&\leq \frac{1}{2} [h(b) - h(a)] \int_{h(a)}^{h(b)} \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz.
\end{aligned} \tag{2.13}$$

If we make the change of variable $t = h^{-1}(z)$, $z \in [h(a), h(b)]$, then $z = h(t)$, $dz = h'(t)\,dt$,

$$\begin{aligned}
\int_{h(a)}^{h(b)} \left\| f \circ h^{-1}(z) \right\| \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\| dz &= \int_a^b \| f(t) \| \left\| \frac{f'(t)}{h'(t)} \right\| h'(t)\,dt \\
&= \int_a^b \| f'(t) \| \| f(t) \| \, dt,
\end{aligned}$$

$$\begin{aligned}
\int_{h(a)}^{h(b)} (z - h(a)) \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz &= \int_a^b [h(t) - h(a)] \left\| \frac{f'(t)}{h'(t)} \right\|^2 h'(t)\,dt \\
&= \int_a^b [h(t) - h(a)] \frac{\| f'(t) \|^2}{h'(t)} dt
\end{aligned}$$

$$\begin{aligned}
\int_{h(a)}^{h(b)} (h(b) - z) \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz &= \int_{h(a)}^{h(b)} [h(b) - h(t)] \left\| \frac{f'(t)}{h'(t)} \right\|^2 h'(t)\,dt \\
&= \int_a^b [h(b) - h(t)] \frac{\| f'(t) \|^2}{h'(t)} dt
\end{aligned}$$

and

$$\int_{h(a)}^{h(b)} \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz = \int_a^b \left\| \frac{f'(t)}{h'(t)} \right\|^2 h'(t)\,dt = \int_a^b \frac{\| f'(t) \|^2}{h'(t)} dt.$$

By utilising (2.13), we then get the desired inequality (2.10).

(ii). By using the inequality (2.2) for the function $u = f \circ h^{-1}$ on the interval $[h(a), h(b)]$, then we get

$$
\begin{aligned}
(2.14) \qquad & \int_{h(a)}^{h(b)} \left\| f \circ h^{-1}(z) \right\| \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\| dz \\
& \leq \left[\int_{h(a)}^{h(b)} \left(\frac{1}{2}\left(h(b) - h(a)\right) - \left| \frac{h(a) + h(b)}{2} - z \right| \right) \right. \\
& \quad \left. \times \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz \right]^{1/2} \\
& \quad \times \left[\int_a^b \left| \frac{h(a) + h(b)}{2} - z \right| \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz \right]^{1/2} \\
& \leq \frac{1}{4}\left[h(b) - h(a)\right] \int_{h(a)}^{h(b)} \left\| \frac{\left(f' \circ h^{-1}\right)(z)}{\left(h' \circ h^{-1}\right)(z)} \right\|^2 dz.
\end{aligned}
$$

If we make the change of variable $t = h^{-1}(z)$, $z \in [h(a), h(b)]$, then by (2.14) we get the desired result (2.11). □

If $w : [a, b] \to \mathbb{R}$ is continuous and positive on the interval $[a, b]$, then the function $W : [a, b] \to [0, \infty)$, $W(x) := \int_a^x w(s)\, ds$ is strictly increasing and differentiable on (a, b). We have $W'(x) = w(x)$ for any $x \in (a, b)$.

Corollary 1. *Assume that $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$ and that $f : [a, b] \subset \mathbb{R} \to X$ is a strongly differentiable vector valued function on the interval $[a, b]$ and such that $\frac{f'}{w^{1/2}} \in L_2([a, b], X)$.*

(i) *If $f(a) = 0$ or $f(b) = 0$, then*

$$
\begin{aligned}
(2.15) \qquad \int_a^b \left\| f'(t) \right\| \left\| f(t) \right\| dt & \leq \left(\int_a^b \left(\int_a^t w(s)\, ds \right) \frac{\left\| f'(t) \right\|^2}{w(t)} dt \right)^{1/2} \\
& \quad \times \left(\int_a^b \left(\int_t^b w(s)\, ds \right) \frac{\left\| f'(t) \right\|^2}{w(t)} dt \right)^{1/2} \\
& \leq \frac{1}{2} \int_a^b w(s)\, ds \int_a^b \frac{\left\| f'(t) \right\|^2}{w(t)} dt.
\end{aligned}
$$

(ii) *If* $f(a) = f(b) = 0$, *then*

$$\int_a^b \|f'(t)\| \|f(t)\| dt \leq \frac{1}{2}\left[\int_a^b w(s) ds \int_a^b \frac{\|f'(t)\|^2}{w(t)} dt - \int_a^b \left|\int_t^b w(s) ds - \int_a^t w(s) ds\right| \frac{\|f'(t)\|^2}{w(t)} dt\right]^{1/2} \times \left[\int_a^b \left|\int_t^b w(s) ds - \int_a^t w(s) ds\right| \frac{\|f'(t)\|^2}{w(t)} dt\right]^{1/2} \leq \frac{1}{4}\int_a^b w(s) ds \int_a^b \frac{\|f'(t)\|^2}{w(t)} dt. \tag{2.16}$$

We give now some examples.

Consider the function $h(t) = \ln t$, $t \in [a,b] \subset (0,\infty)$. Assume that $f : [a,b] \subset \mathbb{R} \to X$ is an absolutely continuous vector valued function on the interval $[a,b]$ and such that $\ell^{1/2} f' \in L_2([a,b], X)$, where $\ell(t) = t$.

If $f(a) = 0$ or $f(b) = 0$, then

$$\int_a^b \|f'(t)\| \|f(t)\| dt \leq \left(\int_a^b t \ln\left(\frac{t}{a}\right) \|f'(t)\|^2 dt\right)^{1/2} \left(\int_a^b t \ln\left(\frac{b}{t}\right) \|f'(t)\|^2 dt\right)^{1/2} \leq \frac{1}{2}\ln\left(\frac{b}{a}\right) \int_a^b t \|f'(t)\|^2 dt. \tag{2.17}$$

If $f(a) = f(b) = 0$, then

$$\int_a^b \|f'(t)\| \|f(t)\| dt \leq \left[\ln\left(\sqrt{\frac{b}{a}}\right) \int_a^b t \|f'(t)\|^2 dt - \int_a^b \left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right| t \|f'(t)\|^2 dt\right]^{1/2} \times \left[\int_a^b \left|\ln\left(\frac{\sqrt{ab}}{t}\right)\right| t \|f'(t)\|^2 dt\right]^{1/2} \leq \frac{1}{4}\ln\left(\frac{b}{a}\right) \int_a^b t \|f'(t)\|^2 dt. \tag{2.18}$$

Consider the function $h(t) = \frac{t^2}{2}$, $t \in [a,b] \subset (0,\infty)$. Assume that $f : [a,b] \subset \mathbb{R} \to X$ is an absolutely continuous vector valued function on the interval $[a,b]$ and such that $\frac{f'}{\ell^{1/2}} \in L_2([a,b], X)$, where $\ell(t) = t$.

If $f(a)=0$ or $f(b)=0$, then

$$\begin{aligned}(2.19)\qquad & \int_a^b \|f'(t)\|\,\|f(t)\|\,dt \\ & \leq \frac{1}{2}\left(\int_a^b \frac{(t^2-a^2)\,\|f'(t)\|^2}{t}dt\right)^{1/2}\left(\int_a^b \frac{(b^2-t^2)\,\|f'(t)\|^2}{t}dt\right)^{1/2} \\ & \leq \frac{1}{4}\left(b^2-a^2\right)\int_a^b \frac{\|f'(t)\|^2}{t}dt.\end{aligned}$$

If $f(a)=f(b)=0$, then

$$\begin{aligned}(2.20)\qquad & \int_a^b \|f'(t)\|\,\|f(t)\|\,dt \\ & \leq \frac{1}{2}\left[\frac{b^2-a^2}{2}\int_a^b \frac{\|f'(t)\|^2}{t}dt - \int_a^b \left|\frac{a^2+b^2}{2}-t^2\right|\frac{\|f'(t)\|^2}{t}dt\right]^{1/2} \\ & \times\left[\int_a^b \left|\frac{a^2+b^2}{2}-t^2\right|\frac{\|f'(t)\|^2}{t}dt\right]^{1/2} \leq \frac{1}{8}\left(b^2-a^2\right)\int_a^b \frac{\|f'(t)\|^2}{t}dt.\end{aligned}$$

3. The Case of Semi-Inner Products

In what follows, we assume that X is a linear space over the real or complex number field $\mathbb{K}$.

The following concept was introduced in 1961 by G. Lumer [9] but the main properties of it were discovered by J. R. Giles [10], P. L. Papini [21], P. M. Miličić [15]–[17], I. Roşca [23], B. Nath [18] and others, see [5].

In this introductory section we give the definition of this concept and point out the main facts which are derived directly from the definition.

Definition 1. *The mapping $[\cdot,\cdot]: X\times X\to\mathbb{K}$ will be called the semi-inner product in the sense of Lumer-Giles or L-G-s.i.p., for short, if the following properties are satisfied:*

(i) $[x+y,z]=[x,z]+[y,z]$ *for all* $x,\ y,\ z\in X$;
(ii) $[\lambda x,y]=\lambda[x,y]$ *for all* $x,\ y\in X$ *and* λ *a scalar in* $\mathbb{K}$;
(iii) $[x,x]\geq 0$ *for all* $x\in X$ *and* $[x,x]=0$ *implies that* $x=0$;
(iv) $|[x,y]|^2\leq [x,x]\,[y,y]$ *for all* $x,\ y\in X$;
(v) $[x,\lambda y]=\bar{\lambda}[x,y]$ *for all* $x,\ y\in X$ *and* λ *a scalar in* $\mathbb{K}$.

The following results collects some fundamental facts concerning the connection between the semi-inner products and norms.

Proposition 1. *Let X be a linear space and $[\cdot,\cdot]$ a L-G-s.i.p on X. Then the following statements are true:*

(i) *The mapping* $X\ni x\overset{\|\cdot\|}{\longrightarrow}[x,x]^{\frac{1}{2}}\in\mathbb{R}_+$ *is a norm on* X;
(ii) *For every* $y\in X$ *the functional* $X\ni x\overset{f_y}{\longrightarrow}[x,y]\in\mathbb{K}$ *is a continuous linear functional on* X *endowed with the norm generated by the L-G-s.i.p. Moreover, one has the equality* $\|f_y\|=\|y\|$.

Definition 2. *The mapping $J : X \to 2^{X^*}$, where X^* is the dual space of X, given by:*

$$J(x) := \{x^* \in X^* | \langle x^*, x\rangle = \|x^*\| \|x\|, \ \ \|x^*\| = \|x\|\}, \ \ x \in X$$

will be called the normalised duality mapping of normed linear space $(X, \|\cdot\|)$.

Definition 3. *A mapping $\tilde{J} : X \to X^*$ will be called a section of normalised duality mapping if $\tilde{J}(x) \in J(x)$ for all x in X.*

The following theorem due to I. Roşca [23] establishes a natural connection between the normalised duality mapping and the semi-inner products in the sense of Lumer-Giles.

Theorem 5. *Let $(X, \|\cdot\|)$ be a normed space. Then every L-G-s.i.p. which generates the norm $\|\cdot\|$ is of the form*

$$[x, y] = \left\langle \tilde{J}(y), x \right\rangle \ \ \text{for all } x, y \text{ in } X,$$

where $\tilde{J}$ is a section of the normalised duality mapping.

The following proposition is a natural consequence of Roşca's result.

Proposition 2. *Let $(X, \|\cdot\|)$ be a normed linear space. Then the following statements are equivalent:*

(i) *X is smooth;*
(ii) *There exists a unique L-G-s.i.p. which generates the norm $\|\cdot\|$.*

Since, in general a semi-inner product $[\cdot, \cdot]$ is not continuous in the second variable, the existence of the Lebesgue integral $\int_a^b [f'(t), f(t)]\, dt$ must be postulated.

Proposition 3. *Let $(X, \|\cdot\|)$ be a normed linear space and $[\cdot, \cdot]$ a L-G-s.i.p. which generates the norm $\|\cdot\|$. Assume that $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$ and that $f : [a, b] \subset \mathbb{R} \to X$ is a strongly differentiable vector valued function on the interval $[a, b]$ such that $[f'(\cdot), f(\cdot)] \in L([a, b], \mathbb{C})$ and $\frac{f'}{w^{1/2}} \in L_2([a, b], X)$. Then*

$$\begin{aligned}
(3.1) \qquad \int_a^b |[f'(t), f(t)]|\, dt &\leq \left(\int_a^b \left(\int_a^t w(s)\, ds \right) \frac{\|f'(t)\|^2}{w(t)} dt \right)^{1/2} \\
&\quad \times \left(\int_a^b \left(\int_t^b w(s)\, ds \right) \frac{\|f'(t)\|^2}{w(t)} dt \right)^{1/2} \\
&\leq \frac{1}{2} \int_a^b w(s)\, ds \int_a^b \frac{\|f'(t)\|^2}{w(t)} dt,
\end{aligned}$$

provided that either $f(a) = 0$ or $f(b) = 0$.

If $f(a) = f(b) = 0$, *then*

$$\int_a^b |[f'(t), f(t)]|\, dt \le \frac{1}{2}\left[\int_a^b w(s)\, ds \int_a^b \frac{\|f'(t)\|^2}{w(t)} dt - \int_a^b \left|\int_t^b w(s)\, ds - \int_a^t w(s)\, ds\right| \frac{\|f'(t)\|^2}{w(t)} dt\right]^{1/2} \times \left[\int_a^b \left|\int_t^b w(s)\, ds - \int_a^t w(s)\, ds\right| \frac{\|f'(t)\|^2}{w(t)} dt\right]^{1/2} \le \frac{1}{4}\int_a^b w(s)\, ds \int_a^b \frac{\|f'(t)\|^2}{w(t)} dt. \tag{3.2}$$

The proof follows by Corollary 1 and by observing that, by Schwarz inequality

$$\int_a^b |[f'(t), f(t)]|\, dt \le \int_a^b \|f'(t)\| \|f(t)\|\, dt.$$

4. The Case Lower and Upper Semi-Inner Products

Let X be a real linear space, $x, y \in X$, $x \neq y$ and let $[x, y] := \{(1-\lambda)x + \lambda y,\ \lambda \in [0,1]\}$ be the *segment* generated by x and y. We consider the function $f : [x, y] \to \mathbb{R}$ and the attached function $\varphi_{(x,y)} : [0,1] \to \mathbb{R}$, $\varphi_{(x,y)}(t) := f[(1-t)x + ty]$, $t \in [0,1]$.

It is well known that f is convex on $[x, y]$ iff $\varphi(x, y)$ is convex on $[0,1]$, and the following lateral derivatives exist and satisfy

(i) $\varphi'_{\pm(x,y)}(s) = \nabla_{\pm} f_{(1-s)x+sy}(y - x)$, $s \in [0,1)$,
(ii) $\varphi'_{+(x,y)}(0) = \nabla_{+} f_x(y - x)$,
(iii) $\varphi'_{-(x,y)}(1) = \nabla_{-} f_y(y - x)$,

where $\nabla_{\pm} f_x(y)$ are the *Gâteaux lateral derivatives,* we recall that

$$\nabla_{+} f_x(y) := \lim_{h \to 0+} \frac{f(x + hy) - f(x)}{h},$$
$$\nabla_{-} f_x(y) := \lim_{k \to 0-} \frac{f(x + ky) - f(x)}{k}, \quad x,\ y \in X.$$

Now, assume that $(X, \|\cdot\|)$ is a normed linear space. The function $f_0(s) = \frac{1}{2}\|x\|^2$, $x \in X$ is convex and thus the following limits exist

(iv) $\langle x, y\rangle_s := \nabla_{+} f_{0,y}(x) = \lim_{t \to 0+} \frac{\|y+tx\|^2 - \|y\|^2}{2t}$;
(v) $\langle x, y\rangle_i := \nabla_{-} f_{0,y}(x) = \lim_{s \to 0-} \frac{\|y+sx\|^2 - \|y\|^2}{2s}$;

for any $x,\ y \in X$. They are called the *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

For the sake of completeness we list here some of the main properties of these mappings that will be used in the sequel (see for example [4] or [5]), assuming that $p,\ q \in \{s, i\}$ and $p \neq q$:

(a) $\langle x, x\rangle_p = \|x\|^2$ for all $x \in X$;
(aa) $\langle \alpha x, \beta y\rangle_p = \alpha\beta \langle x, y\rangle_p$ if $\alpha,\ \beta \ge 0$ and $x,\ y \in X$;
(aaa) $\left|\langle x, y\rangle_p\right| \le \|x\| \|y\|$ for all $x,\ y \in X$;
(av) $\langle \alpha x + y, x\rangle_p = \alpha \langle x, x\rangle_p + \langle y, x\rangle_p$ if $x,\ y \in X$ and $\alpha \in \mathbb{R}$;

(v) $\langle -x, y\rangle_p = -\langle x, y\rangle_q$ for all x, $y \in X$;

(va) $\langle x+y, z\rangle_p \leq \|x\| \|z\| + \langle y, z\rangle_p$ for all x, y, $z \in X$;

(vaa) The mapping $\langle \cdot, \cdot\rangle_p$ is continuous and subadditive (superadditive) in the first variable for $p = s$ (or $p = i$);

(vaaa) The normed linear space $(X, \|\cdot\|)$ is smooth at the point $x_0 \in X \setminus \{0\}$ if and only if $\langle y, x_0\rangle_s = \langle y, x_0\rangle_i$ for all $y \in X$; in general $\langle y, x\rangle_i \leq \langle y, x\rangle_s$ for all x, $y \in X$;

(ax) If the norm $\|\cdot\|$ is induced by an inner product $\langle \cdot, \cdot\rangle$, then $\langle y, x\rangle_i = \langle y, x\rangle = \langle y, x\rangle_s$ for all x, $y \in X$.

We also have the following result for *lower* and *upper semi-inner* products associated to the norm $\|\cdot\|$.

Proposition 4. *Let $(X, \|\cdot\|)$ be a normed linear space. Assume that $w : [a, b] \to (0, \infty)$ is continuous on $[a, b]$ and that $f : [a, b] \subset \mathbb{R} \to X$ is a strongly differentiable vector valued function on the interval $[a, b]$ such that $\langle f'(\cdot), f(\cdot)\rangle_p \in L([a, b], \mathbb{R})$ and $\frac{f'}{w^{1/2}} \in L_2([a, b], X)$. Then*

$$
\begin{aligned}
(4.1)\qquad \int_a^b \left|\langle f'(t), f(t)\rangle_p\right| dt &\leq \left(\int_a^b \left(\int_a^t w(s)\, ds\right) \frac{\|f'(t)\|^2}{w(t)} dt\right)^{1/2} \\
&\times \left(\int_a^b \left(\int_t^b w(s)\, ds\right) \frac{\|f'(t)\|^2}{w(t)} dt\right)^{1/2} \\
&\leq \frac{1}{2} \int_a^b w(s)\, ds \int_a^b \frac{\|f'(t)\|^2}{w(t)} dt,
\end{aligned}
$$

provided that either $f(a) = 0$ or $f(b) = 0$.

If $f(a) = f(b) = 0$, then

$$
\begin{aligned}
(4.2)\qquad \int_a^b \left|\langle f'(t), f(t)\rangle_p\right| dt &\leq \frac{1}{2} \left[\int_a^b w(s)\, ds \int_a^b \frac{\|f'(t)\|^2}{w(t)} dt\right. \\
&\left. - \int_a^b \left|\int_t^b w(s)\, ds - \int_a^t w(s)\, ds\right| \frac{\|f'(t)\|^2}{w(t)} dt\right]^{1/2} \\
&\times \left[\int_a^b \left|\int_t^b w(s)\, ds - \int_a^t w(s)\, ds\right| \frac{\|f'(t)\|^2}{w(t)} dt\right]^{1/2} \\
&\leq \frac{1}{4} \int_a^b w(s)\, ds \int_a^b \frac{\|f'(t)\|^2}{w(t)} dt.
\end{aligned}
$$

The proof follows by Corollary 1 and by observing that, by Schwarz inequality

$$
\int_a^b \left|\langle f'(t), f(t)\rangle_p\right| dt \leq \int_a^b \|f'(t)\| \|f(t)\| dt.
$$

References

[1] G. A. Anastassiou, Complex Opial type inequalities. *Rom. J. Math. Comput. Sci.* **9** (2019), no. 2, 93–97

[2] G.A. Anastassiou, Integer and fractional self adjoint operator Opial type inequalities. *J. Comput. Anal. Appl.* **23** (2017), no. 8, 1398–1411.

[3] P. R. Beesack, On an integral inequality of Z. Opial. *Trans. Am. Math. Soc.* **104** (1962), 470–475.

[4] I. Ciorănescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer Academic Publishers, Dordrecht, 1990.

[5] S. S. Dragomir, *Semi-inner Products and Applications*. Nova Science Publishers, Inc., Hauppauge, NY, 2004. x+222 pp. ISBN: 1-59033-947-9

[6] S. S. Dragomir, Some inequalities for semi-inner products on complex Banach spaces, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art. 157. [Online http://rgmia.org/papers/v20/v20a157.pdf].

[7] R. J. Fleming and J. E. Jamison, Hermitian and adjoint Abelian operators on certain Banach spaces, *Pacific J. Math.* **52** (1974), No. 1, 67-84.

[8] D. O. Koehler, A note on some operator theory in certain semi-inner-product spaces. *Proc. Amer. Math. Soc.* **30** 1971 363–366.

[9] G. Lumer, Semi-inner product spaces, *Trans. Amer. Math. Soc.,* **100** (1961), 29-43.

[10] J. R. Giles Classes of semi-inner product spaces, *Trans. Amer. Math. Soc.,* **116** (1967), 436-446.

[11] B. W. Glickfeld, On an inequality of Banach algebra geometry and semi-inner-product space theory, *Illinois J. Math.* **14** (1970), 76–81.

[12] L.-G. Hua, On an inequality of Opial. *Sci. Sinica* 14 (1965), 789–790.

[13] N. Levinson, On an inequality of Opial and Beesack. *Proc. Amer. Math. Soc.* **15** (1964), 565–566.

[14] C. L. Mallows, An even simpler proof of Opial's inequality. *Proc. Amer. Math. Soc.* **16** (1965), 173.

[15] P. M. Miličić, Sur le semi-produit scalaire dans quelques espaces vectorial normés, *Mat. Vesnik,* **8**(**23**) (1971), 181-185.

[16] P. M. Miličić, Une généralisation naturellle du produit scalaire dans une espace normé et son utilisation, *Pub. L'Inst. Mat.* (Belgrade), **42**(**56**) (1987), 63-70.

[17] P. M. Miličić, La fonctionelle g et quelques problèmes des meilleurs approximations dans des espaces normés, *Pub. L'Inst. Mat.* (Belgrade), **48**(**62**) (1990), 110-118.

[18] B. Nath, On a generalization of semi-inner product spaces, *Math. J. Okoyama Univ.,* **15**(1) (1971), 1-6.

[19] C. Olech, A simple proof of a certain result of Z. Opial. *Ann. Polon. Math.* **8** (1960), 61–63.

[20] Z. Opial, Sur une inégalité. *Ann. Polon. Math.* **8** (1960), 29–32.

[21] P. L. Papini, Un' asservatione sui prodotti semi-scalari negli spasi di Banach, *Boll. Un. Mat. Ital.,* **6** (1969), 684-689.

[22] R. N. Pederson, On an inequality of Opial, Beesack and Levinson. *Proc. Amer. Math. Soc.* **16** (1965), 174.

[23] I. Roşca, Semi-produits scalaires et réprésentation du type Riesz pour les fonctionélles linéaires et bornées sur les espace normés, *C.R. Acad. Sci. Paris,* **283** (19), 1976.

[24] S. H. Saker, D. M. Abdou and I. Kubiaczyk, Opial and Pólya type inequalities via convexity. *Fasc. Math.* No. **60** (2018), 145–159.

[25] M. Z. Sarıkaya, On the generalization of Opial type inequality for convex function. *Konuralp J. Math.* **7** (2019), no. 2, 456–461.

[26] G. G. Vrânceanu, On an inequality of Opial. *Bull. Math. Soc. Sci. Math. R.S. Roumanie* (N.S.) 17(65) (1973), 315-316 (1975).

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