SOME OPIAL LIKE INEQUALITIES FOR TWO FUNCTIONS AND APPLICATIONS

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Abstract. In this paper we show among others that, if $f, g : [a,b] \to \mathbb{C}$ are absolutely continuous with $f(a) = g(b) = 0$, then

$$\int_a^b |f(t) g(t)| \, dt \leq \frac{1}{4}(b-a)^2 \int_a^b \left[|f'(t)|^2 + |g'(t)|^2 \right] dt - \frac{1}{4} \int_a^b \left[(t-a)^2 |f'(t)|^2 + (b-t)^2 |g'(t)|^2 \right] dt,$$

provided the integrals in the right side are finite. In particular, if $f(a) = f(b) = 0$, then we have the sharp inequality

$$\int_a^b |f(t)|^2 \, dt \leq \int_a^b (b-t)(t-a) |f'(t)|^2 \, dt.$$

Some trapezoid and Grüss' type inequalities are also given.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u' \in L_2[a,b]$.*

(i) *If $u(a) = u(b) = 0$, then*

$$\int_a^b |u(t) u'(t)| \, dt \leq \frac{1}{4}(b-a) \int_a^b |u'(t)|^2 \, dt, \tag{1.1}$$

with equality if and only if

$$u(t) = \begin{cases} c(t-a) & \text{if } a \leq t \leq \frac{a+b}{2}, \\ c(b-t) & \text{if } \frac{a+b}{2} < t \leq b, \end{cases}$$

where c is an arbitrary constant;

(ii) *If $u(a) = 0$, then*

$$\int_a^b |u(t) u'(t)| \, dt \leq \frac{1}{2}(b-a) \int_a^b |u'(t)|^2 \, dt, \tag{1.2}$$

with equality if and only if $u(t) = c(t-a)$ for some constant c.

The inequality (1.1) was obtained by Olech in [10] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [11].

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Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [1], which is satisfied by those u vanishing only at a.

For various proofs of the above inequalities, see [6]-[9] and [13].

In the recent paper [3] we obtained the following two functions version of Opial inequalities above:

Theorem 2. *Assume that $f,\ g:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$ with $f',\ g'\in L_2[a,b]$.*

(i) *If $g(a)=0$, then*

$$\int_a^b |f'(t)\,g(t)|\,dt \leq \left(\int_a^b (t-a)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)\,|g'(t)|^2\,dt\right)^{1/2} \leq \frac{1}{2}\left[\int_a^b (t-a)\,|f'(t)|^2\,dt+\int_a^b (b-t)\,|g'(t)|^2\,dt\right]. \tag{1.3}$$

(ii) *If $g(b)=0$, then*

$$\int_a^b |f'(t)\,g(t)|\,dt \leq \left(\int_a^b (b-t)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (t-a)\,|g'(t)|^2\,dt\right)^{1/2} \leq \frac{1}{2}\left[\int_a^b (b-t)\,|f'(t)|^2\,dt+\int_a^b (t-a)\,|g'(t)|^2\,dt\right]. \tag{1.4}$$

(iii) *If $g(a)=g(b)=0$, then*

$$\begin{aligned}\int_a^b |f'(t)\,g(t)|\,dt &\leq \left(\int_a^b K(t)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b \left|\frac{a+b}{2}-t\right|\,|g'(t)|^2\,dt\right)^{1/2}\\ &\leq \frac{1}{2}\left[\int_a^b K(t)\,|f'(t)|^2\,dt+\int_a^b \left|\frac{a+b}{2}-t\right|\,|g'(t)|^2\,dt\right],\end{aligned} \tag{1.5}$$

where

$$K(t):=\begin{cases} t-a & \text{if } a\leq t\leq \frac{a+b}{2},\\ b-t & \text{if } \frac{a+b}{2}<t\leq b.\end{cases}$$

We have the following refinement of Opial inequalities (1.1) and (1.2):

Corollary 1. *Assume that $f:[a,b]\to\mathbb{C}$ are absolutely continuous on $[a,b]$ and $f'\in L_2[a,b]$.*

(i) *If either $f(a)=0$ or $f(b)=0$, then*

$$\int_a^b |f'(t)\,f(t)|\,dt \leq \left(\int_a^b (t-a)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b (b-t)\,|f'(t)|^2\,dt\right)^{1/2} \leq \frac{1}{2}(b-a)\int_a^b |f'(t)|^2\,dt. \tag{1.6}$$

(ii) *If* $f(a) = f(b) = 0$, *then*

$$\int_a^b |f'(t) f(t)| \, dt \le \left[\int_a^b K(t) |f'(t)|^2 \, dt \right]^{1/2} \left[\int_a^b \left| \frac{a+b}{2} - t \right| |f'(t)|^2 \, dt \right]^{1/2} \le \frac{1}{4}(b-a) \int_a^b |f'(t)|^2 \, dt. \tag{1.7}$$

2. Main Results

We consider the positive weight

$$w_a(t; a, b) := \frac{1}{2}\left[(b-a)^2 - (t-a)^2\right] = (b-t)\left(\frac{b+t}{2} - a\right),$$

for $t \in [a, b]$.

Theorem 3. *Assume that* f, $g : [a, b] \to \mathbb{C}$ *are absolutely continuous with* $f(a) = g(a) = 0$ *and* f', $g' \in L_{2,w_a}[a, b]$, *then*

$$\int_a^b |f(t) g(t)| \, dt \le \left(\int_a^b w_a(t; a, b) |f'(t)|^2 \, dt \right)^{1/2} \left(\int_a^b w_a(t; a, b) |g'(t)|^2 \, dt \right)^{1/2} \le \frac{1}{2} \int_a^b w_a(t; a, b) \left[|f'(t)|^2 + |g'(t)|^2 \right] dt. \tag{2.1}$$

The inequalities in (2.1) are sharp.

Proof. Since $f(a) = g(a) = 0$, hence $f(t) = \int_a^t f'(s) \, ds$ and $g(t) = \int_a^t g'(s) \, ds$ and we have

$$\int_a^b |f(t) g(t)| \, dt = \int_a^b \left| \int_a^t f'(s) \, ds \right| \left| \int_a^t g'(s) \, ds \right| dt = \int_a^b (t-a)(t-a)^{-1/2} \left| \int_a^t f'(s) \, ds \right| (t-a)^{-1/2} \left| \int_a^t g'(s) \, ds \right| dt =: A. \tag{2.2}$$

By Cauchy-Bunyakowsky-Schwarz integral inequality, we have

$$(t-a)^{-1/2} \left| \int_a^t f'(s) \, ds \right| \le \left(\int_a^t |f'(s)|^2 \, ds \right)^{1/2}$$

and

$$(t-a)^{-1/2} \left| \int_a^t g'(s) \, ds \right| \le \left(\int_a^t |g'(s)|^2 \, ds \right)^{1/2}$$

for all $t \in [a, b]$.

Therefore

$$A \leq \int_a^b (t-a) \left(\int_a^t |f'(s)|^2 ds \right)^{1/2} \left(\int_a^t |g'(s)|^2 ds \right)^{1/2} dt.$$

By utilising Cauchy-Bunyakowsky-Schwarz weighted integral inequality, we have

$$\begin{aligned}
(2.3)\quad & \int_a^b (t-a) \left(\int_a^t |f'(s)|^2 ds \right)^{1/2} \left(\int_a^t |g'(s)|^2 ds \right)^{1/2} dt \\
& \leq \left[\int_a^b (t-a) \left(\left(\int_a^t |f'(s)|^2 ds \right)^{1/2} \right)^2 dt \right]^{1/2} \\
& \times \left[\int_a^b (t-a) \left(\left(\int_a^t |g'(s)|^2 ds \right)^{1/2} \right)^2 dt \right]^{1/2} \\
& = \left[\int_a^b (t-a) \left(\int_a^t |f'(s)|^2 ds \right) dt \right]^{1/2} \left[\int_a^b (t-a) \left(\int_a^t |g'(s)|^2 ds \right) dt \right]^{1/2} \\
& =: B.
\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}
& \int_a^b (t-a) \left(\int_a^t |f'(s)|^2 ds \right) dt \\
& = \int_a^b \left(\int_a^t |f'(s)|^2 ds \right) d\left(\frac{(t-a)^2}{2} \right) \\
& = \left(\int_a^t |f'(s)|^2 ds \right) \left(\frac{(t-a)^2}{2} \right) \Bigg|_a^b - \int_a^b \frac{(t-a)^2}{2} |f'(t)|^2 dt \\
& = \frac{(b-a)^2}{2} \left(\int_a^b |f'(s)|^2 ds \right) - \int_a^b \frac{(t-a)^2}{2} |f'(t)|^2 dt \\
& = \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2} \right] |f'(t)|^2 dt
\end{aligned}$$

and

$$\int_a^b (t-a) \left(\int_a^t |g'(s)|^2 ds \right) dt = \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2} \right] |g'(t)|^2 dt.$$

Therefore

$$\begin{aligned}
(2.4)\quad B & \leq \left(\int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2} \right] |f'(t)|^2 dt \right)^{1/2} \\
& \times \left(\int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2} \right] |g'(t)|^2 dt \right)^{1/2} \\
& = \left(\int_a^b w_a(t;a,b) |f'(t)|^2 dt \right)^{1/2} \left(\int_a^b w_a(t;a,b) |g'(t)|^2 dt \right)^{1/2}.
\end{aligned}$$

By making use of (2.2)-(2.4) we derive the first part in (2.1). The second part follows by the arithmetic mean-geometric mean inequality,

$$\sqrt{\alpha\beta} \leq \frac{\alpha+\beta}{2}, \ \ \alpha, \beta \geq 0.$$

Now, consider $f(t) = g(t) = t - a$. Then

$$\int_a^b |f(t) g(t)| \, dt = \int_a^b (t-a)^2 \, dt = \frac{1}{3}(b-a)^3$$

and

$$\begin{aligned} &\frac{1}{2}\int_a^b w_a(t;a,b)\left[|f'(t)|^2 + |g'(t)|^2\right] dt \\ &= \int_a^b w_a(t;a,b)\, dt = \frac{1}{2}\int_a^b \left[(b-a)^2 - (t-a)^2\right] \\ &= \frac{1}{2}\left[(b-a)^3 - \frac{1}{3}(b-a)^3\right] = \frac{1}{3}(b-a)^3, \end{aligned}$$

which shows that all terms in (2.1) are equal with $\frac{1}{3}(b-a)^3$. □

Remark 1. *Assume that f' is absolutely continuous on $[a,b]$. If $f(a) = f'(a) = 0$ and $f',\ f'' \in L_{2,w_a}[a,b]$, then*

$$\begin{aligned} &\int_a^b |f(t) f'(t)| \, dt \\ &\leq \left(\int_a^b w_a(t;a,b)\,|f'(t)|^2 \, dt\right)^{1/2} \left(\int_a^b w_a(t;a,b)\,|f''(t)|^2 \, dt\right)^{1/2} \\ &\leq \frac{1}{2}\int_a^b w_a(t;a,b)\left[|f'(t)|^2 + |f''(t)|^2\right] dt. \end{aligned} \tag{2.5}$$

The inequality follows by (2.17) for $g = f'$.

Corollary 2. *Assume that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $f(a) = 0$ and $f' \in L_{2,w_a}[a,b]$, then*

$$\int_a^b |f(t)|^2 \, dt \leq \int_a^b w_a(t;a,b)\,|f'(t)|^2 \, dt. \tag{2.6}$$

The inequality in (2.6) is sharp.

Now consider the dual weight

$$w_b(t;a,b) := \frac{1}{2}\left[(b-a)^2 - (b-t)^2\right] = (t-a)\left(b - \frac{a+t}{2}\right),$$

for $t \in [a,b]$.

Theorem 4. *Assume that* $f,\ g:[a,b]\to\mathbb{C}$ *are absolutely continuous with* $f(b)=g(b)=0$ *and* $f',\ g'\in L_{2,w_b}[a,b]$, *then*

$$\begin{aligned}\int_a^b |f(t)\,g(t)|\,dt &\\ \le \left(\int_a^b w_b(t;a,b)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\,|g'(t)|^2\,dt\right)^{1/2} &\\ \le \frac{1}{2}\int_a^b w_b(t;a,b)\left[|f'(t)|^2+|g'(t)|^2\right]dt. &\end{aligned} \tag{2.7}$$

The inequalities in (2.7) are sharp.

Proof. Since $f(b)=g(b)=0$, hence $f(t)=-\int_t^b f'(s)\,ds$ and $g(t)=-\int_t^b g'(s)\,ds$ and we have

$$\begin{aligned}&\int_a^b |f(t)\,g(t)|\,dt\\ &=\int_a^b\left|\int_t^b f'(s)\,ds\right|\left|\int_t^b g'(s)\,ds\right|dt\\ &=\int_a^b (b-t)(b-t)^{-1/2}\left|\int_t^b f'(s)\,ds\right|(b-t)^{-1/2}\left|\int_t^b g'(s)\,ds\right|dt\\ &=:C.\end{aligned} \tag{2.8}$$

By Cauchy-Bunyakowsky-Schwarz integral inequality, we have

$$(b-t)^{-1/2}\left|\int_t^b f'(s)\,ds\right|\le\left(\int_t^b |f'(s)|^2\,ds\right)^{1/2}$$

and

$$(b-t)^{-1/2}\left|\int_t^b g'(s)\,ds\right|\le\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2}$$

for all $t\in[a,b]$.

Therefore

$$C\le\int_a^b (b-t)\left(\int_t^b |f'(s)|^2\,ds\right)^{1/2}\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2}dt.$$

By utilising Cauchy-Bunyakowsky-Schwarz weighted integral inequality, we have

$$\int_a^b (b-t)\left(\int_t^b |f'(s)|^2\,ds\right)^{1/2}\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2}dt \tag{2.9}$$

$$\leq \left[\int_a^b (b-t)\left(\left(\left(\int_t^b |f'(s)|^2 ds\right)^{1/2}\right)^2 dt\right]^{1/2}$$

$$\times \left[\int_a^b (b-t)\left(\left(\left(\int_t^b |g'(s)|^2 ds\right)^{1/2}\right)^2 dt\right]^{1/2}$$

$$= \left[\int_a^b (b-t)\left(\int_t^b |f'(s)|^2 ds\right) dt\right]^{1/2} \left[\int_a^b (b-t)\left(\int_t^b |g'(s)|^2 ds\right) dt\right]^{1/2}$$

$$=: D.$$

Using integration by parts, we have

$$\int_a^b (b-t)\left(\int_t^b |f'(s)|^2 ds\right) dt$$

$$= -\int_a^b \left(\int_t^b |f'(s)|^2 ds\right) d\left(\frac{(b-t)^2}{2}\right)$$

$$= -\left(\int_t^b |f'(s)|^2 ds\right)\left(\frac{(b-t)^2}{2}\right)\Bigg|_a^b - \int_a^b \frac{(b-t)^2}{2}|f'(t)|^2 dt$$

$$= \frac{(b-a)^2}{2}\left(\int_a^b |f'(s)|^2 ds\right) - \int_a^b \frac{(b-t)^2}{2}|f'(t)|^2 dt$$

$$= \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(b-t)^2}{2}\right]|f'(t)|^2 dt$$

and

$$\int_a^b (b-t)\left(\int_t^b |g'(s)|^2 ds\right) dt = \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(b-t)^2}{2}\right]|g'(t)|^2 dt.$$

Therefore

$$D \leq \left(\int_a^b \left[\frac{(b-a)^2}{2} - \frac{(b-t)^2}{2}\right]|f'(t)|^2 dt\right)^{1/2} \tag{2.10}$$

$$\times \left(\int_a^b \left[\frac{(b-a)^2}{2} - \frac{(b-t)^2}{2}\right]|g'(t)|^2 dt\right)^{1/2}$$

$$= \left(\int_a^b w_b(t;a,b)|f'(t)|^2 dt\right)^{1/2} \left(\int_a^b w_b(t;a,b)|g'(t)|^2 dt\right)^{1/2}.$$

By making use of (2.8)-(2.10) we derive the first part in (2.7). The second part follows by the arithmetic mean-geometric mean inequality,

$$\sqrt{\alpha\beta} \leq \frac{\alpha+\beta}{2},\ \alpha,\beta \geq 0. \tag{2.11}$$

The sharpness follows by taking $f(t) = b - t$, $t \in [a,b]$. □

Remark 2. *Assume that f' is absolutely continuous on $[a,b]$. If with $f(b) = f'(b) = 0$ and $f',\ f'' \in L_{2,w_b}[a,b]$, then*

$$\begin{aligned}\int_a^b |f(t) f'(t)|\, dt &\leq \left(\int_a^b w_b(t;a,b) |f'(t)|^2 dt\right)^{1/2} \left(\int_a^b w_b(t;a,b) |f''(t)|^2 dt\right)^{1/2} \\ &\leq \frac{1}{2}\int_a^b w_b(t;a,b) \left[|f'(t)|^2 + |f''(t)|^2\right] dt. \end{aligned} \tag{2.12}$$

Corollary 3. *Assume that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $f(b) = 0$ and $f' \in L_{2,w_b}[a,b]$, then*

$$\int_a^b |f(t)|^2 dt \leq \int_a^b w_b(t;a,b) |f'(t)|^2 dt. \tag{2.13}$$

The inequality in (2.13) is sharp.

We also have:

Theorem 5. *Assume that $f,\ g : [a,b] \to \mathbb{C}$ are absolutely continuous with $f(a) = g(b) = 0$ and $f' \in L_{2,w_a}[a,b]$, $g' \in L_{2,w_b}[a,b]$, then*

$$\begin{aligned}\int_a^b |f(t) g(t)|\, dt &\leq \left(\int_a^b w_a(t;a,b) |f'(t)|^2 dt\right)^{1/2} \left(\int_a^b w_b(t;a,b) |g'(t)|^2 dt\right)^{1/2} \\ &\leq \frac{1}{4}(b-a)^2 \int_a^b \left[|f'(t)|^2 + |g'(t)|^2\right] dt \\ &- \frac{1}{4}\int_a^b \left[(t-a)^2 |f'(t)|^2 + (b-t)^2 |g'(t)|^2\right] dt. \end{aligned} \tag{2.14}$$

The inequalities in (2.7) are sharp.

Proof. Since $f(a) = g(b) = 0$, hence $f(t) = \int_a^t f'(s)\, ds$ and $g(t) = -\int_t^b g'(s)\, ds$. Therefore

$$\begin{aligned}\int_a^b |f(t) g(t)|\, dt &= \int_a^b \left|\int_a^t f'(s)\, ds\right| \left|\int_t^b g'(s)\, ds\right| dt \\ &= \int_a^b (t-a)^{1/2} (b-t)^{1/2} (t-a)^{-1/2} \\ &\times \left|\int_a^t f'(s)\, ds\right| (b-t)^{-1/2} \left|\int_t^b g'(s)\, ds\right| dt \\ &=: E. \end{aligned} \tag{2.15}$$

By Cauchy-Bunyakowsky-Schwarz integral inequality, we have

$$(t-a)^{-1/2} \left|\int_a^t f'(s)\, ds\right| \leq \left(\int_a^t |f'(s)|^2 ds\right)^{1/2}$$

and

$$(b-t)^{-1/2}\left|\int_t^b g'(s)\,ds\right| \le \left(\int_t^b |g'(s)|^2\,ds\right)^{1/2},$$

which imply that

$$\begin{aligned} E &\le \int_a^b (t-a)^{1/2}(b-t)^{1/2}\left(\int_a^t |f'(s)|^2\,ds\right)^{1/2}\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2} \\ &= \int_a^b (t-a)^{1/2}\left(\int_a^t |f'(s)|^2\,ds\right)^{1/2}(b-t)^{1/2}\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2}. \end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz integral inequality, we also have

$$\begin{aligned} &\int_a^b (t-a)^{1/2}\left(\int_a^t |f'(s)|^2\,ds\right)^{1/2}(b-t)^{1/2}\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2} \\ &\le \left(\int_a^b \left[(t-a)^{1/2}\left(\int_a^t |f'(s)|^2\,ds\right)^{1/2}\right]^2 dt\right)^{1/2} \\ &\times \left(\int_a^b \left[(b-t)^{1/2}\left(\int_t^b |g'(s)|^2\,ds\right)^{1/2}\right]^2 dt\right)^{1/2} \\ &= \left(\int_a^b (t-a)\left(\int_a^t |f'(s)|^2\,ds\right)\right)^{1/2}\left(\int_a^b (b-t)\left(\int_t^b |g'(s)|^2\,ds\right)dt\right)^{1/2} \\ &= \left(\int_a^b w_a(t;a,b)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\,|g'(t)|^2\,dt\right)^{1/2}, \end{aligned}$$

which proves the first inequality in (2.14).

By (2.11) inequality, we have that

$$\begin{aligned} &\left(\int_a^b w_a(t;a,b)\,|f'(t)|^2\,dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\,|g'(t)|^2\,dt\right)^{1/2} \\ &\le \frac{1}{2}\left[\int_a^b w_a(t;a,b)\,|f'(t)|^2\,dt + \int_a^b w_b(t;a,b)\,|g'(t)|^2\,dt\right] \\ &= \frac{1}{2}\left[\int_a^b \frac{1}{2}\left[(b-a)^2-(t-a)^2\right]|f'(t)|^2\,dt \right. \\ &\left. + \int_a^b \frac{1}{2}\left[(b-a)^2-(b-t)^2\right]|g'(t)|^2\,dt\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\int_a^b \frac{1}{2}\left[(b-a)^2-(t-a)^2\right]\left|f'(t)\right|^2 dt\right. \\
&\left.+\int_a^b \frac{1}{2}\left[(b-a)^2-(b-t)^2\right]\left|g'(t)\right|^2 dt\right] \\
&= \frac{1}{4}(b-a)^2\int_a^b\left[\left|f'(t)\right|^2+\left|g'(t)\right|^2\right]dt \\
&-\frac{1}{4}\int_a^b\left[(t-a)^2\left|f'(t)\right|^2+(b-t)^2\left|g'(t)\right|^2\right]dt
\end{aligned}
$$

and the second part of inequality (2.14) is proved. □

Remark 3. *Assume that f' is absolutely continuous on $[a,b]$. If $f(a)=f'(b)=0$ and $f'\in L_{2,w_a}[a,b]$, $f''\in L_{2,w_b}[a,b]$, then*

$$
\begin{aligned}
(2.16)\qquad &\int_a^b \left|f(t)f'(t)\right|dt \\
&\le \left(\int_a^b w_a(t;a,b)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\left|f''(t)\right|^2 dt\right)^{1/2} \\
&\le \frac{1}{4}(b-a)^2\int_a^b\left[\left|f'(t)\right|^2+\left|f''(t)\right|^2\right]dt \\
&-\frac{1}{4}\int_a^b\left[(t-a)^2\left|f'(t)\right|^2+(b-t)^2\left|f''(t)\right|^2\right]dt.
\end{aligned}
$$

Corollary 4. *Assume that $f:[a,b]\to\mathbb{C}$ are absolutely continuous with $f(a)=f(b)=0$ and $f'\in L_{2,w_a}[a,b]\cap L_{2,w_b}[a,b]$, then*

$$
\begin{aligned}
(2.17)\qquad &\int_a^b \left|f(t)\right|^2 dt \\
&\le \left(\int_a^b w_a(t;a,b)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\left|f'(t)\right|^2 dt\right)^{1/2} \\
&\le \int_a^b\left[\frac{1}{4}(b-a)^2-\left(t-\frac{a+b}{2}\right)^2\right]\left|f'(t)\right|^2 dt \\
&= \int_a^b (b-t)(t-a)\left|f'(t)\right|^2 dt.
\end{aligned}
$$

The inequalities (2.17) are sharp.

Proof. From (2.14) we get for $g=f$ that

$$
\begin{aligned}
&\int_a^b \left|f(t)\right|^2 dt \\
&\le \left(\int_a^b w_a(t;a,b)\left|f'(t)\right|^2 dt\right)^{1/2}\left(\int_a^b w_b(t;a,b)\left|f'(t)\right|^2 dt\right)^{1/2} \\
&\le \frac{1}{2}(b-a)^2\int_a^b\left|f'(t)\right|^2 dt-\frac{1}{2}\int_a^b\left[(t-a)^2+(b-t)^2\right]\left|f'(t)\right|^2 dt.
\end{aligned}
$$

Since

$$\frac{(t-a)^2+(b-t)^2}{2}=\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2,$$

hence

$$\begin{aligned}&\frac{1}{2}\int_a^b\left[(t-a)^2+(b-t)^2\right]\left|f'(t)\right|^2dt\\&=\int_a^b\left[\frac{1}{4}(b-a)^2+\left(t-\frac{a+b}{2}\right)^2\right]\left|f'(t)\right|^2dt\\&=\frac{1}{4}(b-a)^2\int_a^b\left|f'(t)\right|^2dt+\int_a^b\left(t-\frac{a+b}{2}\right)^2\left|f'(t)\right|^2dt.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{2}(b-a)^2\int_a^b\left|f'(t)\right|^2dt-\frac{1}{2}\int_a^b\left[(t-a)^2+(b-t)^2\right]\left|f'(t)\right|^2dt\\&=\frac{1}{2}(b-a)^2\int_a^b\left|f'(t)\right|^2dt-\frac{1}{4}(b-a)^2\int_a^b\left|f'(t)\right|^2dt\\&-\int_a^b\left(t-\frac{a+b}{2}\right)^2\left|f'(t)\right|^2dt\\&=\int_a^b\left[\frac{1}{4}(b-a)^2-\left(t-\frac{a+b}{2}\right)^2\right]\left|f'(t)\right|^2dt,\end{aligned}$$

which proves the second part of the inequality (2.17).

Now, consider the function

$$f(t)=(t-a)(b-t),\ t\in[a,b].$$

Then $f'(t)=a+b-2t$, $t\in(a,b)$,

$$\int_a^b|f(t)|^2dt=\int_a^b(t-a)^2(b-t)^2dt=\frac{1}{30}(b-a)^5$$

and

$$\begin{aligned}&\int_a^b(b-t)(t-a)\left|f'(t)\right|^2dt\\&=\int_a^b(b-t)(t-a)\left|(a+b)t-2t\right|^2dt\\&=4\int_a^b(b-t)(t-a)\left(t-\frac{a+b}{2}\right)^2dt=\frac{1}{30}(b-a)^5,\end{aligned}$$

which show that the inequalities (2.17) are sharp. □

Theorem 6. *Assume that* $f,\ g:[a,b]\to\mathbb{C}$ *are absolutely continuous with* $f(a)=g(a)=0$ *and* $f(b)=g(b)=0$ *with* $f',\ g'\in L_{2,w_a}[a,b]\cap L_{2,w_b}[a,b]$, *then*

$$\int_a^b|f(t)g(t)|\,dt\le\frac{1}{4}\int_a^bL(t;a,b)\left[\left|f'(t)\right|^2+\left|g'(t)\right|^2\right]dt \tag{2.18a}$$

where

$$L(t;a,b):=\frac{1}{4}(b-a)^2-\begin{cases}(t-a)^2\,,\ t\in\left[0,\frac{a+b}{2}\right],\\[2ex](b-t)^2\,,\ t\in\left(\frac{a+b}{2},b\right].\end{cases}\tag{2.19}$$

Proof. From (2.1) we have

$$\begin{aligned}&\int_a^{\frac{a+b}{2}}|f(t)g(t)|\,dt\\&\le\frac{1}{2}\int_a^{\frac{a+b}{2}}w_a\left(t;a,\frac{a+b}{2}\right)\left[|f'(t)|^2+|g'(t)|^2\right]dt.\\&=\frac{1}{4}\int_a^{\frac{a+b}{2}}\left[\left(\frac{a+b}{2}-a\right)^2-(t-a)^2\right]\left[|f'(t)|^2+|g'(t)|^2\right]dt,\end{aligned}\tag{2.20}$$

while from (2.7) we have

$$\begin{aligned}&\int_{\frac{a+b}{2}}^b|f(t)g(t)|\,dt\\&\le\frac{1}{2}\int_{\frac{a+b}{2}}^b w_b\left(t;\frac{a+b}{2},b\right)\left[|f'(t)|^2+|g'(t)|^2\right]dt\\&=\frac{1}{4}\int_{\frac{a+b}{2}}^b\left[\left(b-\frac{a+b}{2}\right)^2-(b-t)^2\right]\left[|f'(t)|^2+|g'(t)|^2\right]dt\end{aligned}\tag{2.21}$$

If we add these two inequalities, then we get

$$\begin{aligned}\int_a^b|f(t)g(t)|\,dt&\le\frac{1}{4}\int_a^{\frac{a+b}{2}}\left[\frac{1}{4}(b-a)^2-(t-a)^2\right]\left[|f'(t)|^2+|g'(t)|^2\right]dt\\&+\frac{1}{4}\int_{\frac{a+b}{2}}^b\left[\frac{1}{4}(b-a)^2-(b-t)^2\right]\left[|f'(t)|^2+|g'(t)|^2\right]dt\\&=\frac{1}{16}(b-a)^2\int_a^{\frac{a+b}{2}}\left[|f'(t)|^2+|g'(t)|^2\right]dt\\&+\frac{1}{16}(b-a)^2\int_{\frac{a+b}{2}}^b\left[|f'(t)|^2+|g'(t)|^2\right]dt\\&-\frac{1}{4}\int_a^{\frac{a+b}{2}}(t-a)^2\left[|f'(t)|^2+|g'(t)|^2\right]dt\\&-\frac{1}{4}\int_{\frac{a+b}{2}}^b(b-t)^2\left[|f'(t)|^2+|g'(t)|^2\right]dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{16}(b-a)^2 \int_a^b \left[|f'(t)|^2 + |g'(t)|^2 \right] dt \\
&- \frac{1}{4} \int_a^{\frac{a+b}{2}} (t-a)^2 \left[|f'(t)|^2 + |g'(t)|^2 \right] dt \\
&- \frac{1}{4} \int_{\frac{a+b}{2}}^b (b-t)^2 \left[|f'(t)|^2 + |g'(t)|^2 \right] dt \\
&= \frac{1}{4} \int_a^b L(t;a,b) \left[|f'(t)|^2 + |g'(t)|^2 \right] dt,
\end{aligned}
$$

which proves the desired result. □

Corollary 5. *Assume that $f:[a,b]\to\mathbb{C}$ is absolutely continuous with $f(a)=f(b)=0$ with $f'\in L_{2,w_a}[a,b]\cap L_{2,w_b}[a,b]$, then*

$$\int_a^b |f(t)|^2 dt \le \frac{1}{2}\int_a^b L(t;a,b)\,|f'(t)|^2 dt, \tag{2.22}$$

where $L(t;a,b)$ is defined by (2.19). The constant $\frac{1}{2}$ is best possible.

Proof. We consider the function

$$f(t) := \begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right], \\ b-t, & t\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

Then f is absolutely continuous, $|f'(t)|=1$, $t\in(a,b)$,

$$
\begin{aligned}
\int_a^b f^2(t)\,dt &= \int_a^{\frac{a+b}{2}} (t-a)^2 dt + \int_{\frac{a+b}{2}}^b (t-b)^2 dt \\
&= \frac{1}{24}(b-a)^3 + \frac{1}{24}(b-a)^3 = \frac{1}{12}(b-a)^3
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\int_a^b L(t;a,b)\,[f'(t)]^2 dt \\
&= \frac{1}{2}\int_a^b L(t;a,b)\,dt \\
&= \frac{1}{2}\int_a^b \frac{1}{4}(b-a)^2 dt - \frac{1}{2}\left[\int_a^{\frac{a+b}{2}} (t-a)^2 dt + \int_{\frac{a+b}{2}}^b (t-b)^2 dt\right] \\
&= \frac{1}{8}(b-a)^3 - \frac{1}{24}(b-a)^3 = \frac{1}{12}(b-a)^3,
\end{aligned}
$$

which shows that in both sides of (2.22) we get the same quantity $\frac{1}{12}(b-a)^3$.

This proves the sharpness of the constant $\frac{1}{2}$. □

Remark 4. *Assume that f' is absolutely continuous on $[a,b]$. If $f(a)=f'(a)=0$ and $f(b)=f'(b)=0$ with $f',\ f''\in L_{2,w_a}[a,b]\cap L_{2,w_b}[a,b]$, then*

$$\int_a^b |f(t)\,f'(t)|\,dt \le \frac{1}{4}\int_a^b L(t;a,b)\left[|f'(t)|^2 + |f''(t)|^2\right] dt. \tag{2.23}$$

3. Applications

We have the following trapezoid type inequalities:

Proposition 1. *Let $g \in C^1([a,b],\mathbb{C})$. Then*

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right|^2 \tag{3.1}$$
$$\leq \frac{1}{4}\left(\frac{1}{b-a}\int_a^b w_a(t;a,b)\left|g'(t)-g'(a+b-t)\right|^2 dt\right)^{1/2}$$
$$\times\left(\frac{1}{b-a}\int_a^b w_b(t;a,b)\left|g'(t)-g'(a+b-t)\right|^2 dt\right)^{1/2}$$
$$\leq \frac{1}{4}\frac{1}{b-a}\int_a^b (b-t)(t-a)\left|g'(t)-g'(a+b-t)\right|^2 dt.$$

Proof. If $g \in C^1([a,b],\mathbb{C})$, then by taking

$$f(t):=\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2},\ t\in[a,b]$$

we have $f(a)=f(b)=0$,

$$f'(t)=\frac{g'(t)-g'(a+b-t)}{2}$$

and by (2.17) we have

$$\int_a^b\left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^2 dt \tag{3.2}$$
$$\leq \frac{1}{4}\left(\int_a^b w_a(t;a,b)\left|g'(t)-g'(a+b-t)\right|^2 dt\right)^{1/2}$$
$$\times\left(\int_a^b w_b(t;a,b)\left|g'(t)-g'(a+b-t)\right|^2 dt\right)^{1/2}$$
$$\leq \frac{1}{4}\int_a^b (b-t)(t-a)\left|g'(t)-g'(a+b-t)\right|^2 dt.$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$(b-a)\int_a^b\left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^2 dt$$
$$\geq\left|\int_a^b\left[\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right]dt\right|^2$$
$$=\left|\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}(b-a)\right|^2,$$

which implies that

$$\left|\frac{g\left(a\right)+g\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}g\left(t\right)dt\right|^{2} \leq \frac{1}{b-a}\int_{a}^{b}\left|\frac{g\left(t\right)+g\left(a+b-t\right)}{2}-\frac{g\left(a\right)+g\left(b\right)}{2}\right|^{2}dt. \tag{3.3}$$

By utilising (3.2) and (3.3) we derive the desired result (3.1). □

From a different perspective we also have:

Proposition 2. *Let* $g\in C^{1}([a,b],\mathbb{C})$*. Then*

$$\begin{aligned}\left|\frac{g\left(a\right)+g\left(b\right)}{2}-\frac{1}{b-a}\int_{a}^{b}g\left(t\right)dt\right|^{2} &\leq\left(\frac{1}{b-a}\int_{a}^{b}w_{a}\left(t;a,b\right)\left|g'\left(t\right)-\frac{g\left(b\right)-g\left(a\right)}{b-a}\right|^{2}dt\right)^{1/2}\\ &\times\left(\frac{1}{b-a}\int_{a}^{b}w_{b}\left(t;a,b\right)\left|g'\left(t\right)-\frac{g\left(b\right)-g\left(a\right)}{b-a}\right|^{2}dt\right)^{1/2}\\ &\leq\frac{1}{b-a}\int_{a}^{b}\left(b-t\right)\left(t-a\right)\left|g'\left(t\right)-\frac{g\left(b\right)-g\left(a\right)}{b-a}\right|^{2}dt.\end{aligned} \tag{3.4}$$

Proof. If $g\in C^{1}([a,b],\mathbb{C})$, then by taking

$$f\left(t\right):=g\left(t\right)-\frac{g\left(a\right)\left(b-t\right)+g\left(b\right)\left(t-a\right)}{b-a},\ t\in[a,b]$$

we have $f\left(a\right)=f\left(b\right)=0$

$$f'\left(t\right)=g'\left(t\right)-\frac{g\left(b\right)-g\left(a\right)}{b-a}$$

and by (2.17) we get

$$\begin{aligned}\int_{a}^{b}\left|g\left(t\right)-\frac{g\left(a\right)\left(b-t\right)+g\left(b\right)\left(t-a\right)}{b-a}\right|^{2}dt &\leq\left(\int_{a}^{b}w_{a}\left(t;a,b\right)\left|g'\left(t\right)-\frac{g\left(b\right)-g\left(a\right)}{b-a}\right|^{2}dt\right)^{1/2}\\ &\times\left(\int_{a}^{b}w_{b}\left(t;a,b\right)\left|g'\left(t\right)-\frac{g\left(b\right)-g\left(a\right)}{b-a}\right|^{2}dt\right)^{1/2}\\ &\leq\int_{a}^{b}\left(b-t\right)\left(t-a\right)\left|g'\left(t\right)-\frac{g\left(b\right)-g\left(a\right)}{b-a}\right|^{2}dt.\end{aligned} \tag{3.5}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we also have

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right|^2 \le \frac{1}{b-a}\int_a^b \left|g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right|^2 dt. \tag{3.6}$$

By utilising (3.5) and (3.6) we derive the desired result (3.4). □

We also have the following result as well:

Proposition 3. *Let* $g \in C^1([a,b],\mathbb{C})$. *Then*

$$\begin{aligned}
&\left|\frac{b+a}{2}\frac{1}{b-a}\int_a^b g(s)\,ds-\frac{1}{b-a}\int_a^b tg(t)\,dt\right|^2 \\
&\le \left(\frac{1}{b-a}\int_a^b w_a(t;a,b)\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{1/2} \\
&\times \left(\frac{1}{b-a}\int_a^b w_b(t;a,b)\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{1/2} \\
&\le \frac{1}{b-a}\int_a^b (b-t)(t-a)\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt.
\end{aligned} \tag{3.7}$$

Proof. Assume that $g:[a,b]\to\mathbb{C}$ is continuous, then by taking

$$f(t) := \int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds,\ \ t\in[a,b]$$

we have $f(a)=f(b)=0$,

$$f'(t)=g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds$$

and by (2.17) we get

$$\begin{aligned}
&\int_a^b \left|\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^2 dt \\
&\le \left(\int_a^b w_a(t;a,b)\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{1/2} \\
&\times \left(\int_a^b w_b(t;a,b)\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt\right)^{1/2} \\
&\le \int_a^b (b-t)(t-a)\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^2 dt.
\end{aligned} \tag{3.8}$$

Observe that, integrating by parts, we have

$$\begin{aligned}
&\int_a^b \left(\int_a^t g(s)\,ds - \frac{t-a}{b-a} \int_a^b g(s)\,ds \right) dt \\
&= \int_a^b \left(\int_a^t g(s)\,ds \right) dt - \frac{b-a}{2} \int_a^b g(s)\,ds \\
&= b \int_a^b g(s)\,ds - \int_a^b t g(t)\,dt - \frac{b-a}{2} \int_a^b g(s)\,ds \\
&= \frac{b+a}{2} \int_a^b g(s)\,ds - \int_a^b t g(t)\,dt.
\end{aligned}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$\begin{aligned}
&(b-a) \int_a^b \left| \int_a^t g(s)\,ds - \frac{t-a}{b-a} \int_a^b g(s)\,ds \right|^2 dt \\
&\geq \left| \int_a^b \left(\int_a^t g(s)\,ds - \frac{t-a}{b-a} \int_a^b g(s)\,ds \right) dt \right|^2 \\
&= \left| \frac{b+a}{2} \int_a^b g(s)\,ds - \int_a^b t g(t)\,dt \right|^2 .
\end{aligned} \tag{3.9}$$

By making use of (3.8) and (3.9), we derive the desired result (3.7). □

Consider now the *weighted Čebyšev functional*

$$C_w(f,g) := \int_a^b w(t) f(t) g(t)\,dt - \int_a^b w(t) f(t)\,dt \int_a^b w(t) g(t)\,dt \tag{3.10}$$

where $f,\ g,\ w : [a,b] \to \mathbb{R}$ and $w(t) \geq 0$ for a.e. $t \in [a,b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t)\,dt = 1$.

Theorem 7. *Assume that* $w : [a,b] \to (0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds = 1$, $f \in L_2([a,b], \mathbb{C})$ *and* $g \in C^1([a,b], \mathbb{C})$ *is a function with complex values, then*

(3.11)

$$\begin{aligned}
&|C_w(f,g)|^2 \leq \int_a^b |g\prime(t)|^2\,dt \left(\int_a^b w_a(t;a,b) \left| f(t) - \int_a^b f(s) w(s)\,ds \right|^2 w^2(t)\,dt \right)^{1/2} \\
&\times \left(\int_a^b w_b(t;a,b) \left| f(t) - \int_a^b f(s) w(s)\,ds \right|^2 w^2(t)\,dt \right)^{1/2} \\
&\leq \left(\int_a^b |g\prime(t)|^2\,dt \right) \int_a^b (b-t)(t-a) \left| f(t) - \int_a^b f(s) w(s)\,ds \right|^2 w^2(t)\,dt \\
&\leq \frac{1}{4}(b-a)^2 \int_a^b |g\prime(t)|^2\,dt \int_a^b \left| f(t) - \int_a^b f(s) w(s)\,ds \right|^2 w^2(t)\,dt.
\end{aligned}$$

Proof. Integrating by parts, we have

$$\int_a^b \left(\int_a^x f(t)\,w(t)\,dt - \int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds\right) g\prime(x)\,dx$$
$$= \left[\left(\int_a^x f(t)\,w(t)\,dt - \int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds\right) g(x)\right|_a^b$$
$$- \int_a^b g(x)\left(f(x)\,w(x) - w(x)\int_a^b f(s)\,w(s)\,ds\right) dx$$
$$= -\int_a^b f(x)\,g(x)\,w(x)\,dx + \int_a^b f(s)\,w(s)\,ds \int_a^b g(x)\,w(x)\,dx,$$

which gives that

(3.12)
$$C_w(f,g) = \int_a^b \left(\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right) g\prime(x)\,dx.$$

Using (CBS) integral inequality we have

(3.13)
$$|C_w(f,g)|^2$$
$$= \left|\int_a^b \left(\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right) g\prime(x)\,dx\right|^2$$
$$\le \int_a^b \left|\int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt\right|^2 \int_a^b |g\prime(x)|^2\,dx.$$

If we take

$$f(x) := \int_a^x w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^x f(t)\,w(t)\,dt, \;\; x \in [a,b]$$

we observe that $f(a) = f(b) = 0$ and $h \in C^1([a,b],\mathbb{C})$.

Then by (1.3) we get

(3.14)
$$\int_a^b \left|\int_a^t w(s)\,ds \int_a^b f(s)\,w(s)\,ds - \int_a^t f(s)\,w(s)\,ds\right|^2 dt$$
$$\le \left(\int_a^b w_a(t;a,b)\left|f(t) - \int_a^b f(s)\,w(s)\,ds\right|^2 w^2(t)\,dt\right)^{1/2}$$
$$\times \left(\int_a^b w_b(t;a,b)\left|f(t) - \int_a^b f(s)\,w(s)\,ds\right|^2 w^2(t)\,dt\right)^{1/2}$$
$$\le \int_a^b (b-t)(t-a)\left|f(t) - \int_a^b f(s)\,w(s)\,ds\right|^2 w^2(t)\,dt.$$

By (3.13) and (3.14) we obtain the first part of (3.11). The second part is obvious. □

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