SOME p-NORMS OPIAL LIKE INEQUALITIES FOR TWO FUNCTIONS AND APPLICATIONS

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Abstract. In this paper we show among others that, if f, $g : [a,b] \to \mathbb{C}$ are absolutely continuous with $f(a) = g(b) = 0$, then for p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned}
&\int_a^b |f(t)\, g(t)|\, dt \\
&\leq \frac{1}{2}\left(\int_a^b (b-t)^2 |f'(t)|^q\, dt\right)^{1/q} \left(\int_a^b (t-a)^2 |g'(t)|^p\, dt\right)^{1/p} \\
&\leq \frac{1}{4}\int_a^b \left[(b-t)^2 |f'(t)|^q + (t-a)^2 |g'(t)|^p\right] dt,
\end{aligned}$$

provided the integrals from the right side are finite. The inequalities are sharp. Some trapezoid and Grüss' type inequalities are also given.

1. Introduction

We recall the following Opial type inequalities:

Theorem 1. *Assume that $u : [a,b] \subset \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function on the interval $[a,b]$ and such that $u' \in L_2[a,b]$.*

(i) *If $u(a) = u(b) = 0$, then*

$$\int_a^b |u(t)\, u'(t)|\, dt \leq \frac{1}{4}(b-a)\int_a^b |u'(t)|^2\, dt, \tag{1.1}$$

with equality if and only if

$$u(t) = \begin{cases} c(t-a) & \text{if } a \leq t \leq \frac{a+b}{2}, \\ c(b-t) & \text{if } \frac{a+b}{2} < t \leq b, \end{cases}$$

where c is an arbitrary constant;

(ii) *If $u(a) = 0$, then*

$$\int_a^b |u(t)\, u'(t)|\, dt \leq \frac{1}{2}(b-a)\int_a^b |u'(t)|^2\, dt, \tag{1.2}$$

with equality if and only if $u(t) = c(t-a)$ for some constant c.

The inequality (1.1) was obtained by Olech in [10] in which he gave a simplified proof of an inequality originally due in a slightly less general form to Zdzislaw Opial [11].

Embedded in Olech's proof is the half-interval form of Opial's inequality, also discovered by Beesack [1], which is satisfied by those u vanishing only at a.

1991 *Mathematics Subject Classification.* 26D15; 26D10.

Key words and phrases. Opial's inequality, Trapezoid inequality, Grüss' inequality.

RGMIA Res. Rep. Coll. 24 (2021), Art. 57, 14 pp. Received 19/05/21

For various proofs of the above inequalities, see [6]-[9] and [13].

In the recent paper [3] we obtained the following results:

Theorem 2. *Assume that* $f,\ g:[a,b]\to\mathbb{C}$ *are absolutely continuous on* $[a,b]$ *with* $f'\in L_p[a,b]$ *and* $g'\in L_q[a,b]$ *for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

(i) *If* $g(a)=0$, *then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b (t-a)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b (b-t)\,|g'(t)|^q\,dt\right)^{1/q} \le \frac{1}{p}\int_a^b (t-a)\,|f'(t)|^p\,dt+\frac{1}{q}\int_a^b (b-t)\,|g'(t)|^q\,dt. \tag{1.3}$$

(ii) *If* $g(b)=0$, *then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b (b-t)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b (t-a)\,|g'(t)|^q\,dt\right)^{1/q} \le \frac{1}{p}\int_a^b (b-t)\,|f'(t)|^p\,dt+\frac{1}{q}\int_a^b (t-a)\,|g'(t)|^q\,dt. \tag{1.4}$$

(iii) *If* $g(a)=g(b)=0$, *then*

$$\begin{aligned}&\int_a^b |f'(t)\,g(t)|\,dt \\ &\le \left(\int_a^b K(t)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b \left|\frac{a+b}{2}-t\right|\,|g'(t)|^q\,dt\right)^{1/q} \\ &\le \frac{1}{p}\int_a^b K(t)\,|f'(t)|^p\,dt+\frac{1}{q}\int_a^b \left|\frac{a+b}{2}-t\right|\,|g'(t)|^q\,dt,\end{aligned} \tag{1.5}$$

where K *is defined by*

$$K(t):=\begin{cases} t-a \ \text{ if } a\le t\le \frac{a+b}{2}, \\ b-t \ \text{ if } \frac{a+b}{2}<t\le b.\end{cases}$$

In particular, we have the Opial type inequalities [3]:

Corollary 1. *Assume that* $f:[a,b]\to\mathbb{C}$ *are absolutely continuous on* $[a,b]$ *and* $f'\in L_p[a,b]\cap L_q[a,b]$ *for* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

(i) *If* $f(a)=0$, *then*

$$\int_a^b |f'(t)\,f(t)|\,dt \le \left(\int_a^b (t-a)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b (b-t)\,|f'(t)|^q\,dt\right)^{1/q} \le \frac{1}{p}\int_a^b (t-a)\,|f'(t)|^p\,dt+\frac{1}{q}\int_a^b (b-t)\,|f'(t)|^q\,dt. \tag{1.6}$$

(ii) *If* $f(b)=0$, *then*

$$\int_a^b |f'(t)\,g(t)|\,dt \le \left(\int_a^b (b-t)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b (t-a)\,|f'(t)|^q\,dt\right)^{1/q} \le \frac{1}{p}\int_a^b (b-t)\,|f'(t)|^p\,dt + \frac{1}{q}\int_a^b (t-a)\,|f'(t)|^q\,dt. \tag{1.7}$$

(iii) *If* $f(a)=f(b)=0$, *then*

$$\int_a^b |f'(t)\,f(t)|\,dt \le \left(\int_a^b K(t)\,|f'(t)|^p\,dt\right)^{1/p}\left(\int_a^b \left|\frac{a+b}{2}-t\right| |f'(t)|^q\,dt\right)^{1/q} \le \frac{1}{p}\int_a^b K(t)\,|f'(t)|^p\,dt + \frac{1}{q}\int_a^b \left|\frac{a+b}{2}-t\right| |f'(t)|^q\,dt. \tag{1.8}$$

2. Main Results

We consider the positive weight

$$w_a(t;a,b) := \frac{1}{2}\left[(b-a)^2-(t-a)^2\right] = (b-t)\left(\frac{b+t}{2}-a\right),$$

for $t\in[a,b]$.

Theorem 3. *Assume that* $f,\ g:[a,b]\to\mathbb{C}$ *are absolutely continuous with* $f(a)=g(a)=0$ *and* $f'\in L_{q,w_a}[a,b]$, $g'\in L_{p,w_a}[a,b]$, *where* $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\int_a^b |f(t)\,g(t)|\,dt \le \left(\int_a^b w_a(t;a,b)\,|f'(t)|^q\,dt\right)^{1/q}\left(\int_a^b w_a(t;a,b)\,|g'(t)|^p\,dt\right)^{1/p} \le \int_a^b w_a(t;a,b)\left[\frac{1}{q}|f'(t)|^q+\frac{1}{p}|g'(t)|^p\right]dt. \tag{2.1}$$

The inequalities (2.1) are sharp.

Proof. Since $f(a)=g(a)=0$, hence $f(t)=\int_a^t f'(s)\,ds$ and $g(t)=\int_a^t g'(s)\,ds$ and we have

$$\int_a^b |f(t)\,g(t)|\,dt = \int_a^b \left|\int_a^t f'(s)\,ds\right|\left|\int_a^t g'(s)\,ds\right|dt \le \int_a^b \left(\int_a^t |f'(s)|\,ds\right)\left(\int_a^t |g'(s)|\,ds\right)dt =: A. \tag{2.2}$$

Applying twice Hölder's inequality, we get for $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, that

$$\int_a^t |f'(s)|\,ds \le (t-a)^{1/p}\left(\int_a^t |f'(s)|^q\,ds\right)^{1/q}$$

and

$$\int_a^t |g'(s)|\, ds \le (t-a)^{1/q} \left(\int_a^t |g'(s)|^p \, ds \right)^{1/p}.$$

Therefore

$$A \le \int_a^b (t-a) \left(\int_a^t |f'(s)|^q \, ds \right)^{1/q} \left(\int_a^t |g'(s)|^p \, ds \right)^{1/p} dt$$

By applying again Hölder's weighted inequality, we get

$$\begin{aligned}
(2.3)\quad & \int_a^b (t-a) \left(\int_a^t |f'(s)|^q \, ds \right)^{1/q} \left(\int_a^t |g'(s)|^p \, ds \right)^{1/p} dt \\
&\le \left[\int_a^b (t-a) \left(\left(\int_a^t |f'(s)|^q \, ds \right)^{1/q} \right)^q dt \right]^{1/q} \\
&\times \left[\int_a^b (t-a) \left(\left(\int_a^t |g'(s)|^p \, ds \right)^{1/p} \right)^p dt \right]^{1/p} \\
&= \left[\int_a^b (t-a) \left(\int_a^t |f'(s)|^q \, ds \right) dt \right]^{1/q} \left[\int_a^b (t-a) \left(\int_a^t |g'(s)|^p \, ds \right) dt \right]^{1/p} \\
&=: B
\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}
& \int_a^b (t-a) \left(\int_a^t |f'(s)|^q \, ds \right) dt \\
&= \int_a^b \left(\int_a^t |f'(s)|^q \, ds \right) d\left(\frac{(t-a)^2}{2} \right) \\
&= \left(\int_a^t |f'(s)|^q \, ds \right) \left(\frac{(t-a)^2}{2} \right) \Bigg|_a^b - \int_a^b \frac{(t-a)^2}{2} |f'(t)|^q \, q dt \\
&= \frac{(b-a)^2}{2} \left(\int_a^b |f'(s)|^q \, ds \right) - \int_a^b \frac{(t-a)^2}{2} |f'(t)|^q \, dt \\
&= \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2} \right] |f'(t)|^2 \, dt
\end{aligned}$$

and

$$\int_a^b (t-a) \left(\int_a^t |g'(s)|^p \, ds \right) dt = \int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2} \right] |g'(t)|^p \, dt.$$

Therefore

$$B \leq \left(\int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2} \right] |f'(t)|^q \, dt \right)^{1/q} \times \left(\int_a^b \left[\frac{(b-a)^2}{2} - \frac{(t-a)^2}{2} \right] |g'(t)|^p \, dt \right)^{1/p} = \left(\int_a^b w_a(t;a,b) |f'(t)|^q \, dt \right)^{1/q} \left(\int_a^b w_a(t;a,b) |g'(t)|^p \, dt \right)^{1/p}. \tag{2.4}$$

This proves the first part of (2.1).

By utilising Young's inequality

$$\alpha^{1/q}\beta^{1/p} \leq \frac{1}{q}\alpha + \frac{1}{p}\beta, \text{ for } \alpha, \beta \geq 0, \tag{2.5}$$

where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we get

$$\left(\int_a^b w_a(t;a,b) |f'(t)|^q \, dt \right)^{1/q} \left(\int_a^b w_a(t;a,b) |g'(t)|^p \, dt \right)^{1/p} \leq \frac{1}{q} \int_a^b w_a(t;a,b) |f'(t)|^q \, dt + \frac{1}{p} \int_a^b w_a(t;a,b) |g'(t)|^p \, dt = \int_a^b w_a(t;a,b) \left[\frac{1}{q} |f'(t)|^q + \frac{1}{p} |g'(t)|^p \right] dt,$$

which proves the last part of (2.1).

Now, consider $f(t) = g(t) = t - a$. Then

$$\int_a^b |f(t) g(t)| \, dt = \int_a^b (t-a)^2 \, dt = \frac{1}{3} (b-a)^3$$

and

$$\int_a^b w_a(t;a,b) \left[\frac{1}{q} |f'(t)|^q + \frac{1}{p} |g'(t)|^p \right] dt = \int_a^b w_a(t;a,b) \, dt = \frac{1}{2} \int_a^b \left[(b-a)^2 - (t-a)^2 \right] dt = \frac{1}{2} \left[(b-a)^3 - \frac{1}{3} (b-a)^3 \right] = \frac{1}{3} (b-a)^3,$$

which shows that all terms in (2.1) are equal with $\frac{1}{3}(b-a)^3$. □

Remark 1. *Assume that f' is absolutely continuous on $[a,b]$. If $f(a) = f'(a) = 0$ and $f' \in L_{q,w_a}[a,b]$, $f'' \in L_{p,w_a}[a,b]$, where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} &\int_a^b |f(t) f'(t)|\, dt \\ &\le \left(\int_a^b w_a(t;a,b) |f'(t)|^q \, dt \right)^{1/q} \left(\int_a^b w_a(t;a,b) |f''(t)|^p \, dt \right)^{1/p} \\ &\le \int_a^b w_a(t;a,b) \left[\frac{1}{q} |f'(t)|^q + \frac{1}{p} |f''(t)|^p \right] dt. \end{aligned} \tag{2.6}$$

Corollary 2. *Assume that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $f(a) = 0$ and $f' \in L_{q,w_a}[a,b] \cap L_{p,w_a}[a,b]$, where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} &\int_a^b |f(t)|^2 \, dt \\ &\le \left(\int_a^b w_a(t;a,b) |f'(t)|^q \, dt \right)^{1/q} \left(\int_a^b w_a(t;a,b) |f'(t)|^p \, dt \right)^{1/p} \\ &\le \int_a^b w_a(t;a,b) \left[\frac{1}{q} |f'(t)|^q + \frac{1}{p} |f'(t)|^p \right] dt. \end{aligned} \tag{2.7}$$

The inequalities (2.1) are sharp.

Remark 2. *We observe that for $p = q = 2$, we get*

$$\int_a^b |f(t)|^2 \, dt \le \int_a^b w_a(t;a,b) |f'(t)|^2 \, dt, \tag{2.8}$$

with the assumption that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $f(a) = 0$ and $f' \in L_{2,w_a}[a,b]$. The inequality is sharp.

Now consider the dual weight

$$w_b(t;a,b) := \frac{1}{2} \left[(b-a)^2 - (b-t)^2 \right] = (t-a) \left(b - \frac{a+t}{2} \right),$$

for $t \in [a,b]$.

Theorem 4. *Assume that $f, g : [a,b] \to \mathbb{C}$ are absolutely continuous with $f(b) = g(b) = 0$ and $f' \in L_{q,w_b}[a,b]$, $g' \in L_{p,w_b}[a,b]$, where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned} &\int_a^b |f(t) g(t)|\, dt \\ &\le \left(\int_a^b w_b(t;a,b) |f'(t)|^q \, dt \right)^{1/q} \left(\int_a^b w_b(t;a,b) |g'(t)|^p \, dt \right)^{1/p} \\ &\le \int_a^b w_b(t;a,b) \left[\frac{1}{q} |f'(t)|^q + \frac{1}{p} |g'(t)|^p \right] dt. \end{aligned} \tag{2.9}$$

The inequalities in (2.6) are sharp.

Proof. Since $f(b) = g(b) = 0$, hence $f(t) = -\int_b^t f'(s)\,ds$ and $g(t) = -\int_b^t g'(s)\,ds$ and we have

$$\int_a^b |f(t)\,g(t)|\,dt = \int_a^b \left|\int_t^b f'(s)\,ds\right| \left|\int_t^b g'(s)\,ds\right| dt \tag{2.10}$$
$$\leq \int_a^b \left(\int_t^b |f'(s)|\,ds\right)\left(\int_t^b |g'(s)|\,ds\right) dt.$$

Now, by using a similar argument as in the proof of Theorem 3 we deduce the desired inequality (2.9). □

Remark 3. *Assume that f' is absolutely continuous on $[a,b]$. If $f(b) = f'(b) = 0$ and $f' \in L_{q,w_a}[a,b]$, $f'' \in L_{p,w_a}[a,b]$, where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_a^b |f(t)\,f'(t)|\,dt \tag{2.11}$$
$$\leq \left(\int_a^b w_b(t;a,b)\,|f'(t)|^q\,dt\right)^{1/q}\left(\int_a^b w_b(t;a,b)\,|f''(t)|^p\,dt\right)^{1/p}$$
$$\leq \int_a^b w_b(t;a,b)\left[\frac{1}{q}|f'(t)|^q + \frac{1}{p}|f''(t)|^p\right] dt.$$

Corollary 3. *Assume that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $f(b) = 0$ and $f' \in L_{q,w_a}[a,b] \cap L_{p,w_a}[a,b]$, where p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_a^b |f(t)|^2\,dt \tag{2.12}$$
$$\leq \left(\int_a^b w_b(t;a,b)\,|f'(t)|^q\,dt\right)^{1/q}\left(\int_a^b w_b(t;a,b)\,|f'(t)|^p\,dt\right)^{1/p}$$
$$\leq \int_a^b w_b(t;a,b)\left[\frac{1}{q}|f'(t)|^q + \frac{1}{p}|f'(t)|^p\right] dt.$$

The inequalities (2.12) are sharp.

Remark 4. *We observe that for $p = q = 2$, we get*

$$\int_a^b |f(t)|^2\,dt \leq \int_a^b w_b(t;a,b)\,|f'(t)|^2\,dt, \tag{2.13}$$

with the assumption that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $f(b) = 0$ and $f' \in L_{2,w_b}[a,b]$. The inequality is sharp.

We also have:

Theorem 5. *Assume that f, $g : [a,b] \to \mathbb{C}$ are absolutely continuous with $f(a) = g(b) = 0$ and $f' \in L_{q,(b-\ell)^2}[a,b]$, $g' \in L_{p,(\ell-a)^2}[a,b]$, where $\ell(t) = t$, p, $q > 1$ with*

$\frac{1}{p}+\frac{1}{q}=1$, *then*

$$\begin{aligned}\int_a^b |f(t)\,g(t)|\,dt &\leq \frac{1}{2}\left(\int_a^b (b-t)^2\,|f'(t)|^q\,dt\right)^{1/q}\left(\int_a^b (t-a)^2\,|g'(t)|^p\,dt\right)^{1/p} \\ &\leq \frac{1}{4}\int_a^b \left[(b-t)^2\,|f'(t)|^q+(t-a)^2\,|g'(t)|^p\right]dt.\end{aligned} \tag{2.14}$$

The inequalities in (2.6) are sharp.

Proof. Since $f(a)=g(b)=0$, hence $f(t)=\int_a^t f'(s)\,ds$ and $g(t)=-\int_t^b g'(s)\,ds$. Therefore

$$\begin{aligned}\int_a^b |f(t)\,g(t)|\,dt &= \int_a^b \left|\int_a^t f'(s)\,ds\right|\left|\int_t^b g'(s)\,ds\right|dt \\ &= \int_a^b (t-a)^{1/p}(b-t)^{1/q}(t-a)^{-1/p} \\ &\times \left|\int_a^t f'(s)\,ds\right|(b-t)^{-1/q}\left|\int_t^b g'(s)\,ds\right|dt \\ &=: E.\end{aligned} \tag{2.15}$$

By Cauchy-Bunyakowsky-Schwarz integral inequality, we have

$$(t-a)^{-1/p}\left|\int_a^t f'(s)\,ds\right| \leq \left(\int_a^t |f'(s)|^q\,ds\right)^{1/q}$$

and

$$(b-t)^{-1/q}\left|\int_t^b g'(s)\,ds\right| \leq \left(\int_t^b |g'(s)|^p\,ds\right)^{1/p},$$

which imply that

$$\begin{aligned}E &\leq \int_a^b (t-a)^{1/p}(b-t)^{1/q}\left(\int_a^t |f'(s)|^q\,ds\right)^{1/q}\left(\int_t^b |g'(s)|^p\,ds\right)^{1/p} \\ &= \int_a^b (b-t)^{1/q}\left(\int_a^t |f'(s)|^q\,ds\right)^{1/q}(t-a)^{1/p}\left(\int_t^b |g'(s)|^p\,ds\right)^{1/p}.\end{aligned} \tag{2.16}$$

By Hölder's integral inequality, we also have

$$\int_a^b (b-t)^{1/q} \left(\int_a^t |f'(s)|^q \, ds \right)^{1/q} (t-a)^{1/p} \left(\int_t^b |g'(s)|^p \, ds \right)^{1/p} \tag{2.17}$$
$$\leq \left(\int_a^b \left[(b-t)^{1/q} \left(\int_a^t |f'(s)|^q \, ds \right)^{1/q} \right]^q dt \right)^{1/q}$$
$$\times \left(\int_a^b \left[(t-a)^{1/p} \left(\int_t^b |g'(s)|^p \, ds \right)^{1/p} \right]^p dt \right)^{1/p}$$
$$= \left(\int_a^b (b-t) \left(\int_a^t |f'(s)|^q \, ds \right) dt \right)^{1/q}$$
$$\times \left(\int_a^b (t-a) \left(\int_t^b |g'(s)|^p \, ds \right) dt \right)^{1/p}.$$

Using integration by parts, we have

$$\int_a^b (b-t) \left(\int_a^t |f'(s)|^q \, ds \right) dt \tag{2.18}$$
$$= -\int_a^b \left(\int_a^t |f'(s)|^q \, ds \right) d\left(\frac{(b-t)^2}{2} \right)$$
$$= -\left[\left(\int_a^t |f'(s)|^q \, ds \right) \left(\frac{(b-t)^2}{2} \right) \Bigg|_a^b - \int_a^b \frac{(b-t)^2}{2} |f'(t)|^q \, dt \right]$$
$$= \frac{1}{2} \int_a^b (b-t)^2 |f'(t)|^q \, dt$$

and

$$\int_a^b (t-a) \left(\int_t^b |g'(s)|^p \, ds \right) dt \tag{2.19}$$
$$= \int_a^b \left(\int_t^b |g'(s)|^p \, ds \right) dt \left(\frac{(t-a)^2}{2} \right)$$
$$= \left(\int_t^b |g'(s)|^p \, ds \right) \frac{(t-a)^2}{2} \Bigg|_a^b + \int_a^b \frac{(t-a)^2}{2} |g'(t)|^p \, dt$$
$$= \frac{1}{2} \int_a^b (t-a)^2 |g'(t)|^p \, dt,$$

which proves the first inequality in (2.11).

The last part follows by Young's inequality (2.5).

Now, consider $f(t) = t-a$ and $g(t) = t-b$. Then

$$\int_a^b |f(t) g(t)| \, dt = \int_a^b (t-a)(b-t) \, dt = \frac{1}{6} (b-a)^3$$

and

$$\begin{aligned}&\frac{1}{4}\int_a^b \left[(b-t)^2 |f'(t)|^q + (t-a)^2 |g'(t)|^p\right] dt \\ &= \frac{1}{4}\int_a^b \left[(b-t)^2 + (t-a)^2\right] dt = \frac{1}{4}\left(\frac{(b-a)^3}{3} + \frac{(b-a)^3}{3}\right) \\ &= \frac{1}{6}(b-a)^3,\end{aligned}$$

and all terms in (2.14) are equal with $\frac{1}{6}(b-a)^3$. □

Corollary 4. *Assume that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $f(a) = f(b) = 0$ and $f' \in L_{q,(b-\ell)^2}[a,b] \cap L_{p,(\ell-a)^2}[a,b]$, where $\ell(t) = t$, p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\begin{aligned}\int_a^b |f(t)|^2 dt &\le \frac{1}{2}\left(\int_a^b (b-t)^2 |f'(t)|^q dt\right)^{1/q} \left(\int_a^b (t-a)^2 |f'(t)|^p dt\right)^{1/p} \\ &\le \frac{1}{4}\int_a^b \left[(b-t)^2 |f'(t)|^q + (t-a)^2 |f'(t)|^p\right] dt.\end{aligned} \tag{2.20}$$

Remark 5. *Assume that f, $g : [a,b] \to \mathbb{C}$ are absolutely continuous with $f(a) = g(b) = 0$ and $f' \in L_{2,(b-\ell)^2}[a,b]$, $g' \in L_{2,(\ell-a)^2}[a,b]$, where $\ell(t) = t$, then*

$$\begin{aligned}&\int_a^b |f(t)g(t)| dt \\ &\le \frac{1}{2}\left(\int_a^b (b-t)^2 |f'(t)|^2 dt\right)^{1/2} \left(\int_a^b (t-a)^2 |g'(t)|^2 dt\right)^{1/2} \\ &\le \frac{1}{4}\int_a^b \left[(b-t)^2 |f'(t)|^2 + (t-a)^2 |g'(t)|^2\right] dt.\end{aligned} \tag{2.21}$$

The inequalities in (2.6) are sharp.

Also, if $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $f(a) = f(b) = 0$ and $f' \in L_{2,(b-\ell)^2}[a,b] \cap L_{2,(\ell-a)^2}[a,b]$, then

$$\begin{aligned}\int_a^b |f(t)|^2 dt &\le \frac{1}{2}\left(\int_a^b (b-t)^2 |f'(t)|^2 dt\right)^{1/2} \left(\int_a^b (t-a)^2 |f'(t)|^2 dt\right)^{1/2} \\ &\le \frac{1}{4}\int_a^b \left[(b-t)^2 + (t-a)^2\right] |f'(t)|^2 dt.\end{aligned} \tag{2.22}$$

3. Applications

We will use the inequality

$$\begin{aligned}&\int_a^b |f(t)|^2 dt \\ &\le \frac{1}{2}\left(\int_a^b (b-t)^2 |f'(t)|^q dt\right)^{1/q} \left(\int_a^b (t-a)^2 |f'(t)|^p dt\right)^{1/p}\end{aligned} \tag{3.1}$$

that holds for $f:[a,b]\to\mathbb{C}$ which is absolutely continuous with $f(a)=f(b)=0$ and $f'\in L_{q,(b-\ell)^2}[a,b]\cap L_{p,(\ell-a)^2}[a,b]$, where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

We have the following trapezoid type inequalities:

Proposition 1. *Let $g\in C^1([a,b],\mathbb{C})$. Then*

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right|^2 \tag{3.2}$$
$$\leq\frac{1}{8}\left(\frac{1}{b-a}\int_a^b (b-t)^2\left|g'(t)-g'(a+b-t)\right|^q dt\right)^{1/q}$$
$$\times\left(\frac{1}{b-a}\int_a^b (t-a)^2\left|g'(t)-g'(a+b-t)\right|^p dt\right)^{1/p},$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. If $g\in C^1([a,b],\mathbb{C})$, then by taking

$$f(t):=\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2},\ t\in[a,b]$$

we have $f(a)=f(b)=0$,

$$f'(t)=\frac{g'(t)-g'(a+b-t)}{2}$$

and by (2.14) we have

$$\int_a^b\left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^2 dt \tag{3.3}$$
$$\leq\frac{1}{8}\left(\int_a^b (b-t)^2\left|g'(t)-g'(a+b-t)\right|^q dt\right)^{1/q}$$
$$\times\left(\int_a^b (t-a)^2\left|g'(t)-g'(a+b-t)\right|^2 dt\right)^{1/2}.$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$(b-a)\int_a^b\left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^2 dt$$
$$\geq\left|\int_a^b\left[\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right]dt\right|^2$$
$$=\left|\int_a^b g(t)\,dt-\frac{g(a)+g(b)}{2}(b-a)\right|^2,$$

which implies that

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right|^2 \tag{3.4}$$
$$\leq\frac{1}{b-a}\int_a^b\left|\frac{g(t)+g(a+b-t)}{2}-\frac{g(a)+g(b)}{2}\right|^2 dt.$$

By utilising (3.3) and (3.4) we derive the desired result (3.2). □

From a different perspective we also have:

Proposition 2. *Let* $g \in C^1([a,b],\mathbb{C})$. *Then*

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right|^2 \leq \left(\frac{1}{b-a}\int_a^b (b-t)^2\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^q dt\right)^{1/q} \times \left(\frac{1}{b-a}\int_a^b (t-a)^2\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^p dt\right)^{1/p}, \tag{3.5}$$

where $p,\ q>1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

Proof. If $g \in C^1([a,b],\mathbb{C})$, then by taking

$$f(t):=g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a},\ t\in[a,b]$$

we have $f(a)=f(b)=0$

$$f'(t)=g'(t)-\frac{g(b)-g(a)}{b-a}$$

and by (2.14) we get

$$\int_a^b\left|g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right|^2 dt \leq \left(\int_a^b (b-t)^2\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^q dt\right)^{1/q} \times \left(\int_a^b (t-a)^2\left|g'(t)-\frac{g(b)-g(a)}{b-a}\right|^p dt\right)^{1/p}. \tag{3.6}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we also have

$$\left|\frac{g(a)+g(b)}{2}-\frac{1}{b-a}\int_a^b g(t)\,dt\right|^2 \leq \frac{1}{b-a}\int_a^b\left|g(t)-\frac{g(a)(b-t)+g(b)(t-a)}{b-a}\right|^2 dt. \tag{3.7}$$

By utilising (3.6) and (3.7) we derive the desired result (3.5). □

We also have the following result as well:

Proposition 3. *Let $g \in C^1([a,b],\mathbb{C})$. Then*

$$\left|\frac{b+a}{2}\frac{1}{b-a}\int_a^b g(s)\,ds-\frac{1}{b-a}\int_a^b tg(t)\,dt\right|^2 \le \left(\frac{1}{b-a}\int_a^b (b-t)^2\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^q dt\right)^{1/q} \times\left(\frac{1}{b-a}\int_a^b (t-a)^2\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right)^{1/p}, \tag{3.8}$$

where $p,\ q>1$ with $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Assume that $g:[a,b]\to\mathbb{C}$ is continuous, then by taking

$$f(t):=\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds,\ \ t\in[a,b]$$

we have $f(a)=f(b)=0$,

$$f'(t)=g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds$$

and by (2.14) we get

$$\int_a^b\left|\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^2 dt \le\left(\int_a^b (b-t)^2\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^q dt\right)^{1/q} \times\left(\int_a^b (t-a)^2\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|^p dt\right)^{1/p}. \tag{3.9}$$

Observe that, integrating by parts, we have

$$\begin{aligned}&\int_a^b\left(\int_a^t g(s)\,ds-\frac{t-a}{b-a}\int_a^b g(s)\,ds\right)dt\\&=\int_a^b\left(\int_a^t g(s)\,ds\right)dt-\frac{b-a}{2}\int_a^b g(s)\,ds\\&=b\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt-\frac{b-a}{2}\int_a^b g(s)\,ds\\&=\frac{b+a}{2}\int_a^b g(s)\,ds-\int_a^b tg(t)\,dt.\end{aligned}$$

By Cauchy-Bunyakovsky-Schwarz integral inequality we have

$$(b-a)\int_a^b \left|\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right|^2 dt \tag{3.10}$$
$$\geq \left|\int_a^b \left(\int_a^t g(s)\,ds - \frac{t-a}{b-a}\int_a^b g(s)\,ds\right) dt\right|^2$$
$$= \left|\frac{b+a}{2}\int_a^b g(s)\,ds - \int_a^b tg(t)\,dt\right|^2.$$

By making use of (3.9) and (3.10), we derive the desired result (3.8). □

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