RIEMANN-STIELTJES INTEGRAL INEQUALITIES RELATED TO WIRTINGER'S RESULT

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Abstract. In this paper we obtain sharp upper bounds for the Riemann-Stieltjes integral $\int_a^b |f(t)|^2 du(t)$ in the case that $f:[a,b]\to\mathbb{C}$ is absolutely continuous with $f(a)=f(b)=0$, $f'\in L_2[a,b]$ and $u:[a,b]\to\mathbb{R}$ is monotonic nondecreasing. Applications for trapezoid and Grüss' type inequalities are provided. Some extensions to continuous functions of selfadjoint operators and unitary operators in complex Hilbert spaces are also given.

1. Introduction

It is well known that, see for instance [5], or [12], if $u\in C^1([a,b],\mathbb{R})$ satisfies $u(a)=u(b)=0$, then we have *Wirtinger's inequality*

$$\int_a^b u^2(t)\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.1}$$

with the equality holding if and only if $u(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$ for some constant $K\in\mathbb{R}$.

If $u\in C^1([a,b],\mathbb{R})$ satisfies the condition $u(a)=0$, then also

$$\int_a^b u^2(t)\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b [u'(t)]^2\,dt \tag{1.2}$$

and the equality holds if and only if $u(t)=L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$ for some constant $L\in\mathbb{R}$.

If $h\in C^1([a,b],\mathbb{C})$ is a function with complex values and $h(a)=h(b)=0$, then $\operatorname{Re}h(a)=\operatorname{Re}h(b)=0$ and $\operatorname{Im}h(a)=\operatorname{Im}h(b)=0$ and by writing (1.1) for $\operatorname{Re}h$ and $\operatorname{Im}h$ and adding the obtained inequalities, we get

$$\int_a^b |h(t)|^2\,dt \le \frac{(b-a)^2}{\pi^2}\int_a^b |h'(t)|^2\,dt \tag{1.3}$$

with the equality holding if and only if

$$h(t)=K\sin\left[\frac{\pi(t-a)}{b-a}\right]$$

for some complex constant $K\in\mathbb{C}$.

Similarly, if $h\in C^1([a,b],\mathbb{C})$ with $h(a)=0$, then by (1.2) we have

$$\int_a^b |h(t)|^2\,dt \le \frac{4(b-a)^2}{\pi^2}\int_a^b |h'(t)|^2\,dt \tag{1.4}$$

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and the equality holds if and only if

$$h(t) = L\sin\left[\frac{\pi(t-a)}{2(b-a)}\right]$$

for some complex constant $L \in \mathbb{C}$.

For some related Wirtinger type integral inequalities see [1], [3], [5] and [10]-[16].

In the recent paper [8] we obtained the following weighted version of Wirtinger results:

Theorem 1. *Assume that* $w : [a,b] \to (0,\infty)$ *is continuous on* $[a,b]$ *with* $\int_a^b w(s)\,ds = 1$ *and* $f \in C^1([a,b],\mathbb{C})$ *is a function with complex values and* $f(a) = f(b) = 0$*, then*

$$\int_a^b |f(t)|^2 w(t)\,dt \le \frac{1}{\pi^2}\int_a^b \frac{|f'(t)|^2}{w(t)}dt. \tag{1.5}$$

The equality holds in (1.5) iff

$$f(t) = K\sin\left[\pi\int_a^t w(s)\,ds\right],\ K \in \mathbb{C}.$$

If $f(a) = 0$*, then*

$$\int_a^b |f(t)|^2 w(t)\,dt \le \frac{4}{\pi^2}\int_a^b \frac{|f'(t)|^2}{w(t)}dt \tag{1.6}$$

with equality iff

$$f(t) = K\sin\left[\frac{1}{2}\pi\int_a^t w(s)\,ds\right],\ K \in \mathbb{C}.$$

Motivated by the above results, in this paper we obtain sharp upper bounds for the Riemann-Stieltjes integral $\int_a^b |f(t)|^2 du(t)$ in the case that $f : [a,b] \to \mathbb{C}$ is absolutely continuous with $f(a) = f(b) = 0$, $f' \in L_2[a,b]$ and $u : [a,b] \to \mathbb{R}$ is monotonic nondecreasing. Applications for trapezoid and Grüss' type inequalities are provided. Some extensions to continuous functions of selfadjoint operators and unitary operators in complex Hilbert spaces are also given.

2. Main Results

We have the following result:

Theorem 2. *Assume that* $f : [a,b] \to \mathbb{C}$ *is absolutely continuous with* $f(a) = f(b) = 0$ *and* $f' \in L_2[a,b]$. *If* $u : [a,b] \to \mathbb{R}$ *is monotonic nondecreasing, then*

$$\begin{aligned}
&\int_a^b |f(t)|^2 du(t) \\
&\le \frac{1}{2}\int_a^b \left[(t-a)\int_a^t |f'(s)|^2 ds + (b-t)\int_t^b |f'(s)|^2 ds\right] du(t) \\
&\le \frac{1}{2}\left[\frac{1}{2}(b-a)[u(b)-u(a)] + \int_a^b \left|t-\frac{a+b}{2}\right| du(t)\right]\int_a^b |f'(s)|^2 ds \\
&= \frac{1}{2}\left[(b-a)[u(b)-u(a)] - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)u(t)\,dt\right]\int_a^b |f'(s)|^2 ds.
\end{aligned} \tag{2.1}$$

The inequalities are sharp in (2.1).

Proof. Since $f(a) = f(b) = 0$, hence $f(t) = \int_a^t f'(s)\,ds$ and $f(t) = -\int_t^b f'(s)\,ds$ for $t \in [a,b]$.

We have by Cauchy-Bunyakowsky-Schwarz inequality for the Riemann-Stieltjes integral, that

$$\begin{aligned}
&\int_a^b |f(t)|^2\,du(t) \\
&= \int_a^b |f(t)|\,|f(t)|\,du(t) = \int_a^b \left|\int_a^t f'(s)\,ds\right| \left|\int_t^b f'(s)\,ds\right| du(t) \\
&\le \int_a^b \left(\int_a^t |f'(s)|\,ds\right)\left(\int_t^b |f'(s)|\,ds\right) du(t) \\
&\le \left(\int_a^b \left(\int_a^t |f'(s)|\,ds\right)^2 du(t)\right)^{1/2} \left(\int_a^b \left(\int_t^b |f'(s)|\,ds\right)^2 du(t)\right).
\end{aligned}$$

Using the arithmetic mean-geometric mean inequality

$$\sqrt{\alpha\beta} \le \frac{1}{2}(\alpha+\beta),\ \ \alpha,\beta \ge 0,$$

we have

$$\begin{aligned}
&\left(\int_a^b \left(\int_a^t |f'(s)|\,ds\right)^2 du(t)\right)^{1/2} \left(\int_a^b \left(\int_t^b |f'(s)|\,ds\right)^2 du(t)\right) \\
&\le \frac{1}{2}\left[\int_a^b \left(\int_a^t |f'(s)|\,ds\right)^2 du(t) + \int_a^b \left(\int_t^b |f'(s)|\,ds\right)^2 du(t)\right] \\
&= \frac{1}{2}\int_a^b \left[\left(\int_a^t |f'(s)|\,ds\right)^2 + \left(\int_t^b |f'(s)|\,ds\right)^2\right] du(t) =: A.
\end{aligned}$$

By Cauchy-Bunyakowsky-Schwarz inequality we also have

$$\left(\int_a^t |f'(s)|\,ds\right)^2 \le (t-a)\int_a^t |f'(s)|^2\,ds$$

and

$$\left(\int_t^b |f'(s)|\,ds\right)^2 \le (b-t)\int_t^b |f'(s)|^2\,ds,$$

and then

$$\begin{aligned}
&\left(\int_a^t |f'(s)|\,ds\right)^2 + \left(\int_t^b |f'(s)|\,ds\right)^2 \\
&\le (t-a)\int_a^t |f'(s)|^2\,ds + (b-t)\int_t^b |f'(s)|^2\,ds \\
&\le \max\{t-a, b-t\}\int_a^b |f'(s)|^2\,ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
A &\leq \frac{1}{2}\int_a^b \left[(t-a)\int_a^t \left|f'(s)\right|^2 ds + (b-t)\int_t^b \left|f'(s)\right|^2 ds\right] du(t) \\
&\leq \frac{1}{2}\int_a^b \left[\max\{t-a,b-t\}\int_a^b \left|f'(s)\right|^2 ds\right] du(t) \\
&= \frac{1}{2}\int_a^b \left|f'(s)\right|^2 ds \int_a^b \left[\max\{t-a,b-t\}\right] du(t) \\
&= \frac{1}{2}\int_a^b \left|f'(s)\right|^2 ds \int_a^b \left(\frac{1}{2}(b-a) + \left|t-\frac{a+b}{2}\right|\right) du(t) \\
&= \frac{1}{2}\int_a^b \left|f'(s)\right|^2 ds \left[\frac{1}{2}(b-a)\left[u(b)-u(a)\right] + \int_a^b \left|t-\frac{a+b}{2}\right| du(t)\right],
\end{aligned}$$

which proves the desired result (2.1).

Using integration by parts for the Riemann-Stieltjes integral, we have further

$$\begin{aligned}
&\int_a^b \left|t-\frac{a+b}{2}\right| du(t) \\
&= \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2}-t\right) du(t) + \int_{\frac{a+b}{2}}^b \left(t-\frac{a+b}{2}\right) du(t) \\
&= \left(\frac{a+b}{2}-t\right) u(t)\Big|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} u(t)\,dt \\
&+ \left(t-\frac{a+b}{2}\right) u(t)\Big|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b u(t)\,dt \\
&= -\frac{b-a}{2}u(a) + \int_a^{\frac{a+b}{2}} u(t)\,dt + \frac{b-a}{2}u(b) - \int_{\frac{a+b}{2}}^b u(t)\,dt.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{2}(b-a)\left[u(b)-u(a)\right] + \int_a^b \left|t-\frac{a+b}{2}\right| du(t) \\
&= \frac{1}{2}(b-a)\left[u(b)-u(a)\right] \\
&- \frac{b-a}{2}u(a) + \int_a^{\frac{a+b}{2}} u(t)\,dt + \frac{b-a}{2}u(b) - \int_{\frac{a+b}{2}}^b u(t)\,dt \\
&= (b-a)\left[u(b)-u(a)\right] + \int_a^{\frac{a+b}{2}} u(t)\,dt - \int_{\frac{a+b}{2}}^b u(t)\,dt \\
&= (b-a)\left[u(b)-u(a)\right] - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right) u(t)\,dt,
\end{aligned}$$

and the equality is proved.

Now, if f is real valued, then integrating by parts we have

$$\int_a^b f^2(t)\,du(t) = f^2(t)\,u(t)\big|_a^b - \int_a^b 2f(t)\,f'(t)\,u(t)\,dt$$
$$= -2\int_a^b f(t)\,f'(t)\,u(t)\,dt.$$

Now, consider the functions

$$f(t) = \begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right] \\ \\ b-t, & t\in\left(\frac{a+b}{2},b\right] \end{cases}$$

and $u(t) = \operatorname{sgn}\left(t-\frac{a+b}{2}\right)$, $t\in[a,b]$. The function f is absolutely continuous on $[a,b]$ and u is monotonic nondecreasing on $[a,b]$. Also

$$f'(t) = \begin{cases} 1, & t\in\left(a,\frac{a+b}{2}\right) \\ \\ -1, & t\in\left(\frac{a+b}{2},b\right), \end{cases}$$

which gives that $\int_a^b |f'(t)|^2\,dt = b-a$.

Therefore

$$\frac{1}{2}\left[(b-a)\left[u(b)-u(a)\right] - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)u(t)\,dt\right]\int_a^b |f'(s)|^2\,ds$$
$$= \frac{1}{2}\left[2(b-a) - \int_a^b\left(\operatorname{sgn}\left(t-\frac{a+b}{2}\right)\right)^2 dt\right](b-a) = \frac{1}{2}(b-a)^2.$$

Also,

$$-2\int_a^b f(t)\,f'(t)\,u(t)\,dt = 2\int_a^{\frac{a+b}{2}}(t-a)\,dt + 2\int_{\frac{a+b}{2}}^b (b-t)\,dt$$
$$= \frac{(b-a)^2}{4} + \frac{(b-a)^2}{4} = \frac{1}{2}(b-a)^2.$$

This example gives in all sides of (2.1) the same quantity $\frac{1}{2}(b-a)^2$ which proves the sharpness of all inequalities in (2.1). □

Corollary 1. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous with* $f(a) = f(b) = 0$ *and* $f'\in L_2[a,b]$. *If* $w:[a,b]\to(0,\infty)$ *is integrable with* $\int_a^b w(s)\,ds = 1$, *then*

$$\int_a^b |f(t)|^2\,w(t)\,dt \tag{2.2}$$
$$\le \frac{1}{2}\int_a^b\left[(t-a)\int_a^t |f'(s)|^2\,ds + (b-t)\int_t^b |f'(s)|^2\,ds\right]w(t)\,dt$$
$$\le \frac{1}{2}\left[\frac{1}{2}(b-a) + \int_a^b\left|t-\frac{a+b}{2}\right|w(t)\,dt\right]\int_a^b |f'(s)|^2\,ds.$$

The proof follows by (2.1) on taking $u(t) = \int_a^t w(s)\,ds$, $t\in[a,b]$.

The following lemma was obtained by the author in 2007, [7] and is of interest in itself as well (see also [6]):

Lemma 1. *If $p:[a,b]\to\mathbb{C}$ is continuous on $[a,b]$ and $v:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b p(t)\,dv(t)\right| \le \int_a^b |p(t)|\,dV(t) \le \left(\int_a^b |p(t)|^p\,dV(t)\right)^{1/p}\left(\bigvee_a^b(v)\right)^{1/q} \le \max_{t\in[a,b]}|p(t)|\bigvee_a^b(v), \tag{2.3}$$

where $V(t):=\bigvee_a^t(v)$ is the total variation of v on $[a,t]$ with $t\in[a,b]$.

The function V is nondecreasing on $[a,b]$ with $V(a)=0$ and $V(b)=\bigvee_a^b(v)$.

Theorem 3. *Assume that $h:[a,b]\to\mathbb{C}$ is absolutely continuous with $h(a)=h(b)=0$ and $h'\in L_2[a,b]$. If $v:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$\left|\int_a^b h(t)\,dv(t)\right|^2 \le \frac{1}{2}\left[(b-a)\bigvee_a^b(v)-\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right] \times\bigvee_a^b(v)\int_a^b |h'(s)|^2\,ds. \tag{2.4}$$

Proof. Using Lemma 1 and the Cauchy-Bunyakowsky-Schwarz integral inequality for the Riemann-Stieltjes integral, we have

$$\left|\int_a^b h(t)\,dv(t)\right|^2 \le \left(\int_a^b |h(t)|\,dV(t)\right)^2 \le [V(b)-V(a)]\int_a^b |h(t)|^2\,dV(t) = \bigvee_a^b(v)\int_a^b |h(t)|^2\,dV(t) \tag{2.5}$$

From (2.1) we get

$$\begin{aligned}\int_a^b |h(t)|^2\,dV(t) &= \frac{1}{2}\left[(b-a)[V(b)-V(a)]-\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)V(t)\,dt\right]\int_a^b |h'(s)|^2\,ds \\ &= \frac{1}{2}\left[(b-a)\bigvee_a^b(v)-\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right]\int_a^b |h'(s)|^2\,ds,\end{aligned} \tag{2.6}$$

which proves the desired result (2.4). □

Corollary 2. *Assume that* $h:[a,b]\to\mathbb{C}$ *is absolutely continuous with* $h(a)=h(b)=0$ *and* $h'\in L_2[a,b]$. *If* $g:[a,b]\to\mathbb{C}$ *is continuous on* $[a,b]$, *then*

$$\begin{aligned}(2.7)\qquad &\left|\int_a^b h(t)\,g(t)\,dt\right|^2\\ &\leq \frac{1}{2}\left[(b-a)\int_a^b |g(s)|\,ds-\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\left(\int_a^t |g(s)|\,ds\right)dt\right]\\ &\times\int_a^b |g(s)|\,ds\int_a^b |h'(s)|^2\,ds.\end{aligned}$$

It follows by (2.4) for $v(t)=\int_a^t g(s)\,ds$, $t\in[a,b]$.

3. Trapezoid and Grüss' Type Inequalities

We have the following equalities:

Lemma 2. *Let* $f,\ v:[a,b]\to\mathbb{C}$ *be such that one is continuous and the other is of bounded variation. Then*

$$\begin{aligned}(3.1)\qquad &T(f,v;[a,b])\\ &:=\int_a^b f(t)\,dv(t)\\ &-f(b)\left[v(b)-\frac{1}{b-a}\int_a^b v(t)\,dt\right]-f(a)\left[\frac{1}{b-a}\int_a^b v(t)\,dt-v(a)\right]\\ &=\frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt-\int_a^b v(t)\,df(t)\\ &=\int_a^b\left[f(t)-\frac{f(a)(b-t)+f(b)(t-a)}{b-a}\right]dv(t).\end{aligned}$$

Proof. Integrating by parts in the Riemann-Stieltjes integral, we have

$$\begin{aligned}&\int_a^b\left[f(t)-\frac{f(a)(b-t)+f(b)(t-a)}{b-a}\right]dv(t)\\ &=\int_a^b f(t)\,dv(t)-\int_a^b\frac{f(a)(b-t)+f(b)(t-a)}{b-a}dv(t)\end{aligned}$$

$$\begin{aligned}&=\int_a^b f(t)\,dv(t)-\left.\frac{f(a)(b-t)+f(b)(t-a)}{b-a}v(t)\right|_a^b\\ &+\frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt\\ &=\int_a^b f(t)\,dv(t)-f(b)\,v(b)+f(a)\,v(a)+\frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt\\ &=\int_a^b f(t)\,dv(t)\\ &-f(b)\left[v(b)-\frac{1}{b-a}\int_a^b v(t)\,dt\right]-f(a)\left[\frac{1}{b-a}\int_a^b v(t)\,dt-v(a)\right].\end{aligned}$$

Integrating by parts again, we also have

$$\int_a^b f(t)\,dv(t) - f(b)\,v(b) + f(a)\,v(a) + \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt$$
$$= \frac{f(b)-f(a)}{b-a}\int_a^b v(t)\,dt - \int_a^b v(t)\,df(t).$$

These prove the required identities. □

Theorem 4. *Assume that $f:[a,b]\to\mathbb{C}$ is absolutely continuous and $h'\in L_2[a,b]$. If $v:[a,b]\to\mathbb{C}$ is of bounded variation on $[a,b]$, then*

$$|T(f,v;[a,b])|^2 \le \frac{1}{2}\left[(b-a)\bigvee_a^b(v) - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right] \times \bigvee_a^b(v)\int_a^b\left|f'(t)-\frac{f(b)-f(b)}{b-a}\right|^2 ds. \tag{3.2}$$

Proof. Let

$$h(t) := f(t) - \frac{f(a)(b-t)+f(b)(t-a)}{b-a},\ \ t\in[a,b].$$

Observe that $h(a)=h(b)=0$, $h'\in L_2[a,b]$ and

$$h'(t) = f'(t) - \frac{f(b)-f(b)}{b-a},\ \ t\in(a,b).$$

Then by (2.4) we get

$$\left|\int_a^b\left(f(t)-\frac{f(a)(b-t)+f(b)(t-a)}{b-a}\right)dv(t)\right|^2$$
$$\le \frac{1}{2}\left[(b-a)\bigvee_a^b(v) - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right]$$
$$\times\bigvee_a^b(v)\int_a^b\left|f'(t)-\frac{f(b)-f(b)}{b-a}\right|^2 ds.$$

By making use of representation (3.1) we obtain the desired result (3.2). □

Corollary 3. *Assume that $f:[a,b]\to\mathbb{C}$ is absolutely continuous and $h'\in L_2[a,b]$. If $w:[a,b]\to(0,\infty)$ is integrable with $\int_a^b w(s)\,ds=1$, then*

$$\left|\int_a^b f(t)\,w(t)\,dt - \frac{f(a)(b-E(w,[a,b]))+f(b)(E(w,[a,b])-a)}{b-a}\right|^2 \le \frac{1}{2}\left[\frac{1}{2}(b-a)+\int_a^b\left|t-\frac{a+b}{2}\right|w(t)\,dt\right]\int_a^b\left|f'(t)-\frac{f(b)-f(b)}{b-a}\right|^2 ds, \tag{3.3}$$

where $E(w,[a,b]) := \int_a^b tw(t)\,dt$.

For a function $h:[a,b]\to\mathbb{C}$ we consider the *symmetrical transform* $\widehat{h}$ defined by

$$\widehat{h}(t) := \frac{1}{2}\left[h(t)+h(a+b-t)\right],\ \ t\in[a,b]$$

and the *antisymmetrical transform* $\widetilde{h}$ defined by

$$\widetilde{h}(t) := \frac{1}{2}\left[h(t) - h(a+b-t)\right], \ t \in [a,b].$$

Proposition 1. *Assume that f is absolutely continuous on $[a,b]$ and v is of bounded variation on $[a,b]$, then*

$$B(f,v;[a,b]) := \int_a^b \widehat{f}(t)\,dv(t) - \frac{f(a)+f(b)}{2}\left[v(b)-v(a)\right] = \int_a^b f(t)\,d\widetilde{v}(t) - \frac{f(a)+f(b)}{2}\left[v(b)-v(a)\right] \tag{3.4}$$

and we have the inequality

$$|B(f,v;[a,b])| \le \frac{1}{2}\left[(b-a)\bigvee_a^b(v) - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right] \times \bigvee_a^b(v)\int_a^b \left|\widetilde{f}'(s)\right|^2 ds. \tag{3.5}$$

Proof. Consider the function $g:[a,b]\to\mathbb{C}$ defined by

$$g(t) := \widehat{f}(t) - \frac{f(a)+f(b)}{2}, \ t\in[a,b].$$

Then g is absolutely continuous on $[a,b]$, $g(a)=g(b)=0$,

$$g'(t) = \frac{f'(t)-f'(a+b-t)}{2} = \widetilde{f}'(t) \text{ for a.e. } t\in[a,b]$$

and

$$\int_a^b g(t)\,dv(t) = \int_a^b \left(\widehat{f}(t) - \frac{f(a)+f(b)}{2}\right)dv(t) = \int_a^b \widehat{f}(t)\,dv(t) - \frac{f(a)+f(b)}{2}\left[v(b)-v(a)\right].$$

Using the change of variable formula for the Riemann-Stieltjes integral, see for instance [2, p. 144], we have

$$\begin{aligned}
\int_a^b \widehat{f}(t)\,dv(t) &= \frac{1}{2}\int_a^b \left[f(t)+f(a+b-t)\right]dv(t)\\
&= \frac{1}{2}\left[\int_a^b f(t)\,dv(t) + \int_a^b f(a+b-t)\,dv(t)\right]\\
&= \frac{1}{2}\left[\int_a^b f(t)\,dv(t) + \int_b^a f(u)\,dv(a+b-u)\right]\\
&= \frac{1}{2}\left[\int_a^b f(t)\,dv(t) - \int_a^b f(u)\,dv(a+b-u)\right] = \int_a^b f(t)\,d\widetilde{v}(t),
\end{aligned}$$

which proves the equality (3.4).

Now by using (2.4) for $h(t)=\widehat{f}(t)-\frac{f(a)+f(b)}{2}$, $t\in[a,b]$, we get

$$\begin{aligned}&\left|\int_a^b\left[\widehat{f}(t)-\frac{f(a)+f(b)}{2}\right]dv(t)\right|^2\\&\leq\frac{1}{2}\left[(b-a)\bigvee_a^b(v)-\int_a^b\operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right]\\&\times\bigvee_a^b(v)\int_a^b\left|\widetilde{f'}(s)\right|^2ds,\end{aligned}$$

which proves the desired result (3.5). □

Corollary 4. *Assume that* $f:[a,b]\to\mathbb{C}$ *is absolutely continuous and* $h'\in L_2[a,b]$. *If* $w:[a,b]\to(0,\infty)$ *is integrable with* $\int_a^b w(s)\,ds=1$, *then*

$$\begin{aligned}&\left|\int_a^b\widehat{f}(t)\,w(t)\,dt-\frac{f(a)+f(b)}{2}\right|\\&\leq\frac{1}{2}\left[\frac{1}{2}(b-a)+\int_a^b\left|t-\frac{a+b}{2}\right|w(t)\,dt\right]\int_a^b\left|\widetilde{f'}(s)\right|^2ds.\end{aligned}\tag{3.6}$$

For two *Lebesgue integrable* functions $f,\ g:[a,b]\to\mathbb{R}$, consider the *Čebyšev functional*:

$$C(f,g):=\frac{1}{b-a}\int_a^b f(t)g(t)dt-\frac{1}{(b-a)^2}\int_a^b f(t)dt\int_a^b g(t)dt.\tag{3.7}$$

In 1935, Grüss [11] showed that

$$|C(f,g)|\leq\frac{1}{4}(M-m)(N-n),\tag{3.8}$$

provided that there exists the real numbers m, M, n, N such that

$$m\leq f(t)\leq M\quad\text{and}\quad n\leq g(t)\leq N\quad\text{for a.e. }t\in[a,b].\tag{3.9}$$

The constant $\frac{1}{4}$ is best possible in (2.3) in the sense that it cannot be replaced by a smaller quantity.

Theorem 5. *Assume that* $h:[a,b]\to\mathbb{C}$ *is integrable on* $[a,b]$ *and* $v:[a,b]\to\mathbb{C}$ *of bounded variation on* $[a,b]$, *then*

$$\begin{aligned}|C(h,v)|^2&\leq\frac{1}{2}\left[\bigvee_a^b(v)-\frac{1}{(b-a)}\int_a^b\operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right]\\&\times\bigvee_a^b(v)\left(\frac{1}{b-a}\int_a^b|h(t)|^2-\left|\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2\right).\end{aligned}\tag{3.10}$$

Proof. Using the integration by parts for the Riemann-Stieltjes integral, we have

$$\begin{aligned}
&\int_a^b \left(\int_a^t h(s)\,ds - \frac{t-a}{b-a}\int_a^b h(s)\,ds \right) dv(t) \\
&= \left(\int_a^t h(s)\,ds - \frac{t-a}{b-a}\int_a^b h(s)\,ds \right) v(t) \Big|_a^b \\
&- \int_a^b v(t)\, d\left(\int_a^t h(s)\,ds - \frac{t-a}{b-a}\int_a^b h(s)\,ds \right) \\
&= -\int_a^b v(t)\,h(t)\,dt + \frac{1}{b-a}\int_a^b h(s)\,ds \int_a^b v(t)\,dt,
\end{aligned}$$

which gives that

$$C(h,v) = \frac{1}{b-a}\int_a^b \left(\frac{t-a}{b-a}\int_a^b h(s)\,ds - \int_a^t h(s)\,ds \right) dv(t)\,. \tag{3.11}$$

Consider

$$g(t) := \frac{t-a}{b-a}\int_a^b h(s)\,ds - \int_a^t h(s)\,ds,\ \ t \in [a,b]\,,$$

then g is absolutely continuous, $g(a) = g(b) = 0$,

$$g'(t) := \frac{1}{b-a}\int_a^b h(s)\,ds - h(t)\,,\ \ t \in [a,b]$$

and by (2.4) we get

$$\begin{aligned}
&\left| \int_a^b \left(\frac{t-a}{b-a}\int_a^b h(s)\,ds - \int_a^t h(s)\,ds \right) dv(t) \right|^2 \\
&\le \frac{1}{2}\left[(b-a)\bigvee_a^b (v) - \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2} \right) \bigvee_a^t (v)\,dt \right] \\
&\times \bigvee_a^b (v) \int_a^b \left| \frac{1}{b-a}\int_a^b h(s)\,ds - h(t) \right|^2 dt.
\end{aligned} \tag{3.12}$$

Since

$$\begin{aligned}
&\frac{1}{(b-a)}\int_a^b \left| \frac{1}{b-a}\int_a^b h(s)\,ds - h(t) \right|^2 dt \\
&= \frac{1}{b-a}\int_a^b |h(t)|^2 - \left| \frac{1}{b-a}\int_a^b h(s)\,ds \right|^2,
\end{aligned}$$

hence

$$\begin{aligned}
&\left|\int_a^b \left(\frac{t-a}{b-a}\int_a^b h(s)\,ds - \int_a^t h(s)\,ds\right) dv(t)\right|^2 \\
&\le \frac{1}{2}(b-a)\left[(b-a)\bigvee_a^b(v) - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right] \\
&\times \bigvee_a^b(v)\left(\frac{1}{b-a}\int_a^b |h(t)|^2 - \left|\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&|C(h,v)|^2 \\
&= \frac{1}{(b-a)^2}\left|\int_a^b \left(\frac{t-a}{b-a}\int_a^b h(s)\,ds - \int_a^t h(s)\,ds\right) dv(t)\right|^2 \\
&\le \frac{1}{(b-a)^2}\frac{1}{2}(b-a)\left[(b-a)\bigvee_a^b(v) - \int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right] \\
&\times \bigvee_a^b(v)\left(\frac{1}{b-a}\int_a^b |h(t)|^2 - \left|\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2\right) \\
&= \frac{1}{2}\left[\bigvee_a^b(v) - \frac{1}{(b-a)}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t(v)\,dt\right] \\
&\times \bigvee_a^b(v)\left(\frac{1}{b-a}\int_a^b |h(t)|^2 - \left|\frac{1}{b-a}\int_a^b h(s)\,ds\right|^2\right)
\end{aligned}$$

and the inequality (3.10) is proved. □

4. Applications for Selfadjoint and Unitary Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H;\langle\cdot,\cdot\rangle)$. Let $A\in\mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda\in\mathbb{R}$ as follows

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda\in\mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{4.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see for instance [13, p. 256]:

Theorem 6 (Spectral Representation Theorem). *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a=\min\{\lambda\,|\lambda\in\operatorname{Sp}(A)\}=:\min\operatorname{Sp}(A)$ and*

$b = \max\{\lambda \,|\, \lambda \in \mathrm{Sp}(A)\} =: \max \mathrm{Sp}(A)$. *Then there exists a family of projections* $\{E_\lambda\}_{\lambda\in\mathbb{R}}$, *called the spectral family of* A, *with the following properties*

a) $E_\lambda \le E_{\lambda'}$ *for* $\lambda \le \lambda'$;

b) $E_{a-0} = 0, E_b = 1_H$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$*;*

c) *We have the representation*

$$A = \int_{a-0}^{b} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ *defined on* $\mathbb{R}$ *there exists a unique operator* $\varphi(A) \in \mathcal{B}(H)$ *such that for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *satisfying the inequality*

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi(\lambda_k') \left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < ... < \lambda_{n-1} < \lambda_n = b, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda, \tag{4.2}$$

where the integral is of Riemann-Stieltjes type.

Corollary 5. *With the assumptions of Theorem 6 for* A, E_λ *and* φ *we have the representations*

$$\varphi(A)x = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(A)x, y\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \quad \text{for all } x,\ y \in H. \tag{4.3}$$

In particular,

$$\langle \varphi(A)x, x\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)x\|^2 = \int_{a-0}^{b} |\varphi(\lambda)|^2\, d\|E_\lambda x\|^2 \quad \text{for all } x \in H.$$

We have the following result:

Theorem 7. *Assume that* $f : I \to \mathbb{C}$ *is locally absolutely continuous with* $[a,b] \subset \mathring{I}$ *(the interior of* I*),* $f(a) = f(b) = 0$ *and* $f' \in L_2[a,b]$. *Let* A *be a bounded* *selfadjoint operator on the Hilbert space* H *and let* $a = \min\{\lambda \,|\lambda \in \mathrm{Sp}(A)\} =: \min \mathrm{Sp}(A)$ *and* $b = \max\{\lambda \,|\lambda \in \mathrm{Sp}(A)\} =: \max \mathrm{Sp}(A)$. *Then*

$$|f(A)|^2 \le \frac{1}{2}\left(\int_a^b |f'(s)|^2 ds\right)\left[\frac{1}{2}(b-a)1_H + \left|A - \frac{a+b}{2}1_H\right|\right] \le \frac{1}{2}(b-a)\left(\int_a^b |f'(s)|^2 ds\right)1_H, \tag{4.4}$$

in the operator order of $\mathcal{B}(H)$.

Proof. Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of A, $x \in H$ and small $\varepsilon > 0$. Consider the function

$$f_\varepsilon(t) := \begin{cases} 0, & t \in [a-\varepsilon, a) \\ f(t), & t \in [a,b]. \end{cases},$$

Then $f_\varepsilon(a-\varepsilon) = f(b) = 0$ and by (2.1) on the interval $[a-\varepsilon, b]$ we get

$$\int_{a-\varepsilon}^b |f_\varepsilon(t)|^2 d\langle E_t x, x\rangle \le \frac{1}{2}\left[\frac{1}{2}(b-a+\varepsilon)\left[\langle E_b x, x\rangle - \langle E_{a-\varepsilon}x, x\rangle\right] + \int_{a-\varepsilon}^b \left|t - \frac{a-\varepsilon+b}{2}\right| d\langle E_t x, x\rangle\right]\int_{a-\varepsilon}^b |f'(s)|^2 ds. \tag{4.5}$$

By taking the limit over $\varepsilon \to 0+$ in (4.5) and using Corollary 5, then we get

$$\left\langle |f(A)|^2 x, x\right\rangle \le \frac{1}{2}\left[\frac{1}{2}(b-a)\langle x, x\rangle + \left\langle\left|A - \frac{a+b}{2}1_H\right| x, x\right\rangle\right]\int_a^b |f'(s)|^2 ds,$$

namely

$$\left\langle |f(A)|^2 x, x\right\rangle \le \frac{1}{2}\left[\left\langle\left[\frac{1}{2}(b-a)1_H + \left|A - \frac{a+b}{2}1_H\right|\right] x, x\right\rangle\right]\int_a^b |f'(s)|^2 ds,$$

for all $x \in H$, which is equivalent, in the operator order, to (4.4). □

We say that the bounded linear operator $U : H \to H$ on the Hilbert space H is *unitary* iff $U^* = U^{-1}$.

It is well known that (see for instance [13, p. 275-p. 276]), if U is a unitary operator, then there exists a family of *projections* $\{E_\lambda\}_{\lambda\in[0,2\pi]}$, called the *spectral family* of U with the following properties:

a) $E_\lambda \le E_\mu$ for $0 \le \lambda \le \mu \le 2\pi$;
b) $E_0 = 0$ and $E_{2\pi} = 1_H$ (the *identity operator* on H);
c) $E_{\lambda+0} = E_\lambda$ for $0 \le \lambda < 2\pi$;
d) $U = \int_0^{2\pi} e^{i\lambda} dE_\lambda$, where the integral is of Riemann-Stieltjes type.

Moreover, if $\{F_\lambda\}_{\lambda\in[0,2\pi]}$ is a family of projections satisfying the requirements a)-d) above for the operator U, then $F_\lambda = E_\lambda$ for all $\lambda \in [0, 2\pi]$.

Also, for every continuous complex valued function $g : \mathcal{C}(0,1) \to \mathbb{C}$ on the complex unit circle $\mathcal{C}(0,1)$, we have

$$g(U) = \int_0^{2\pi} g\left(e^{i\lambda}\right) dE_\lambda \tag{4.6}$$

where the integral is taken in the Riemann-Stieltjes sense.

In particular, we have the equalities

$$\langle g(U)x, y\rangle = \int_0^{2\pi} g\left(e^{i\lambda}\right) d\langle E_\lambda x, y\rangle \tag{4.7}$$

and

$$\|g(U)x\|^2 = \int_0^{2\pi} \left|g\left(e^{i\lambda}\right)\right|^2 d\|E_\lambda x\|^2 = \int_0^{2\pi} \left|g\left(e^{i\lambda}\right)\right|^2 d\langle E_\lambda x, x\rangle, \tag{4.8}$$

for any $x, y \in H$.

Consider the function $f(z) = \operatorname{Log}(z)$ where $\operatorname{Log}(z) = \ln|z| + i\operatorname{Arg}(z)$ and $\operatorname{Arg}(z)$ is such that $-\pi < \operatorname{Arg}(z) \le \pi$. Log is called the *"principal branch"* of the complex logarithmic function. The function f is analytic on all of $\mathbb{C}_\ell := \mathbb{C}\setminus\{x + iy : x \le 0,\ y = 0\}$ and

$$f^{(k)}(z) = \frac{(-1)^{k-1}(k-1)!}{z^k},\ k \ge 1,\ z \in \mathbb{C}_\ell.$$

We also have the identity

$$\operatorname{Log}\left(e^{is}\right) = is \text{ for } -\pi < s \le \pi.$$

Theorem 8. *Assume that $g : \mathcal{C}(0,1) \to \mathbb{C}$ is of class C^1 on the circle $\mathcal{C}(0,1)$ with $g(1) = 0$. If $U : H \to H$ is unitary, then*

$$\begin{aligned} |g(U)|^2 &\le \frac{1}{2}\left(\int_0^{2\pi} \left|g'\left(e^{is}\right)\right|^2 ds\right)\left(\pi 1_H + |\operatorname{Log}(U)|\right) \\ &\le \pi\left(\int_0^{2\pi} \left|g'\left(e^{is}\right)\right|^2 ds\right) 1_H. \end{aligned} \tag{4.9}$$

Proof. Let $\{E_\lambda\}_{\lambda\in[0,2\pi]}$ be the *spectral family* of U and $x \in H$. Consider the function $f(t) = g\left(e^{it}\right),\ t \in [0, 2\pi]$. We have that $f(0) = g(1) = f(2\pi) = 0$ and $f'(t) = ig'\left(e^{it}\right),\ t \in (0, 2\pi)$. By (2.1) we then have

$$\begin{aligned} &\int_0^{2\pi} \left|g\left(e^{it}\right)\right|^2 d\langle E_t x, x\rangle \\ &\le \frac{1}{2}\left[\pi\langle 1_H x, x\rangle + \int_0^{2\pi} |t - \pi|\, d\langle E_t x, x\rangle\right] \int_0^{2\pi} \left|g'\left(e^{is}\right)\right|^2 ds \\ &\le \pi\langle 1_H x, x\rangle \int_0^{2\pi} \left|g'\left(e^{is}\right)\right|^2 ds. \end{aligned}$$

Now, observe that

$$\begin{aligned} \int_0^{2\pi} |t - \pi|\, d\langle E_t x, x\rangle &= \int_0^{2\pi} |i(t - \pi)|\, d\langle E_t x, x\rangle = \int_0^{2\pi} \left|\operatorname{Log}\left(e^{i(t-\pi)}\right)\right| d\langle E_t x, x\rangle \\ &= \int_0^{2\pi} \left|\operatorname{Log}\left(e^{it}\right)\right| d\langle E_t x, x\rangle = \langle|\operatorname{Log}(U)|\, x, x\rangle \end{aligned}$$

and by (4.8) we derive

$$\begin{aligned} \|g(U)x\|^2 &\le \frac{1}{2}\left[\pi\langle 1_H x, x\rangle + \langle|\operatorname{Log}(U)|\, x, x\rangle\right] \int_0^{2\pi} \left|g'\left(e^{is}\right)\right|^2 ds \\ &\le \left\langle\left(\pi\int_0^{2\pi} \left|g'\left(e^{is}\right)\right|^2 ds\right) 1_H x, x\right\rangle. \end{aligned}$$

Since

$$\|g(U)x\|^2 = \langle g(U)x, g(U)x\rangle = \langle (g(U))^* g(U)x, x\rangle = \left\langle |g(U)|^2 x, x\right\rangle,$$

hence

$$\begin{aligned}\left\langle |g(U)|^2 x, x\right\rangle &\le \left\langle \frac{1}{2}\left(\int_0^{2\pi} \left|g'\left(e^{is}\right)\right|^2 ds\right)(\pi 1_H + |\mathrm{Log}\,(U)|)\,x, x\right\rangle \\ &\le \left\langle \left(\pi \int_0^{2\pi} \left|g'\left(e^{is}\right)\right|^2 ds\right) 1_H x, x\right\rangle\end{aligned}$$

for all $x \in H$, which is, in the operator order, the desired inequality (4.9). □

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